ON THE *k*-FREE VALUES OF THE POLYNOMIAL $xy^k + C$

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Abstract. Consider the polynomial $f(x, y) = xy^k + C$ for $k \geq 2$ and any nonzero integer constant C. We derive an asymptotic formula for the k-free values of $f(x, y)$ when $x, y \leq H$. We also prove a similar result for the k-free values of $f(p, q)$ when $p, q \leq H$ are primes, thus extending Erdős' conjecture for our specific polynomial. The strongest tool we use is a recent generalization of the determinant method due to Reuss.

1. Introduction

Let k and n be integers and $k \geq 2$. We say that n is *k-free* if there is no prime p such that $p^k \mid n$. Consider the irreducible polynomial $f(n) \in \mathbb{Z}[x]$ of degree d. Let also $f(n)$ have no fixed *k-th power prime divisor*, i.e. there is no prime p for which $p^k \mid f(n)$ for every $n \in \mathbb{Z}$. Then we expect that the set $f(\mathbb{Z}) = \{f(n), n \in \mathbb{Z}\}$ contains infinitely many k-free values. The first one who obtained a result in this direction was Ricci [19], who derived an asymptotic formula for the quantity

$$(1.1) \qquad \#\{n \leq H : f(n) \text{ is } k\text{-free}\}$$

when $k \geq d$. Later, Erdős [3] proved the conjecture in the case $k = d - 1$ and $d \geq 3$, and Hooley [11] obtained an asymptotic formula for each such k. Some of the recent results on this conjecture are asymptotic formulas for the k-free values due to Browning [1] when $k \geq (3d+1)/4$ and $d \geq 3$, and to Heath-Brown [9] for the polynomial $f(x) = x^d + C$ when $k \geq (5d+3)/9$ for $d \geq 3$. In general the problem is harder with d increasing, but easier the closer k is to d.

Similarly for irreducible multi-variable polynomials $f(t_1, \ldots, t_r)$ of degree d, which do not have a fixed k-th power prime divisor, we conjecture that the quantity

$$(1.2) \qquad \#\big\{ t_1, \ldots, t_r \leq H : f(t_1, \ldots, t_r) \text{ is } k\text{-free} \big\}$$

has a positive density. Recent results in this direction are due to Tolev [20] and Le Boudec [15] who investigate carefully asymptotic formulas for specific polynomials. Earlier result of Poonen [16], which lays on the *abc* conjecture, shows that the number of square-free values of any $f(t_1, \ldots, t_r)$ has a positive density, which, however, is differently defined than the traditional density which arises when we evaluate asymptotically (1.2).

Naturally the most extensively studied multi-variate polynomials are the bivariate polynomials, where the situation differs in the homogeneous and inhomogeneous case. While the power-free values of binary forms were investigated already by Greaves [5], it is only recently that results were obtained for inhomogeneous bivariate polynomials by Hooley [13]. Later Browning [1] achieved the smallest $k > 39d/64$ up to date. Note that the latter papers derive not an asymptotic formula but a positive lower bound of the expected true order of magnitude for the quantity (1.2). Nevertheless, in the case $k = d - 1$ for bivariate polynomials, as in Hooley's papers [11] and [12] for univariate polynomials, it is expected that we can derive an asymptotic formula.

In this paper we consider the inhomogeneous polynomial $f(x, y) = xy^k + C$ for $k \geq 2$ and any nonzero integer constant C. We derive an asymptotic formula for its k-free values when $x, y \leq H$.

THEOREM 1. *Let $f(x, y) = xy^k + C \in \mathbb{Z}[x, y]$ for $k \geq 2$ and $C \neq 0$. Let $S(H)$ count the k-free values of $f(x, y)$ when $1 \leq x, y \leq H$. Then, for some real $\delta = \delta(k, f) > 0$, we have*

$$S(H) = c_f H^2 + \mathcal{O}\big(H^{2-\delta} \big),$$

where

$$c_f = \prod_p \left(1 - \frac{\rho(p^k)}{p^{2k}} \right)$$

and

$$\rho(m) = \#\big\{ (\mu, \nu) \in (\mathbb{Z}/m\mathbb{Z})^2 : f(\mu, \nu) \equiv 0 \;(\mathrm{mod}\; m) \big\}.$$

REMARK 1. We will show that the infinite product defining c_f is convergent and is not zero. For $k \geq 3$ we can take $\delta = 1/(7k)$ and for $k = 2$ we can take $\delta = 2 - \varepsilon - G_2$, where G_k is the expression (4.8) and $\varepsilon > 0$ is arbitrary small. This way for $k = 2$ we can get an error term $\mathcal{O}(H^{1.979})$. We do

not guarantee that this way δ is the best possible, but it is close to optimal by the choices made during the proof.

The determinant method of Heath-Brown [7] has been already successfully used for counting power-free values of univariate polynomials by Heath-Brown himself in [9] and [8], by Browning [1] and Reuss [18]. In this paper we apply a major result of Reuss [17] which generalizes the approximate determinant method developed by Heath-Brown in [8]. As we reduce our problem of counting k-free values of the polynomial $f(x, y) = xy^k + C$ to the problem of counting solutions of a Diophantine equation inside a certain box, we can directly use a result of Reuss [17, Theorem 1], supplying one more application of his powerful result. Note that for our particular polynomial it would suffice to use Lemma 5 from Dietmann and Marmon [4], where again by determinant method they give upper bound of the solutions of the equation $ax^k - by^k = 1$ with restricted sizes of the variables. Reuss however resolves the more general equation $a^l x^k - b^l y^k = C$ for any integers k, l, C with $1 \leq l < k$ and $C \neq 0$.

