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SEVERAL EXPLICIT FORMULAE OF CAUCHY POLYNOMIALS IN TERMS OF r-STIRLING NUMBERS

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Abstract. The integer values of Cauchy polynomials are expressed in terms of r-Stirling numbers of the first kind. Several relations between the integral values of Bernoulli polynomials and those of Cauchy polynomials are obtained in terms of r-Stirling numbers of both kinds. Also, we find a relation between the Cauchy polynomials and hyperharmonic numbers.

1. Introduction

Cauchy polynomials (of the first kind) $c_n(x)$ are defined by the generating function

$$\frac{t}{(1+t)^x \ln(1+t)} = \sum_{n=0}^{\infty} c_n(x) \frac{t^n}{n!} \quad (|t| < 1) \tag{1}$$

[8,10,11]. (Note that x is replaced by $-x$ in [6].) The polynomials $b_n(x) := c_n(x)/n!$ are sometimes called Bernoulli polynomials of the second kind (see, e.g., [3]). When $x = 0$, $c_n(0) = c_n$ are the classical Cauchy numbers of the

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first kind ([5,7,11]). Similarly, Cauchy polynomials of the second kind $\widehat{c}_n(x)$ are defined by the generating function

$$\frac{t(1+t)^x}{(1+t)\ln(1+t)}=\sum_{n=0}^{\infty}\widehat{c}_n(x)\frac{t^n}{n!}\quad(|t|<1) \tag{2}$$

[8,10,11]. Note that x is replaced by $-x$ in [6]. When $x=0$, $\widehat{c}_n(0)=\widehat{c}_n$ are the classical Cauchy numbers of the second kind [5,7,11].

In [2], the concept of r-Stirling numbers was introduced as a generalization of the classical Stirling numbers. The (unsigned) r-Stirling numbers of the first kind, denoted by $\left[{n \atop m}\right]_r$, are defined by the number of permutations of the set $\{1,\dots,n\}$ having m cycles such that the numbers 1, 2, …, r are in distinct cycles. The r-Stirling numbers of the second kind, denoted by $\left\{{n \atop m}\right\}_r$, are defined by the number of partitions of the set $\{1,\dots,n\}$ into m non-empty disjoint subsets, such that the numbers 1, 2, …, r are in distinct subsets. Hence, the classical Stirling numbers can be expressed as

$$\begin{bmatrix}n\\m\end{bmatrix}=\begin{bmatrix}n\\m\end{bmatrix}_0,\quad \begin{Bmatrix}n\\m\end{Bmatrix}=\begin{Bmatrix}n\\m\end{Bmatrix}_0,$$

and also as

$$\begin{bmatrix}n\\m\end{bmatrix}=\begin{bmatrix}n\\m\end{bmatrix}_1,\quad \begin{Bmatrix}n\\m\end{Bmatrix}=\begin{Bmatrix}n\\m\end{Bmatrix}_1\quad(n>0)$$

with

$$\begin{bmatrix}0\\0\end{bmatrix}_r=\begin{Bmatrix}0\\0\end{Bmatrix}_r=1,\quad \begin{bmatrix}n\\0\end{bmatrix}_r=\begin{Bmatrix}n\\0\end{Bmatrix}_r=0\quad(n>0).$$

2. Some basic results

The generating functions of r-Stirling numbers of the first kind $\left[{n+r \atop m+r}\right]_r$ and of the second kind $\left\{{n+r \atop m+r}\right\}_r$ are given by

$$\frac{\big(-\ln(1-t)\big)^m}{m!(1-t)^r}=\sum_{n=0}^{\infty}\begin{bmatrix}n+r\\m+r\end{bmatrix}_r\frac{t^n}{n!} \tag{3}$$

and

$$\frac{e^{rt}(e^t-1)^m}{m!}=\sum_{n=0}^{\infty}\begin{Bmatrix}n+r\\m+r\end{Bmatrix}_r\frac{t^n}{n!}, \tag{4}$$

respectively ([2, Theorem 15, Theorem 16]).

For an integer r, Cauchy polynomials of the first kind $c_n(r)$ can be expressed in terms of the r-Stirling numbers of the first kind $\left[{n+r \atop m+r}\right]_r$.

THEOREM 1. *For nonnegative integers n and r, we have*

$$c_n(r)=\sum_{m=0}^{n}\left[{n+r \atop m+r}\right]_r\frac{(-1)^{n-m}}{m+1}\,. \tag{5}$$

REMARK. If $r=0$, the identity (5) is reduced to

$$c_n=\sum_{m=0}^{n}\left[{n \atop m}\right]\frac{(-1)^{n-m}}{m+1}$$

(see [5, Ch. VII], [7, Theorem 1], [11, p.1908]).

