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SOME EQUIVALENCE THEOREMS ON ABSOLUTE SUMMABILITY METHODS

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Abstract. We obtained necessary and sufficient conditions for the equivalence of two general summability methods. Some new and known results are also obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$(1) \qquad P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1).$$

The sequence-to-sequence transformation

$$(2) \qquad t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [10]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [7])

$$(3) \qquad \sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty.$$

If we set $\delta = 0$, then we obtain $|\bar{N}, p_n|_k$ summability (see [2]). If we take $p_n = 1$ for all values of n, then we get $|C, 1; \delta|_k$ summability (se [9]). Finally, if we set $\delta = 0$ and $k = 1$, then we get $|\bar{N}, p_n|$ summability.

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2. Known results

We say that two summability methods are equivalent if they sum the same set of series (not necessarily to the same sums). In the special case $k = 1$ Sunouchi has proved the following theorem.

THEOREM A [11]. *Let (p_n) and (q_n) be positive sequences (where $Q_n = \sum_{v=0}^{n} q_v$). In order that every $|\bar{N}, p_n|$ summable series should be $|\bar{N}, q_n|$ summable it is sufficient that*

$$\frac{q_n P_n}{Q_n p_n} = O(1). \tag{4}$$

Bosanquet observed that (4) is also necessary for the conclusion and so completed Theorem A in necessary and sufficient form (see [8]).

THEOREM B [4]. *Let (p_n) and (q_n) be positive sequences and $k \geq 1$. In order that $|\bar{N}, p_n|_k$ should be equivalent to $|\bar{N}, q_n|_k$ it is sufficient that* (4) *and*

$$\frac{p_n Q_n}{P_n q_n} = O(1) \tag{5}$$

hold.

In this theorem, if we take $q_n = 1$ for $n \in N$, then we get a result of Bor (see [3]).

THEOREM C [6]. *Let (p_n) and (q_n) be positive sequences and $k \geq 1$. In order that every $|\bar{N}, p_n|_k$ summable series be $|\bar{N}, q_n|_k$ summable it is necessary that* (4) *holds. If* (5) *holds then* (4) *is also sufficient for the conclusion.*

THEOREM D [6]. *Let (p_n) and (q_n) be positive sequences and $k \geq 1$. In order that $|\bar{N}, p_n|_k$ be equivalent to $|\bar{N}, q_n|_k$ it is necessary and sufficient that* (4) *and* (5) *hold.*

3. Main result

The aim of this paper is to generalize Theorem C and Theorem D for the general summability methods. Now, we shall prove the following theorems.

THEOREM 1. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (p_n) and (q_n) be positive sequences, and let*

$$\sum_{n=v+1}^{m+1} \Big(\frac{Q_n}{q_n}\Big)^{\delta k-1} \frac{1}{Q_{n-1}} = O\bigg(\Big(\frac{Q_v}{q_v}\Big)^{\delta k} \frac{1}{Q_v}\bigg) \quad as \quad m \to \infty. \tag{6}$$

In order that every $|\bar{N},p_n;\delta|_k$ *summable series be* $|\bar{N},q_n;\delta|_k$ *summable it is necessary that* (4) *holds. If* (5) *holds then* (4) *is also sufficient for the conclusion.*

It should be noted that if we take δ=0, then Theorem 1 reduces to Theorem C. In this case the condition (6) reduces to

$$\sum_{n=v+1}^{m+1} \frac{q_n}{Q_n Q_{n-1}} = O\Big(\frac{1}{Q_v}\Big) \quad \text{as} \quad m \to \infty, \tag{7}$$

which always exists.

It is also remarked that if we take $(q_n) = 1$ for all values of n, then the condition (6) fulfils.

We use the following lemma in the proof of Theorem 1.

LEMMA 1 [5]. *Let* $k \geq 1$ *and* $A = (a_{nv})$ *be an infinite matrix. In order that* $A \in (l^k, l^k)$ *it is necessary that* $a_{nv} = O(1)$ *for all* $n, v \geq 0$.