In [3] Erdős conjectured that similarly the set $f(\mathbb{P}) = \{f(p), p \text{ prime}\}$ for a polynomial of degree d with no fixed $(d-1)$-th power prime divisor contains infinitely many $(d-1)$-free values. Heath-Brown's strategy for finding an asymptotic formula for the quantity (1.1) was successfully used for this problem as well, e.g. in [1], [9]. Finally Reuss [18] settled the conjecture for all $d \geq 3$, demonstrating the strength of this (mostly) analytic method over other approaches, e.g. of Helfgott [10].

We extend Erdős' conjecture for the set

$$f(\mathbb{P}, \mathbb{P}) = \{f(p, q), \ p \text{ and } q \text{ primes}\}$$

for our specific bivariate polynomial.

THEOREM 2. *Let $f(x, y) = xy^k + C \in \mathbb{Z}[x, y]$ for $k \geq 2$ and $C \neq 0$. Let $S'(H)$ count the k-free values of $f(p, q)$ for prime numbers $1 < p, q \leq H$. Then, for any real $K > 2$, we have the asymptotic formula*

$$S'(H) = c'_f \pi(H)^2 + \mathcal{O}\left(\frac{H^2}{(\log H)^K}\right),$$

where

$$c'_f = \prod_p \left(1 - \frac{\rho'(p^k)}{\varphi(p^k)^2}\right)$$

and

$$\rho'(m) = \#\big\{ (\mu, \nu) \in (\mathbb{Z}/m\mathbb{Z})^2 :$$
$$(\mu, m) = (\nu, m) = 1 \text{ and } f(\mu, \nu) \equiv 0 \ (\mathrm{mod} \ m)\big\}.$$

REMARK 2. We will show that the constant c_f' makes sense, i.e. the infinite product is convergent, and $c_f' \neq 0$.

The motivation for considering power-free values of the polynomial $f(x,y) = xy^k + C$ comes from our previous works [2] and [14]. There we resolve the class number one problem for real quadratic fields with discriminant $(an)^2 + 4a$. It is natural to ask if the polynomial $(an)^2 + 4a$ takes square-free values with positive density. We answer affirmatively this question for the irreducible factor $f(x,y) = xy^2 + 4$. Actually with Theorem 2 we give an asymptotic formula for the square-free values of $(xy)^2 + 4x$ when both x and y are primes, presumably a harder problem than the one when both factors x and $xy^2 + 4$ are square-free.

2. Notations

We reserve the letters p and q for primes and ε, ε for arbitrary small positive numbers, not necessarily the same in different occasions. By $[t]$ we denote the integer part of the real number t, and (a, b) denotes the greatest common divisor of the integers a and b. By $\bar{y}$ we denote the multiplicative inverse of y modulo some integer which should be clear by the context. The dependences in the Landau symbol $\mathcal{O}$ and the Vinogradov symbol $\ll$ usually will be omitted. Sometimes in the summation sign we write only the variables which we sum, but again the intervals they lie in should be clear by the context.

By $\pi(H)$ as usual we denote the prime-counting function, and by $\pi(H; q, a)$ the prime-counting function in the arithmetic progression $\{a + nq, n \geq 0\}$. The Euler function is denoted by $\varphi(n)$. By $\tau(n)$ we denote the number of divisors function. We write $x \sim X$ to say that $X < x < 2X$ and we write $x \asymp X$ to say that there exist positive constants A, B, independent of X such that $AX \leq |x| \leq BX$.

In the next two sections we prove Theorem 1, where the methods used in Section 3 are elementary and in Section 4 we formulate and apply Reuss' theorem. We prove Theorem 2 in Section 5 .

3. Proof of Theorem 1: elementary tools

Let

$$\mu_k(n) = \begin{cases} 1, & \text{if } n \text{ is } k\text{-free} \\ 0, & \text{otherwise} \end{cases}$$

be the characteristic function of the k-free numbers. Just like with the Möbius function we have the identity

$$(3.1) \qquad \mu_k(n) = \sum_{d^k|n} \mu(d).$$

In order to count the k-free values of the polynomial $f(x,y) = xy^k + C$, where $k \geq 2$ is an integer and C is a non-zero integer, and $1 \leq x, y \leq H$, we can consider the sum

$$(3.2) \qquad S := S(H) = \sum_{1 \leq x, y \leq H} \mu_k(xy^k + C).$$

Using the equation (3.1) and changing the order of summation we transform the sum S.

$$S = \sum_{1 \leq x, y \leq H} \sum_{d^k|f(x,y)} \mu(d) = \sum_{1 \leq d \leq f(H,H)^{1/k}} \mu(d) \sum_{\substack{1 \leq x, y \leq H \\ f(x,y) \equiv 0 \,(\mathrm{mod}\ d^k)}} 1$$

$$= \sum_{1 \leq d \ll H^{(k+1)/k}} \mu(d) S(d^k, H),$$

where we denote

$$(3.3) \qquad S(m, H) := \sum_{\substack{1 \leq x, y \leq H \\ f(x,y) \equiv 0 \,(\mathrm{mod}\ m)}} 1.$$

We split the sum S into four parts:

$$(3.4) \qquad S = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 := \sum_{1 \leq d \leq z_1} \mu(d) S(d^k, H)$$

for a parameter z_1, and the summation is correspondingly over $z_1 < d \leq z_2$ for S_2, $z_2 < d \leq z_3$ for S_3 and $z_3 < d \ll H^{(k+1)/k}$ for S_4. The parameters z_1, z_2, z_3 are to be chosen soon in a convenient way. Essentially the estimates of S_1, S_2 and S_4 will be trivial, and for S_3 we will apply [17, Theorem 1].