PROOF. Put

$$G_{n,r}(x)=\sum_{m=0}^{n}\left[{n+r \atop m+r}\right]_r x^m\,.$$

From (3), we have

$$\sum_{n=0}^{\infty}G_{n,r}(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[{n+r \atop m+r}\right]_r x^m\frac{t^n}{n!}=\sum_{m=0}^{\infty}x^m\sum_{n=m}^{\infty}\left[{n+r \atop m+r}\right]_r\frac{t^n}{n!}$$
$$=\sum_{m=0}^{\infty}x^m\frac{\big(-\ln(1-t)\big)^m}{m!\,(1-t)^r}=\frac{1}{(1-t)^r}e^{-(\ln(1-t))x}=\frac{1}{(1-t)^r(1-t)^x}\,.$$

By integrating with respect to x from 0 to -1 on both sides, we have

$$\int_0^{-1}\sum_{n=0}^{\infty}G_{n,r}(x)\frac{t^n}{n!}\,dx=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[{n+r \atop m+r}\right]_r\frac{t^n}{n!}\int_0^{-1}x^m\,dx$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[{n+r \atop m+r}\right]_r\frac{(-1)^{m+1}}{m+1}\frac{t^n}{n!}$$

and by (1)

$$\int_0^{-1}\frac{1}{(1-t)^r(1-t)^x}\,dx=\frac{t}{(1-t)^r\ln(1-t)}=-\sum_{n=0}^{\infty}c_n(r)\frac{(-1)^n t^n}{n!}\,.$$

Comparing the coefficients of both sides, we get the identity (5). □

For an integer r, Cauchy polynomials of the second kind $\widehat{c}_n(r)$ can also be expressed in terms of the r-Stirling numbers of the first kind $\left[{n+r \atop m+r}\right]_r$.

THEOREM 2. *For nonnegative integers n and r, we have*

$$\widehat{c}_n(-r) = (-1)^n \sum_{m=0}^{n} \left[{n+r \atop m+r}\right]_r \frac{1}{m+1}\,. \tag{6}$$

REMARK. If $r = 0$, the identity (6) is reduced to

$$\widehat{c}_n = (-1)^n \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{1}{m+1}$$

(see [5, Ch. VII], [7, Theorem 4], [11]).

PROOF. From (4), we have

$$\sum_{n=0}^{\infty} G_{n,r}(-x)\frac{t^n}{n!} = \frac{1}{(1-t)^r} e^{\left(\ln(1-t)\right)x} = \frac{(1-t)^x}{(1-t)^r}\,.$$

By integrating with respect to x from 0 to -1 on both sides, we have

$$\int_0^{-1} \sum_{n=0}^{\infty} G_{n,r}(-x)\frac{t^n}{n!}\,dx = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \left[{n+r \atop m+r}\right]_r \frac{t^n}{n!} \int_0^{-1} (-x)^m\,dx$$

$$= -\sum_{n=0}^{\infty}\sum_{m=0}^{n} \left[{n+r \atop m+r}\right]_r \frac{1}{m+1}\frac{t^n}{n!}$$

and by (2)

$$\int_0^{-1} \frac{(1-t)^x}{(1-t)^r}\,dx = \frac{t}{(1-t)^r(1-t)\ln(1-t)} = -\sum_{n=0}^{\infty} \widehat{c}_n(-r)\frac{(-1)^n t^n}{n!}\,.$$

Comparing the coefficients of both sides, we get the identity (6). $\square$

3. Some further identities

There exist orthogonality and inverse relations for r-Stirling numbers ([2, Theorem 5, Theorem 6]). Indeed, from the orthogonal relations

$$\sum_{l=m}^{n} (-1)^{n-l} \left[{n \atop l}\right]_r \left\{{l \atop m}\right\}_r = \sum_{l=m}^{n} (-1)^{n-l} \left\{{n \atop l}\right\}_r \left[{l \atop m}\right]_r = \delta_{m,n} \quad (n \ge r)\,,$$

where $\delta_{m,n}$ is the Kronecker delta, we obtain the inverse relations

$$(-1)^n f_n=\sum_{m=0}^n(-1)^m\begin{bmatrix}n+r\\m+r\end{bmatrix}_r g_m \iff g_n=\sum_{m=0}^n\begin{Bmatrix}n+r\\m+r\end{Bmatrix}_r f_m\,. \tag{7}$$

Applying these identities to Theorems 1 and 2, we immediately obtain the following result.

THEOREM 3. *For Cauchy polynomials with an integral value r, we have*

$$\sum_{m=0}^n\begin{Bmatrix}n+r\\m+r\end{Bmatrix}_r c_m(r)=\frac{1}{n+1} \tag{8}$$

and

$$\sum_{m=0}^n\begin{Bmatrix}n+r\\m+r\end{Bmatrix}_r \widehat{c}_m(-r)=\frac{(-1)^n}{n+1}\,. \tag{9}$$

REMARK. If $r=0$, then Theorem 3 is reduced to the results in [11] and the special case in [7]. It is well-known that Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi)\,. \tag{10}$$

When $x=0$, $B_n(0)=B_n$ are the Bernoulli numbers with $B_1=-1/2$. In [12, p. 232], for an integer r, Bernoulli polynomials $B_n(r)$ with an integer value r are expressed as

$$B_n(r)=\sum_{m=0}^n\begin{Bmatrix}n+r\\m+r\end{Bmatrix}_r\frac{(-1)^m m!}{m+1}\,. \tag{11}$$

It immediately follows that

$$\sum_{m=0}^n(-1)^m\begin{bmatrix}n+r\\m+r\end{bmatrix}_r B_m(r)=\frac{n!}{n+1}\,. \tag{12}$$

There are relations between Bernoulli polynomials and Cauchy polynomials.

