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CONVERGENCE RATE IN THE PETROV SLLN FOR DEPENDENT RANDOM VARIABLES

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Abstract. We study the rate of convergence in the strong law of large numbers expressed in terms of complete convergence of Baum–Katz type for sequences of random variables satisfying Petrov's condition.

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of random variables. Put $S_n = \sum_{i=1}^n X_i$. We say that the sequence $X_1, X_2, \dots$ satisfies a law of large numbers if

$$\frac{S_n - ES_n}{n} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

We say that this sequence satisfies a weak law of large numbers (the convergence in probability), a strong law of large numbers (the almost sure convergence) or a strong law of large numbers of Hsu-Robbins type (the complete convergence).

Markov showed that if $\operatorname{Var}(S_n) = o(n^2)$, then the sequence $X_1, X_2, \dots$ satisfies the weak law of large numbers without any additional assumptions about independence.

Petrov [6] strengthened Markov's condition and proved the analogous strong law of large numbers for independent random variables.

THEOREM 1 [6]. *Let $\{X_n,\ n \geqq 1\}$ be a sequence of independent random variables with finite variances. If*

$$\operatorname{Var}(S_n) = O\left(\frac{n^2}{\psi(n)}\right) \tag{1}$$

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for any positive function $\psi(x)$, non-decreasing in the domain $x > x_0$ for some $x_0 > 0$ and such that

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)} < \infty, \tag{2}$$

then

$$\frac{S_n - ES_n}{n} \longrightarrow 0 \quad a.s., \quad as \quad n \longrightarrow \infty. \tag{3}$$

The set of all positive functions $\psi(x)$, non-decreasing in the domain $x > x_0$ (for some $x_0 > 0$) and satisfying (2), was denoted by Petrov as $\mathbf{\Psi_c}$.

Condition (1) was called Petrov's condition. It is easy to see that condition (1) for independent random variables is equivalent to

$$\sum_{i=1}^{n} \operatorname{Var}(X_i) = O\left(\frac{n^2}{\psi(n)}\right). \tag{4}$$

Korchevsky [3] showed that (2) implies the classical Kolmogorov's condition

$$\sum_{n=1}^{\infty} \frac{\operatorname{Var}(X_n)}{n^2} < \infty,$$

so Petrov's theorem is a corollary of Kolmogorov's theorem.

Petrov [7] formulated some additional conditions under which Petrov's condition (1) is sufficient for the strong law of large numbers for random variables without any assumption about independence.

THEOREM 2 [7]. *Let $\{X_n,\ n \geqq 1\}$ be a sequence of nonnegative random variables with finite variances satisfying Petrov's condition* (1) *and*

$$E(S_n - S_m) \leqq C(n - m) \quad \textit{for sufficiently large} \quad n - m,$$

where C is some positive constant. Then (3) *holds.*

Petrov and Korchevsky [4] generalized the above theorem replacing Petrov's condition (1) by a more general condition

$$E|S_n - ES_n|^p = O\left(\frac{n^p}{\psi(n)}\right) \tag{5}$$

for some function $\psi \in \mathbf{\Psi_c}$ and some $p \geqq 1$.

The next step in the Petrov strong law of large numbers research has been done by Korchevsky [2]. He used an arbitrary norming sequence in place of the classical normalization.

Theorem 3 [2]. *Let $\{X_n,\ n \geqq 1\}$ be a sequence of nonnegative random variables with finite absolute moment of order $p \geqq 1$. Assume that $\{a_n,\ n \geqq 1\}$ is non-decreasing unbounded sequence of positive numbers. If $ES_n = O(a_n)$ and $E|S_n - ES_n|^p = O\big(\frac{a_n^p}{\psi(a_n)}\big)$ for some function $\psi \in \mathbf{\Psi_c}$, then*

$$\frac{S_n - ES_n}{a_n} \longrightarrow 0, \quad a.s., \qquad as \quad n \longrightarrow \infty.$$

In all the above theorems there is a function $\psi \in \mathbf{\Psi_c}$. Petrov [6] showed that the set $\mathbf{\Psi_c}$ is optimal for the strong law of large numbers in the sense of condition (1). In this situation, it is natural to ask about the subset of $\mathbf{\Psi_c}$, i.e., about the set of all functions ψ, for which Petrov's condition (1) (or its generalization (5)) is sufficient for the strong law of large numbers of Hsu-Robbins type.

In this work we study the rate of convergence in the strong law of large numbers expressed in terms of complete convergence of Baum–Katz type for sequences of random variables satisfying Petrov's condition (5).