3.1. Estimation of S_1. The sum S_1 will contribute to the main term in our asymptotic formula. Introduce the notation

$$(3.5) \qquad M(\alpha, m, H) := \sum_{\substack{1 \leq x \leq H \\ x \equiv \alpha \,(\mathrm{mod}\ m)}} 1.$$

Then

$$S(m, H) = \sum_{\substack{1 \le \mu, \nu \le m \\ f(\mu, \nu) \equiv 0 \ (\text{mod } m)}} M(\mu, m, H) M(\nu, m, H) \,.$$

Clearly

$$M(\alpha, m, H) = \frac{H}{m} + \mathcal{O}(1) \,,$$

therefore

$$(3.6) \qquad S(m, H) = H^2 \frac{\rho(m)}{m^2} + \mathcal{O}\left(H \frac{\rho(m)}{m}\right) + \mathcal{O}(\rho(m)) \,,$$

where $\rho(m)$ was defined in Theorem 1. Then

$$(3.7) \qquad S_1 = \sum_{1 \le d \le z_1} \mu(d) \left[H^2 \frac{\rho(d^k)}{d^{2k}} + \mathcal{O}\left(H \frac{\rho(d^k)}{d^k}\right) + \mathcal{O}(\rho(d^k)) \right]$$

$$= H^2 \sum_{1 \le d \le z_1} \mu(d) \frac{\rho(d^k)}{d^{2k}} + \mathcal{O}\left(H \sum_{d \le z_1} \frac{\rho(d^k)}{d^k}\right) + \mathcal{O}\left(\sum_{d \le z_1} \rho(d^k)\right)$$

$$= H^2 \sum_{d=1}^{\infty} \mu(d) \frac{\rho(d^k)}{d^{2k}} - H^2 \sum_{d > z_1} \mu(d) \frac{\rho(d^k)}{d^{2k}}$$

$$+ \mathcal{O}\left(H \sum_{d \le z_1} \frac{\rho(d^k)}{d^k}\right) + \mathcal{O}\left(\sum_{d \le z_1} \rho(d^k)\right) \,.$$

Hooley's lemma [13, Lemma 1] claims that the function $\rho(m)$ is multiplicative and $\rho(p^k) = \mathcal{O}(p^{2k-2})$ for $k \ge 2$. Therefore for square-free d we have $\rho(d^k) = \mathcal{O}(d^{2k-2+\varepsilon})$ for $\varepsilon > 0$ arbitrary small. Then the infinite sum

$$(3.8) \qquad c_f := \sum_{d=1}^{\infty} \mu(d) \frac{\rho(d^k)}{d^{2k}} = \prod_p \left(1 - \frac{\rho(p^k)}{p^{2k}}\right)$$

is convergent because its common term

$$\mu(d)\rho(d^k)/d^{2k} = \mathcal{O}(d^{2k-2+\varepsilon})/d^{2k} = \mathcal{O}(d^{-2+\varepsilon}).$$

Furthermore, $c_f \ne 0$. Indeed, $c_f = 0$ only if it has a zero factor, i.e. $\rho(p^k) = p^{2k}$ for some p. Take some $\sigma \in \mathbb{Z}/p^k\mathbb{Z}$ such that $\sigma \not\equiv -C \ (\text{mod } p^k)$. Then $f(\sigma, 1) = \sigma \cdot 1^k + C \not\equiv 0 \ (\text{mod } p^k)$, so surely $\rho(p^k) < p^{2k}$.

Further, using that in all summands in (3.7) the variable d is square-free, thus again $\rho(d^k) = \mathcal{O}(d^{2k-2+\varepsilon})$, we get

$$(3.9) \quad \begin{cases} \displaystyle\sum_{d>z_1} \mu(d)\frac{\rho(d^k)}{d^{2k}} \ll \sum_{d>z_1} d^{-2+\varepsilon} \ll z_1^{-1+\varepsilon}, \\[2ex] \displaystyle\sum_{d\leq z_1} \frac{\rho(d^k)}{d^k} \ll \sum_{d\leq z_1} d^{k-2+\varepsilon} \ll z_1^{k-1+\varepsilon}, \\[2ex] \displaystyle\sum_{d\leq z_1} \rho(d^k) \ll \sum_{d\leq z_1} d^{2k-2+\varepsilon} \ll z_1^{2k-1+\varepsilon}. \end{cases}$$

Putting all these into (3.7) we get

$$(3.10) \qquad S_1 = c_f H^2 + \mathcal{O}\big(H^2 z_1^{-1+\varepsilon}\big) + \mathcal{O}\big(H z_1^{k-1+\varepsilon}\big) + \mathcal{O}\big(z_1^{2k-1+\varepsilon}\big).$$

Let us choose an appropriate value for z_1. Write $z_1 = H^\alpha$ for some real number $0 < \alpha < 1$. For the first error term in (3.10) we always have

$$2 - \alpha + \varepsilon < 2$$

as long as $\varepsilon > 0$ is small enough. In the second error term we want to have $1 + \alpha(k-1+\varepsilon) < 2$, i.e. $\alpha < 1/(k-1+\varepsilon) < 1/(k-1)$. In the third error term we require $\alpha(2k-1+\varepsilon) < 2$, i.e. $\alpha < 2/(2k-1+\varepsilon) < 1/(k-1)$. Then $\alpha = 1/k$ is a good choice and we get

$$S_1 = c_f H^2 + \mathcal{O}\big(H^{2-1/k+\varepsilon}\big) + \mathcal{O}\big(H^{1+(k-1)/k+\varepsilon}\big) + \mathcal{O}\big(H^{(2k-1)/k+\varepsilon}\big),$$

or finally

$$(3.11) \qquad S_1 = \sum_{1\leq d\leq H^{1/k}} \mu(d)S(d^k, H) = c_f H^2 + \mathcal{O}\big(H^{2-1/k+\varepsilon}\big).$$

3.2. Estimation of S_2. It turns out that for the estimation of the second sum trivial arguments yield better results than more elaborate tools like exponential sums and their Weil-type estimates, e.g. like the ones used in [15] and [20]. The gain comes if we straight choose the parameter $z_2 = H^{1-\delta}$ for a small $\delta > 0$, to be specified later. Then

$$S_2 = \sum_{H^{1/k}<d\leq H^{1-\delta}} \mu(d) \sum_{\substack{x,y\leq H \\ xy^k+C\equiv 0 \,(\mathrm{mod}\ d^k)}} 1 \ll \sum_{H^{1/k}<d\leq H^{1-\delta}} \sum_{\substack{x,y\leq H \\ xy^k\equiv -C \,(\mathrm{mod}\ d^k)}} 1.$$