THEOREM 4. *For any integers n and r with $n\ge r\ge 0$, we have*

$$B_n(r)=\sum_{l=0}^n\sum_{m=0}^n m!\begin{Bmatrix}n+r\\m+r\end{Bmatrix}_r\begin{Bmatrix}m+r\\l+r\end{Bmatrix}_r c_l(r)\,,$$

$$= \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! \left\{ {n+r \atop m+r} \right\}_r \left\{ {m+r \atop l+r} \right\}_r \widehat{c}_l(-r)\,,$$

$$c_n(r) = \sum_{l=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-m+l}}{m!} \left[{n+r \atop m+r} \right]_r \left[{m+r \atop l+r} \right]_r B_l(r)\,,$$

$$\widehat{c}_n(-r) = \sum_{l=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-l}}{m!} \left[{n+r \atop m+r} \right]_r \left[{m+r \atop l+r} \right]_r B_l(r)\,.$$

REMARK. If $r = 0$, then Theorem 4 is reduced to the results in [9].

PROOF. We shall prove the first and the fourth identities. The others can be proven similarly. By (8) in Theorem 3, and using (11), we have

$$\sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{m} m! \left\{ {n+r \atop m+r} \right\}_r \left\{ {m+r \atop l+r} \right\}_r c_l(r)$$

$$= \sum_{m=0}^{n} (-1)^{m} m! \left\{ {n+r \atop m+r} \right\}_r \sum_{l=0}^{m} \left\{ {m+r \atop l+r} \right\}_r c_l(r)$$

$$= \sum_{m=0}^{n} \left\{ {n+r \atop m+r} \right\}_r \frac{(-1)^m m!}{m+1} = B_n(r)\,.$$

By (12) in Theorem 3, and using (6), we have

$$\sum_{l=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-l}}{m!} \left[{n+r \atop m+r} \right]_r \left[{m+r \atop l+r} \right]_r B_l(r)$$

$$= \sum_{m=0}^{n} \frac{(-1)^{n}}{m!} \left[{n+r \atop m+r} \right]_r \sum_{l=0}^{m} (-1)^l \left[{m+r \atop l+r} \right]_r B_l(r)$$

$$= (-1)^n \sum_{m=0}^{n} \left[{n+r \atop m+r} \right]_r \frac{1}{m+1} = \widehat{c}_n(-r)\,. \qquad \square$$

4. Hyperharmonic numbers

The hyperharmonic numbers are recursive sums of the classical harmonic numbers $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ $(H_0 = 0)$, defined by

$$H_n^{(1)} := H_n, \quad \text{and} \quad H_n^{(r)} = H_1^{(r-1)} + \dots + H_n^{(r-1)}$$

together with the initial value $H_0^{(r)} = 0$. The generating function of the hyperharmonics reads as

$$\sum_{n=0}^{\infty} H_n^{(r)} t^n = -\frac{\ln(1-t)}{(1-t)^r} \quad (|t| < 1). \tag{13}$$

More on these numbers together with a nice combinatorial interpretation can be found in the work of Benjamin et al. [1].

It can be seen immediately that the generating functions of

$$(-1)^n c_n(-r)/n! \quad \text{and} \quad H_n^{(r)}$$

are inverses of each other:

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n c_n(-r)}{n!} t^n\right)\left(\sum_{n=0}^{\infty} H_n^{(r)} t^n\right)$$
$$= \left(\frac{-t}{(1-t)^{-r}\ln(1-t)}\right)\left(\frac{-\ln(1-t)}{(1-t)^r}\right) = t.$$

The Cauchy product then leads to the following proposition.

Proposition 1. *We have*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} c_k(-r) H_{n-k}^{(r)} = \delta_{1,n}\,.$$

The hyperharmonic numbers can be extended to real r by the generating function (13). This way the generalization of Proposition 1 comes easily.

Proposition 2. *For any real x and r we have*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} c_k(x) H_{n-k}^{(r)} = \begin{cases} \binom{n+x+r-2}{n-1}, & \text{if } n \geq 1; \\ 0, & \text{if } n = 0. \end{cases}$$

Proof. The statement follows from the identity

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n c_n(x)}{n!} t^n\right)\left(\sum_{n=0}^{\infty} H_n^{(r)} t^n\right)$$
$$= \left(\frac{-t}{(1-t)^{x}\ln(1-t)}\right)\left(\frac{-\ln(1-t)}{(1-t)^r}\right) = \frac{t}{(1-t)^{x+r}}$$

after comparing the coefficients of both sides. □

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