PROOF OF THEOREM 1. Firstly we prove the necessity. Let (t_n) denote the $(\overline{N}, p_n)$ mean of the series $\sum a_n$. Then, by definition, we have

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v. \tag{8}$$

If the series $\sum a_n$ is summable $|\overline{N}, p_n;\delta|_k$, then

$$\sum_{n=1}^{\infty}\Big(\frac{P_n}{p_n}\Big)^{\delta k+k-1}|\Delta t_{n-1}|^k < \infty. \tag{9}$$

Since,

$$\Delta t_{n-1} = \Big(-\frac{1}{P_{n-1}} + \frac{1}{P_n}\Big)\sum_{v=0}^{n} P_{v-1}a_v$$

$$= -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v, \quad n \geq 1,\ (P_{-1} = 0),$$

we have

$$P_{n-1}a_n = -\frac{P_n P_{n-1}}{p_n}\Delta t_{n-1} + \frac{P_{n-1}P_{n-2}}{p_{n-1}}\Delta t_{n-2}\,. \tag{10}$$

That is

$$a_n = -\frac{P_n}{p_n}\Delta t_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta t_{n-2}\,. \tag{11}$$

If T_n denotes the $(\overline{N}, q_n)$ mean of the series $\sum a_n$, similarly we have that

$$T_n = \frac{1}{Q_n}\sum_{v=0}^{n} q_v s_v = \frac{1}{Q_n}\sum_{v=0}^{n}(Q_n - Q_{v-1})a_v \,. \tag{12}$$

Hence

$$\Delta T_{n-1} = -\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1}a_v, \quad n \geq 1. \tag{13}$$

Since

$$a_v = -\frac{P_v}{p_v}\Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2},$$

by (11), we have that

$$\Delta T_{n-1} = -\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1}\Big(-\frac{P_v}{p_v}\Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2}\Big)$$

$$= \frac{q_n P_n}{p_n Q_n}\Delta t_{n-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1} Q_{v-1}\frac{P_v}{p_v}\Delta t_{v-1} - \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1} Q_v\frac{P_{v-1}}{p_v}\Delta t_{v-1}$$

$$= \frac{q_n P_n}{p_n Q_n}\Delta t_{n-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{p_v}(Q_{v-1}P_v - Q_v P_{v-1}).$$

Also,

$$Q_{v-1}P_v - Q_v P_{v-1} = Q_{v-1}P_v - Q_v(P_v - p_v) = Q_{v-1}P_v - Q_v P_v + p_v Q_v$$
$$= (Q_{v-1} - Q_v)P_v + p_v Q_v = -q_v P_v + p_v Q_v,$$

so that

$$\Delta T_{n-1} = \frac{q_n P_n}{p_n Q_n}\Delta t_{n-1} - \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{p_v}q_v\Delta t_{v-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1} Q_v\Delta t_{v-1}$$
$$= T_{n,1} + T_{n,2} + T_{n,3}.$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\Big(\frac{Q_n}{q_n}\Big)^{\delta k+k-1}|T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3. \tag{14}$$

Firstly, we have

$$\sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}|T_{n,1}|^k=\sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}\left|\frac{q_nP_n}{p_nQ_n}\Delta t_{n-1}\right|^k$$

$$=O(1)\sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k}\left(\frac{P_n}{p_n}\right)^{k}\frac{q_n}{Q_n}|\Delta t_{n-1}|^k.$$

Since $\frac{q_n}{Q_n}=O(\frac{p_n}{P_n})$ and $\frac{Q_n}{q_n}=O(\frac{P_n}{p_n})$, by (4) and (5), we have that

$$\sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}|T_{n,1}|^k$$

$$=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\Delta t_{n-1}|^k=O(1)\quad\text{as}\quad m\to\infty$$

by (9). Now applying Hölder's inequality, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}|T_{n,2}|^k=\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{p_v}q_v\Delta t_{v-1}\right|^k$$

$$\le\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{\delta k-1}\frac{1}{Q_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}q_v|\Delta t_{v-1}|\right\}^k$$

$$\le\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{\delta k-1}\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k q_v|\Delta t_{v-1}|^k\left\{\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\right\}^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{\delta k-1}\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k q_v|\Delta t_{v-1}|^k$$

$$=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k q_v|\Delta t_{v-1}|^k\sum_{n=v+1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{\delta k-1}\frac{1}{Q_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}\left(\frac{Q_v}{q_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)^k\frac{q_v}{Q_v}|\Delta t_{v-1}|^k$$