If we fix d and y, the innermost sum counts those $x \leq H < d^k$ which satisfy the congruence $xy^k \equiv -C \pmod{d^k}$. This congruence is solvable in x

if and only if $(y^k, d^k) \mid C$, and in this case in a full residue system modulo d^k there are exactly (y^k, d^k) solutions. If $(y, d)^k \mid C$, we have $(y, d)^k \ll 1$. Then

$$(3.12) \qquad S_2 \ll \sum_{H^{1/k} < d \leq H^{1-\delta}} \sum_{\substack{y \leq H \\ (y,d)^k \mid C}} (y, d)^k \ll H^{1-\delta} \cdot H \ll H^{2-\delta} .$$

3.3. Estimation of S_4. To estimate the tail of the sum S we will use a well-known upper bound for the number of divisors function, i.e. for any positive $\varepsilon > 0$ we have $\tau(n) \ll n^\varepsilon$ ([6, Theorem 315]). Then

$$S(d^k, H) = \sum_{\substack{x, y \leq H \\ xy^k \equiv -C \,(\mathrm{mod}\ d^k)}} 1 \leq \sum_{\substack{n \ll H^{k+1} \\ n \equiv -C \,(\mathrm{mod}\ d^k)}} \sum_{n = xy^k} 1 \ll \sum_{\substack{n \ll H^{k+1} \\ n \equiv -C \,(\mathrm{mod}\ d^k)}} \tau(n)$$

$$\ll H^{(k+1)\varepsilon} \sum_{\substack{n \ll H^{k+1} \\ n \equiv -C \,(\mathrm{mod}\ d^k)}} 1 \ll H^\varepsilon \frac{H^{k+1}}{d^k} = \frac{1}{d^k} H^{k+1+\varepsilon} .$$

Then for S_4 we get

$$S_4 = \sum_{z_3 < d \ll H^{1+1/k}} \mu(d) S(d^k, H) \ll \sum_{z_3 < d \ll H^{1+1/k}} S(d^k, H)$$

$$\ll \sum_{z_3 < d \ll H^{1+1/k}} \frac{1}{d^k} H^{k+1+\varepsilon} \ll H^{k+1+\varepsilon} \sum_{z_3 < d} \frac{1}{d^k} \ll H^{k+1+\varepsilon} z_3^{-k+1} .$$

We set $z_3 = H^{1+\delta}$, and for simplicity take this $\delta < 1/k$ the same as in the definition of $z_2 = H^{1-\delta}$, also choose $\varepsilon = \delta(k-1)/2$. Then we have the upper bound

$$(3.13) \qquad\qquad S_4 \ll H^{2-(k-1)\delta/2} .$$

3.4. Estimation of S_3. We will narrow the intervals of summation in S_3 by extracting more trivial estimates. This would make possible the application of [17, Theorem 1], which still does not apply directly to the sum

$$(3.14) \quad S_3 = \sum_{H^{1-\delta} < d \leq H^{1+\delta}} \mu(d) S(d^k, H) = \sum_{H^{1-\delta} < d \leq H^{1+\delta}} \mu(d) \sum_{\substack{1 \leq x, y \leq H \\ xy^k + C \equiv 0 \,(\mathrm{mod}\ d^k)}} 1 .$$

Introduce the notation

$$(3.15) \qquad S_3(K_1, K_2; L_1, L_2) := \sum_{H^{1-\delta} < d \leq H^{1+\delta}} \sum_{K_1 \leq x \leq K_2} \sum_{\substack{L_1 \leq y \leq L_2 \\ xy^k + C \equiv 0 \,(\mathrm{mod}\ d^k)}} 1 .$$

Clearly $S_3 \ll S_3(1, H; 1, H)$.

3.4.1. The sum $S_3(1, H; 1, H^{1-\theta})$. Let us first estimate the contribution to S_3 when the variable y lies in the interval $\left[1, H^{1-\theta}\right)$ for some $\theta > 0$. We fix d and y and want to solve the congruence $xy^k \equiv -C \pmod{d^k}$, in x, for $1 \leq x \leq H$. Note that $d^k > H^{k-k\delta} > H$ if $\delta < 1 - 1/k$. The latter holds as we already want $\delta < 1/k$ and for $k \geq 2$ we have $1/k \leq 1 - 1/k$. So, just like in §3.2, we will have $\mathcal{O}\left((y, d)^k\right) = \mathcal{O}(1)$ solutions in x. Then

$$(3.16) \qquad S_3(1, H; 1, H^{1-\theta}) \ll \sum_{H^{1-\delta} < d \leq H^{1+\delta}} \sum_{1 \leq y < H^{1-\theta}} 1 \ll H^{2-(\theta-\delta)},$$

where clearly we must choose $1 > \theta > \delta$.

3.4.2. The sum $S_3(1, H^\eta; H^{1-\theta}, H)$. Let $0 < \eta < 1$ be a parameter which we will choose later. We are looking closely at the contribution to the sum S_3 when $1 \leq x \leq H^\eta$. We can write the congruence $xy^k + C \equiv 0 \pmod{d^k}$ in the equation form

$$(3.17) \qquad\qquad\qquad xy^k + C = ad^k$$

for a positive integer a. As $ad^k = xy^k + C \ll H^\eta H^k$ and $d^k > H^{k(1-\delta)}$, we should have $a \ll H^{\eta+k}/H^{k(1-\delta)} = H^{\eta+k\delta}$.