This problem was considered by Stoica [9]. He presented the Baum–Katz type theorem in a special form, i.e., considered the convergence of the series

$$\sum_{n=1}^{\infty} c_n P\big[|S_n - ES_n| > \varepsilon b_n\big]$$

for the norming sequence $b_n = ES_n,\ n \geqq 1$ and the coefficients $c_n = (ES_n)^{p-2}$, $n \geqq 1$, $1 < p \leqq 2$ under the assumption analogous to (5) and some additional assumption describing change rate of the sequence $\{X_n,\ n \geqq 1\}$.

Here we recall the Baum–Katz type theorem in the version given in the paper of Gut and Stadtmüller [1].

Theorem 4 [1]. *Let $p > 0$, $\alpha > 1/2$ and $\alpha p \geqq 1$. Suppose that $\{X, X_n,\ n \geqq 1\}$ are independent and identically distributed random variables. If*

$$E|X|^p < \infty \qquad and,\ if \quad p \geqq 1 \quad EX = 0,$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\big[|S_n| > n^{\alpha}\varepsilon\big] < \infty, \quad for\ all \quad \varepsilon > 0$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big[\max_{1 \leqq k \leqq n} |S_k| > n^{\alpha}\varepsilon\Big] < \infty, \quad for\ all \quad \varepsilon > 0.$$

If $\alpha p > 1$, *we also have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left[\sup_{k \geqq n}\left|\frac{S_k}{k^\alpha}\right| > \varepsilon\right] < \infty, \quad \textit{for all} \quad \varepsilon > 0.$$

Conversely, if one of the sums is finite for all $\varepsilon > 0$, *then so are the others (for appropriate values* p *and* α*),* $E|X|^p < \infty$ *and if* $p \geqq 1$, *then* $EX = 0$.

2. Main results

Denote by $\mathbf{\Psi_c}^{(r)}$ the set of positive functions ψ, non-decreasing in the domain $x > x_0$ for some $x_0 > 0$ and such that

$$\sum_{n=1}^{\infty} \frac{n^{r-2}}{\psi(n)} < \infty.$$

Note that for $r = 1$, $\mathbf{\Psi_c}^{(1)} = \mathbf{\Psi_c}$, and if $r_1 > r_2$, then $\mathbf{\Psi_c}^{(r_1)} \subset \mathbf{\Psi_c}^{(r_2)}$.

In the proofs concerning almost sure convergence and Petrov's condition, the following lemma plays the key role.

LEMMA 1. *If* $\psi \in \mathbf{\Psi_c}$ *then the series* $\sum_{n=1}^{\infty} 1/\psi(b^n)$ *is convergent for any* $b > 1$.

Inspired by methods used by Petrov and Korchevsky, we will prove theorems concerning complete convergence using the following generalization of Lemma 1.

LEMMA 2. *If* $\psi \in \mathbf{\Psi_c}^{(r)}$ *for some* $r \geqq 1$, *then the series* $\sum_{n=1}^{\infty} (b^n)^{r-1}/\psi(b^n)$ *is convergent for any* $b > 1$.

In the proof of Lemma 2 we will need the following result.

LEMMA 3. *Let* $\{a_n,\ n \geqq 1\}$ *be a sequence of nonnegative numbers,* $A_n = \sum_{k=1}^{n} a_k$ *and* $A_n \to \infty$. *Then the series* $\sum_{n=1}^{\infty} a_n (A_n)^{r-2}/\psi(A_n)$ *converges for any function* $\psi \in \mathbf{\Psi_c}^{(r)}$, $r \geqq 1$.

PROOF OF LEMMA 3. Let $\psi \in \mathbf{\Psi_c}^{(r)}$ and let n_0 be such that $A_{n_0} > 0$ and $\psi(A_{n_0}) > 0$.

The series $\sum_{n=1}^{\infty} n^{r-2}/\psi(n)$ is convergent for $r \geqq 1$, so the integral

$$I = \int_{A_{n_0}}^{\infty} \frac{x^{r-2}}{\psi(x)}\, dx$$

converges too.