$$=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k+k-1}|\Delta t_{v-1}|^k=O(1)\quad\text{as}\quad m\to\infty$$

by (6) and (9). Finally, as in $T_{n,2}$, we have that

$$\sum_{n=2}^{m+1}\Big(\frac{Q_n}{q_n}\Big)^{\delta k+k-1}|T_{n,3}|^k=\sum_{n=2}^{m+1}\Big(\frac{Q_n}{q_n}\Big)^{\delta k+k-1}\Big|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}Q_v\Delta t_{v-1}\Big|^k$$

$$=\sum_{n=2}^{m+1}\Big(\frac{Q_n}{q_n}\Big)^{\delta k-1}\frac{1}{Q_{n-1}^k}\Big|\sum_{v=1}^{n-1}\frac{Q_v}{q_v}q_v\Delta t_{v-1}\Big|^k$$

$$\le\sum_{n=2}^{m+1}\Big(\frac{Q_n}{q_n}\Big)^{\delta k-1}\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\Big(\frac{Q_v}{q_v}\Big)^k q_v|\Delta t_{v-1}|^k\Big\{\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\Big\}^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\Big(\frac{Q_n}{q_n}\Big)^{\delta k-1}\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\Big(\frac{Q_v}{q_v}\Big)^k q_v|\Delta t_{v-1}|^k$$

$$=O(1)\sum_{v=1}^{m}\Big(\frac{Q_v}{q_v}\Big)^k q_v|\Delta t_{v-1}|^k\sum_{n=v+1}^{m+1}\Big(\frac{Q_n}{q_n}\Big)^{\delta k-1}\frac{1}{Q_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}\Big(\frac{P_v}{p_v}\Big)^{\delta k+k-1}|\Delta t_{v-1}|^k=O(1)\quad\text{as}\quad m\to\infty.$$

This completes the proof of sufficiency of Theorem 1. For the proof of the necessity, we consider the series to series version of (2) i.e. for $n\ge 1$, let

$$b_n=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v\,,\quad c_n=\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n}Q_{v-1}a_v\,.$$

A simple calculation shows that for $n\ge 1$

$$c_n=\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{b_v}{P_v}\big(P_vQ_{v-1}-P_{v-1}Q_v\big)+\frac{q_nP_n}{Q_nP_n}b_n.$$

From this we can write down at once the matrix A that transforms $\big((\frac{P_n}{p_n})^{\frac{\delta k+k-1}{k}}b_n\big)$ into $\big((\frac{Q_n}{q_n})^{\frac{\delta k+k-1}{k}}c_n\big)$. Thus every $|\bar N,p_n;\delta|_k$ summable series is $|\bar N,q_n;\delta|_k$ summable if and only if $A\in(l^k,l^k)$. By Lemma 1, it is necessary that the diagonal terms of A must be bounded, which gives that (4) must hold. □

It should be remarked that Bennett has given necessary and sufficient conditions for certain classes of matrices to belong to (l^k,l^k) (see [1, (19)]). Our matrix A is not quite of this form, but by removing the first row and the main diagonal it is possible, using the results in [1], to obtain complicated conditions that are both necessary and sufficient for Theorem1 to hold.

THEOREM 2. *Let* (p_n) *and* (q_n) *be positive sequences satisfying the condition* (6), $k \geq 1$, *and* $0 \leq \delta < 1/k$. *In order that* $|\bar{N}, p_n; \delta|_k$ *be equivalent to* $|\bar{N}, q_n; \delta|_k$ *it is necessary and sufficient that* (4) *and* (5) *hold.*

It should be remarked that if we set δ=0, then Theorem 2 reduces to Theorem D.

PROOF OF THEOREM 2. Interchange the roles of (p_n) and (q_n) in Theorem 1.

If we take $p_n = 1$ (resp. $q_n = 1$) for all values of n, then we obtain two new equivalence results dealing with the $|C, 1; \delta|_k$ and $|\bar{N}, q_n; \delta|_k$ (resp. $|\bar{N}, p_n; \delta|_k$ and $|C, 1; \delta|_k$) summability methods. □

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