Let us now fix x and d. Observe that the solutions in a of the congruence $ad^k \equiv C \pmod{x}$ are more than the solutions of the original equation (3.17). Also the number of solutions is $(d^k, x) \ll 1$ in case $(d^k, x) \mid C$. Then we can bound from above in the following way.

$$(3.18) \qquad S_3(1, H^\eta; H^{1-\theta}, H) = \sum_{H^{1-\delta} < d \leq H^{1+\delta}} \sum_{1 \leq x \leq H^\eta} \sum_{\substack{y, a \\ ad^k = xy^k + C}} 1 \ll$$

$$\sum_{d, x} \sum_{\substack{1 \leq a \ll H^{\eta+k\delta} \\ ad^k \equiv C(x)}} 1 \ll \sum_{d \leq H^{1+\delta}} \sum_{x \leq H^\eta} \sum_{\substack{a \ll H^{\eta+k\delta} \\ (d^k, x) \mid C}} (d^k, x) \ll \sum_{d \leq H^{1+\delta}} \sum_{x \leq H^\eta} \sum_{a \ll H^{\eta+k\delta}} 1 \,.$$

We get

$$(3.19) \qquad\qquad S_3(1, H^\eta; H^{1-\theta}, H) \ll H^{1+2\eta+(k+1)\delta},$$

as long as $\eta > 0$ is chosen in such a way that $2\eta + (k+1)\delta < 1$. From this condition we also need to have $\delta < 1/(k+1)$.

4. Finishing the proof of Theorem 1: application of Reuss' theorem

In this section we apply [17, Theorem 1] which generalizes the approximate determinant method developed by Heath-Brown in [8]. For sake of

completeness, and for sake of scrutinizing the conditions of the theorem, we formulate it here.

THEOREM 3 (Reuss [17]). *Let $D, E, z > 1$ and $\varepsilon > 0$. Let k, l, h be integers such that $1 \le l < k$ and $h \ne 0$. Let*

$$\mathcal{N}(z; D, E)$$

$$:= \#\big\{(d, e, u, v) \in \mathbb{N}^4 : d \sim D, \, e \sim E, \, u \sim U, \, v \sim V, \, v^l e^k - u^l d^k = h\big\},$$

where

$$U = \frac{z^{1/l}}{D^{k/l}}, \quad and \quad V = \frac{z^{1/l}}{E^{k/l}}.$$

Let $M > 1$ be defined by

$$\log M = \frac{9}{8} \frac{\log(DE) \log(UV)}{\log z},$$

and suppose the following conditions are satisfied:
  (1) $\log(DE) \asymp \log(UV) \asymp \log z$;
  (2) $l \ge 2$, or $DE \gg_{k,l,h} z^{1/k}$.
  Then, if z is large enough in terms of ε,

$$\mathcal{N}(z; D, E) \ll_{\varepsilon, k, l, h} z^\varepsilon \min\big\{(DEM)^{1/2} + D + E, \, (UVM)^{1/2} + U + V\big\}.$$

We write $\widetilde{S}_3 := S_3(H^\eta, H; H^{1-\theta}, H)$. Putting together the estimates (3.11), (3.12), (3.13), (3.16), (3.19) we get

$$(4.1) \qquad S(H) = c_f H^2 + \mathcal{O}\big(\widetilde{S}_3\big) + \mathcal{O}\big(H^{2-1/k+\varepsilon}\big) + \mathcal{O}\big(H^{2-\delta}\big)$$

$$+ \mathcal{O}\big(H^{2-(k-1)\delta/2}\big) + \mathcal{O}\big(H^{2-(\theta-\delta)}\big) + \mathcal{O}\big(H^{1+2\eta+(k+1)\delta}\big),$$

and now we examine the last sum $\widetilde{S}_3$.

Note that by dyadic subdivision of the intervals of the variables d, x and y we can write

$$\widetilde{S}_3 \ll \sum_{H^{1-\delta} < d \le H^{1+\delta}} \sum_{H^\eta < x \le H} \sum_{H^{1-\theta} < y \le H} \sum_{xy^k + C = ad^k} 1$$

$$\ll (\log H)^3 \max_{D, X, Y} \sum_{\substack{d \sim D \\ x \sim X}} \sum_{y \sim Y} \sum_{xy^k - ad^k = -C} 1.$$

Let $Z := XY^k$. It is easy to see that $a \asymp Z/D^k$ and there are $\mathcal{O}(1)$ dyadic subintervals $\widetilde{A} < a < 2\widetilde{A}$ with $\widetilde{A} \asymp Z/D^k$. Hence

$$\widetilde{S}_3 \ll (\log H)^3 \max_{D,X,Y,\widetilde{A}} \sum_{\substack{d \sim D \\ x \sim X}} \sum_{\substack{y \sim Y \\ a \sim \widetilde{A}}} \sum_{xy^k - ad^k = -C} 1 \,.$$

It would be enough to apply Theorem 3 with $Z = XY^k$ and $A := Z/D^k$, so that $\widetilde{A} \asymp A$. Then, using an even finer subdivision in d, x, y we get

$$(4.2) \qquad\qquad \widetilde{S}_3 \ll (\log H)^3 \max_{D,X,Y} \mathcal{N}(Z; D, Y) \,,$$

where $\mathcal{N}(Z; D, Y)$ is the number of solutions of the equation $xy^k - ad^k = -C$ in the box $x \sim X, y \sim Y, a \sim A, d \sim D$, as defined in Theorem 3.

Note that we have the following inequalities

$$(4.3) \qquad \begin{cases} H^{1-\delta} < D \leq H^{1+\delta}, \quad H^\eta < X \leq H, \\ H^{1-\theta} < Y \leq H, \quad 1 \leq A \leq H^{k+1}/D^k \leq H^{1+k\delta} \,. \end{cases}$$

Now we assure that the conditions of Theorem 3 indeed hold. First, we should see whether the first condition $\log(DY) \asymp \log(AX) \asymp \log Z$ holds. Using (4.3) we consecutively obtain $H^{\eta + k(1-\theta)} \leq XY^k = Z \leq H^{k+1}$, therefore

$$(\eta + k(1 - \theta)) \log H \leq \log Z \leq (k + 1) \log H \,.$$

Then $H^{1-\delta+1-\theta} \leq DY \leq H^{1+\delta+1}$, so

$$(2 - (\delta + \theta)) \log H \leq \log(DY) \leq (2 + \delta) \log H \,.$$

Further from $H^\eta \leq AX \leq H^{1+k\delta+1}$ we get

$$\eta \log H \leq \log(AX) \leq (2 + k\delta) \log H \,.$$

We finally get

$$\frac{\eta}{k+1} \log Z \leq \log(AX) \leq \frac{2 + k\delta}{\eta + k(1-\theta)} \log Z$$

and

$$(4.4) \qquad \frac{2 - (\delta + \theta)}{k+1} \log Z \leq \log(DY) \leq \frac{2 + \delta}{\eta + k(1-\theta)} \log Z \,,$$

so indeed condition (1) holds. Observe that we changed earlier the interval of definition for x from $x \geq 1$ to $x \geq H^\eta$ namely because of this condition. In

case $\eta = 0$ we have $\log(AX) \geq 0$ and we cannot assure existence of a positive constant C_1 such that $C_1 \log Z \leq \log(AX)$.

Condition (2) of Theorem 3 requires $l \geq 2$, or $DY \gg Z^{1/k}$. Our Diophantine equation is $xy^k - ad^k = -C$ with $l = 1$, so we need to verify $DY \gg Z^{1/k}$. Let us again impose a condition on the parameter δ: $\delta + \theta < 1 - 1/k$. Then from (4.3) it follows

$$DY \geq H^{2-(\delta+\theta)} > H^{1+1/k} \geq (XY^k)^{1/k} = Z^{1/k}$$

and in this case the second condition is also fulfilled. Therefore Theorem 3 does apply.

In such case we obtain

$$\mathcal{N}(Z; D, Y) \ll Z^\varepsilon \min\left\{(DYM)^{1/2} + D + Y, (AXM)^{1/2} + X + A\right\},$$

where

$$\log M = \frac{9}{8}\frac{\log(DY)\log(AX)}{\log Z}.$$

Since $Z = XY^k = AD^k$, we have

$$Z^2 = AX(DY)^k \quad \text{and} \quad 2\log Z = \log(AX) + k\log(DY).$$

Then we can transform

$$\log M = \frac{9}{8}\frac{\log(DY)}{\log Z}\big(2\log Z - k\log(DY)\big) = \frac{9}{8}\log(DY)\left(2 - k\frac{\log(DY)}{\log Z}\right).$$

From (4.4) we see that

$$\frac{2 - (\delta + \theta)}{k+1} \leq \frac{\log(DY)}{\log Z},$$

so we can bound from above

$$\log M \leq \frac{9}{8}\log(DY)\left(2 - k\frac{2-(\delta+\theta)}{k+1}\right) \leq \frac{9}{8}\left(2 - k\frac{2-(\delta+\theta)}{k+1}\right)(2+\delta)\log H.$$

We denote

$$(4.5) \qquad g = g(k, \delta, \theta) := \frac{9}{8}\left(2 - k\frac{2-(\delta+\theta)}{k+1}\right)(2+\delta).$$

Then $M \leq H^g$ and

$$(4.6) \qquad \mathcal{N}(Z; D, Y) \ll H^{(k+1)\varepsilon}\big((DYM)^{1/2} + D + Y\big)$$

$$\ll H^{(k+1)\varepsilon}\big(H^{1+\delta/2}H^{g/2} + H^{1+\delta}\big) \ll H^{\varepsilon}\big(H^{1+\delta/2+g/2} + H^{1+\delta}\big).$$

Here $\varepsilon > 0$ from Theorem 3 is arbitrary small and is present in the Vinogradov symbol dependence.

First we will assure that $1 + \delta/2 + g/2 < 2$ with a suitable choice of δ and $\theta > \delta$. Take $\theta = 2\delta$. Then

$$G(k,\delta) := 1 + \frac{\delta}{2} + \frac{g}{2} = 1 + \frac{\delta}{2} + \frac{9}{8}\Big(2 - k\frac{2 - 3\delta}{k+1}\Big)\Big(1 + \frac{\delta}{2}\Big)$$

$$= \Big(1 + \frac{\delta}{2}\Big)\Big[1 + \frac{9}{8}\Big(\frac{2}{k+1} + \frac{3k}{k+1}\delta\Big)\Big]$$

From the form of the error terms in (4.1) it is clear that we would be happy with a δ as big as possible. Let us write $\delta = 1/(\alpha k)$ for some integer α. From the conditions on δ and θ, which we want to hold for any $k \geq 2$, e.g. $\delta + \theta = 3\delta < 1 - 1/k$ and $\delta < 1/(k+1)$, we should have $\alpha \geq 4$. We look at $G(k, 1/(\alpha k))$.

$$G\Big(k, \frac{1}{\alpha k}\Big) = \Big(1 + \frac{1}{2\alpha k}\Big)\Big[1 + \frac{9}{8}\Big(\frac{2}{k+1} + \frac{3k}{k+1}\frac{1}{\alpha k}\Big)\Big]$$

$$= \frac{2\alpha k + 1}{2\alpha k}\frac{1}{8\alpha(k+1)}\big(8\alpha(k+1) + 18\alpha + 27\big)$$

We want to choose α so that $2 > G(k, 1/(\alpha k))$, i.e. $2(2\alpha k)(8\alpha(k+1)) > (2\alpha k + 1)(8\alpha k + 26\alpha + 27)$. After performing the operations in the expression we get

$$32\alpha^2 k^2 + 32\alpha^2 k > 16\alpha^2 k^2 + 52\alpha^2 k + 62\alpha k + 26\alpha + 27.$$

We regroup the summands and arrive at the problem of finding parameter α for which the following quadratic form is positive for any $k \geq 2$:

$$(4.7) \qquad \mathcal{Q}(k, \alpha) := 16\alpha^2 k^2 - (20\alpha^2 + 62\alpha)k - (26\alpha + 27).$$

We calculate the zeros of $\mathcal{Q}(k, \alpha)$ for a fixed $\alpha \in \{4, 5, 6, 7\}$ and we see that the first value for which the larger zero of the quadratic equation is smaller than 2 is $\alpha = 7$. That is why we choose $\delta = 1/(7k)$, in which case $\mathcal{Q}(k, 7) > 0$ for any $k \geq 2$, i.e. $G(k, 1/(7k)) < 2$.

More precisely we consider

$$(4.8) \qquad G_k := G\Big(k, \frac{1}{7k}\Big) = \Big(1 + \frac{1}{14k}\Big)\Big(1 + \frac{9 \cdot 17}{7 \cdot 8}\frac{1}{k+1}\Big).$$

Then from (4.2) and (4.6), and $1 + 1/(7k) < G_k$, it follows that

$$(4.9) \qquad \widetilde{S}_3 = \mathcal{O}\big(H^{\varepsilon + G_k}\big)\,.$$

Now in (4.1) we choose $\eta = 1/5$. From (4.1) and (4.9) we conclude that

$$S(H) = c_f H^2 + \mathcal{O}(H^{2 - 1/(7k)}) + \mathcal{O}(H^{2 - 1/14 + 1/(14k)}) + \mathcal{O}\big(H^{\varepsilon + G_k}\big)\,.$$

Note that for $k \geq 3$ we can write

$$S(H) = c_f H^2 + \mathcal{O}\big(H^{2 - 1/(7k)}\big)$$

because the dominating error term is the latter one. For $k = 2$ we have $G_2 > 2 - 1/14 + 1/(14 \cdot 2) > 2 - 1/(7 \cdot 2)$, so we content ourselves to write down

$$S(H) = c_f H^2 + \mathcal{O}\big(H^{2 - \delta}\big)\,,$$

for some real $\delta = \delta(k, f) > 0$. This proves Theorem 1. $\square$

5. Proof of Theorem 2

The method used to attack Erdős' conjecture from [1], [9] and [18] can be extended to the analogous conjecture for our bivariate polynomial. In a similar way like in (3.4) we split the sum $S'(H)$ into few parts. Define

$$S'(m, H) := \sum_{\substack{p,q \leq H \\ f(p,q) \equiv 0 \,(\mathrm{mod}\ m)}} 1\,.$$

We can write

$$(5.1) \qquad S'(H) = S_1' + S_2' + S_3'\,,$$

where

$$S_1' := \sum_{d \leq w} \mu(d) S'(d^k, H)$$

for a positive parameter $w < H^{1/k}$ which we will choose very soon. Further

$$S_2' := \sum_{w < d \leq H^{1/k}} \mu(d) S'(d^k, H)\,.$$

We can directly estimate the last sum S_3' using (3.12), (3.13), (3.16), (3.19) and (4.9):

$$(5.2) \qquad S_3' := \sum_{H^{1/k} < d \ll H^{1 + 1/k}} \mu(d) S'(d^k, H) \ll \sum_{H^{1/k} < d} S(d^k, H) \ll H^{2 - \delta}\,,$$

where $\delta > 0$ is the constant from Theorem 1.

5.1. Estimation of S_1'. We will derive the main term in the desired asymptotic formula from S_1'. Recall the definition of the function $\rho(m)$ from Theorem 1 and let $(\mu_1, \nu_1), \ldots, (\mu_r, \nu_r)$ be the solutions of the congruence $f(\mu, \nu) \equiv 0 \pmod{d^k}$, where $r = \rho(d^k)$. Then

$$S'(d^k, H) = \sum_{i \leq r} \#\big\{(p, q) \in [1, H]^2 : p \equiv \mu_i \pmod{d^k}, \ q \equiv \nu_i \pmod{d^k}\big\}$$

$$= \sum_{i \leq r} \#\big\{p \leq H : p \equiv \mu_i \pmod{d^k}\big\} \cdot \#\big\{q \leq H : q \equiv \nu_i \pmod{d^k}\big\}$$

$$= \sum_{\substack{i \leq r \\ (\mu_i, d) = 1 \\ (\nu_i, d) = 1}} \pi(H; d^k, \mu_i)\pi(H; d^k, \nu_i) + \mathcal{O}\big(\rho(d^k)\big)$$

$$+ \sum_{\substack{i \leq r \\ (\mu_i, d) = 1 \\ (\nu_i, d) > 1}} \pi(H; d^k, \mu_i) \cdot \mathcal{O}(1) + \sum_{\substack{i \leq r \\ (\nu_i, d) = 1 \\ (\mu_i, d) > 1}} \pi(H; d^k, \nu_i) \cdot \mathcal{O}(1)\,.$$

If $d \leq (\log H)^N$ for arbitrary large N, then by Siegel–Walfisz theorem there exists a positive constant c_N depending only on N such that

$$S'(d^k, H) = \sum_{\substack{i \leq r \\ (\mu_i, d) = 1 \\ (\nu_i, d) = 1}} \left[\frac{\pi(H)}{\varphi(d^k)} + \mathcal{O}\big(He^{-c_N\sqrt{\log H}}\big)\right]^2$$

$$+ \mathcal{O}\big(\rho(d^k)\big) + \mathcal{O}\Big(\rho(d^k)\Big(\frac{\pi(H)}{\varphi(d^k)} + \mathcal{O}\big(He^{-c_N\sqrt{\log H}}\big)\Big)\Big)\,.$$

By Hooley [13, Lemma 1] we get $\rho'(d^k) \leq \rho(d^k) \ll d^{2k-2+\varepsilon}$ for $k \geq 2$, and $\varphi(n) \gg_\varepsilon n^{1-\varepsilon}$ [6, Theorem 327]). So there is a positive constant $c > 0$ which might vary in its different occurences, such that for $d \leq (\log H)^N$ we obtain

$$(5.3) \qquad S'(d^k, H) = \rho'(d^k)\frac{\pi(H)^2}{\varphi(d^k)^2} + \mathcal{O}\big(H^2 e^{-c\sqrt{\log H}}\big).$$

Let us then choose $w = (\log H)^N$ so that we apply (5.3). We get

$$(5.4) \quad S_1' = \sum_{d \leq w} \mu(d)S'(d^k, H) = \pi(H)^2 \sum_{d \leq w} \frac{\mu(d)\rho'(d^k)}{\varphi(d^k)^2} + \mathcal{O}\big(wH^2 e^{-c\sqrt{\log H}}\big)$$

$$(5.5) \qquad = c'_f \pi(H)^2 + \mathcal{O}\Big(\pi(H)^2 \sum_{d>w} \frac{\rho'(d^k)}{\varphi(d^k)^2}\Big) + \mathcal{O}\big(H^2 e^{-c\sqrt{\log H}}\big).$$

First of all we pay attention to the constant

$$c'_f = \sum_{d=1}^{\infty} \frac{\mu(d)\rho'(d^k)}{\varphi(d^k)^2}.$$

By $\rho'(p^k) \leq \rho(p^k) \ll p^{2k-2+\varepsilon}$ for $k \geq 2$, and $\varphi(n) \gg_\varepsilon n^{1-\varepsilon'}$, we get

$$c'_f = \sum_{d=1}^{\infty} \frac{\mu(d)\rho'(d^k)}{\varphi(d^k)^2} \ll \sum_{d=1}^{\infty} \frac{d^{2k-2+\varepsilon}}{d^{2k-\varepsilon'}} = \sum_{d=1}^{\infty} d^{-2+\varepsilon} < \infty.$$

It is well-known that $\rho'(m)$ is multiplicative, so we can write

$$c'_f = \prod_p \Big(1 - \frac{\rho'(p^k)}{\varphi(p^k)^2}\Big).$$

We also note that $c'_f \neq 0$. We need $\rho'(p^k) < \varphi(p^k)^2$, but even $\rho'(p^k) \leq \varphi(p^k)$ holds. Indeed, as $\rho'(p^k)$ counts $(\mu, \nu) \in (\mathbb{Z}/p^k\mathbb{Z})^2$ such that $(\mu, p) = (\nu, p) = 1$, for a fixed such ν, which can take $\varphi(p^k)$ values, we have only one solution of the congruence $\mu \equiv -C\bar\nu^k \pmod{p^k}$.

Let us go back to the formula (5.4). For the first error term we first use that

$$(5.6) \qquad \sum_{d>w} \frac{\rho'(d^k)}{\varphi(d^k)^2} \leq \sum_{d>w} \frac{\rho(d^k)}{\varphi(d^k)^2} \ll \sum_{d>w} \frac{d^{2k-2+\varepsilon}}{d^{2k-\varepsilon}}$$

$$= \sum_{d>w} d^{-2+\varepsilon} \ll w^{-1+\varepsilon} = (\log H)^{N(\varepsilon-1)}.$$

Then the error term is itself bounded from above by

$$(5.7) \qquad \frac{H^2}{(\log H)^2} \cdot \frac{1}{(\log H)^{N(1-\varepsilon)}} \ll \frac{H^2}{(\log H)^K},$$

where $K > 2$ is any real number, as long as we choose N and ε appropriately after we have already fixed K. For example we can choose $N = 2K$.

For the second error term in (5.4) we clearly have

$$(5.8) \qquad H^2 e^{-c\sqrt{\log H}} \ll \frac{H^2}{(\log H)^K}.$$

From the formula (5.4) and the error terms estimates (5.7) and (5.8) we conclude that for any $K > 2$ we have

$$(5.9) \qquad S_1' = \sum_{d \leq (\log H)^{2K}} \mu(d) S'(d^k, H) = c_f' \pi(H)^2 + \mathcal{O}\Big(\frac{H^2}{(\log H)^K}\Big).$$

5.2. Estimation of S_2'. It is enough to estimate the sum S_2' trivially. Recall that we have already done similarly in (3.6). Applying again [13, Lemma 1] for square-free d, and using that $1 \ll H/d^k \ll H^2/d^{2k}$ if $d \leq H^{1/k}$, we get

$$(d^k, H) = \rho(d^k)\Big[\frac{H^2}{d^{2k}} + \mathcal{O}\Big(\frac{H}{d^k}\Big) + \mathcal{O}(1)\Big] \ll \rho(d^k)\frac{H^2}{d^{2k}} \ll H^2 d^{-2+\varepsilon}.$$

Then

$$(5.10) \quad S_2' \ll H^2 \sum_{(\log H)^{2K} < d \leq H^{1/k}} d^{-2+\varepsilon} \ll H^2 \cdot (\log H)^{2K(-1+\varepsilon)} \ll \frac{H^2}{(\log H)^K}.$$

Finally Theorem 2 follows from formula (5.1) and the estimates (5.2), (5.9) and (5.10). $\square$

6. Some speculations

It is plausible that a similar asymptotic formula as in Theorem 1 holds for the number of the $(d-1)$-free values of any irreducible bivariate polynomial $f(x,y) \in \mathbb{Z}[x,y]$ of degree d, which does not have a fixed $(d-1)$-th power prime divisor. With the same straightforward approach we get to the problem of estimating the number of solutions of the equation $f(x,y) = ab^{d-1}$, which is hard even in the case $x^l y^k + C = ab^{d-1}$ for general $l + k = d$. It is not clear if the determinant method can be used in such more general setting.

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