By the mean-value theorem, we have

$$\int_{A_{n-1}}^{A_n} \frac{x^{r-2}}{\psi(x)}\,dx = (A_n - A_{n-1})c_n, \quad n > n_0,$$

where

$$\frac{(A_n)^{r-2}}{\psi(A_n)} \leqq c_n \leqq \frac{(A_{n-1})^{r-2}}{\psi(A_{n-1})}.$$

Moreover, we see that $A_n - A_{n-1} = a_n$ and then

$$\sum_{n=n_0+1}^{\infty} \frac{a_n(A_n)^{r-2}}{\psi(A_n)} \leqq \sum_{n=n_0+1}^{\infty} (A_n - A_{n-1}) \cdot c_n = \sum_{n=n_0+1}^{\infty} \int_{A_{n-1}}^{A_n} \frac{x^{r-2}}{\psi(x)}\,dx = I < \infty,$$

which ends the proof of Lemma 3. $\square$

Proof of Lemma 2. Let $b > 1$. Put $a_n = b^{n-1}$. Then

$$A_n = 1 + b + \dots + b^{n-1} \quad \text{and} \quad \lim_{n\to\infty} \frac{A_n}{b^{n-1}} = 1.$$

Therefore, the convergence of the series $\sum_{n=1}^{\infty} \frac{a_n(A_{n-1})^{r-2}}{\psi(A_n)}$ is equivalent to the convergence of the series $\sum_{n=1}^{\infty} \frac{(b^{n-1})^{r-1}}{\psi(b^{n-1})}$.

By Lemma 3, we get the thesis of Lemma 2. $\square$

Theorem 5. *Let $p \geqq 1$, $\alpha > 1/2$ and $\alpha p \geqq 1$. Suppose that $\{X_n,\ n \geqq 1\}$ is a sequence of nonnegative random variables satisfying conditions*

$$\frac{ES_n}{n^\alpha} \longrightarrow A \quad \text{as} \quad n \longrightarrow \infty \tag{6}$$

and (5) *for some function $\psi \in \mathbf{\Psi_c}^{(p)}$. Then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\big[|S_n - ES_n| > n^\alpha \varepsilon\big] < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Proof. Let $b > 1$, $\varepsilon > 0$,

$$m_1 = \inf\big\{m \geqq 0 : b^m \leqq n < b^{m+1} \text{ for some } n\big\}$$

and

$$m_l = \inf\big\{m > m_{l-1} : b^m \leqq n < b^{m+1} \text{ for some } n\big\}, \quad l \geqq 2,$$

where n and m in the above formulas are nonnegative integers.

Note that $\{m_l,\ l \geqq 1\}$ is a sequence of nonnegative integers such that

$$0 \leqq m_1 < m_2 \dots \quad \text{and} \quad m_l \longrightarrow \infty, \ \text{ as } \ l \longrightarrow \infty.$$

Moreover, for any positive integers n there exists a positive integer $l = l(n)$ such that

$$b^{m_{l(n)}} \leqq n < b^{m_{l(n)}+1}.$$

Now for any $l \geqq 1$, we define the sequences

$$k_l^{(1)} = \inf\left\{k : \, b^{m_l} \leqq k < b^{m_l+1}\right\} \quad \text{and} \quad k_l^{(2)} = \sup\left\{k : \, b^{m_l} \leqq k < b^{m_l+1}\right\}.$$

By definition of $k_l^{(1)}$ and $k_l^{(2)}$, we have

$$b^{m_{l(n)}} \leqq k_{l(n)}^{(1)} \leqq n \leqq k_{l(n)}^{(2)} < b^{m_{l(n)}+1}. \tag{7}$$

Hence, we get

$$\frac{k_{l(n)}^{(i)}}{n} < \frac{b^{m_{l(n)}+1}}{b^{m_{l(n)}}} = b, \quad i = 1, 2 \tag{8}$$

and

$$\frac{k_{l(n)}^{(i)}}{n} > \frac{b^{m_{l(n)}}}{b^{m_{l(n)}+1}} = \frac{1}{b} \quad i = 1, 2. \tag{9}$$

Moreover, by (6), we have that for sufficiently large n

$$\left| \frac{ES_n}{n^\alpha} - \frac{ES_{k_{l(n)}^{(i)}}}{\left(k_{l(n)}^{(i)}\right)^\alpha} \right| < \varepsilon, \quad i = 1, 2 \tag{10}$$

and that there exists a constant C such that

$$0 < \frac{ES_n}{n^\alpha} < C.$$

Put

$$b_1 = \left(1 + \frac{\varepsilon}{C - \varepsilon}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad b_2 = \left(1 + \frac{\varepsilon}{C}\right)^{\frac{1}{\alpha}}. \tag{11}$$

We see that for $b = b_1$ and $b = b_2$, the estimations (7), (8) and (9) also hold.

Moreover, for sufficiently large n, we have

$$\frac{S_n - ES_n}{n^\alpha} \leqq \frac{S_{k^{(2)}_{l(n)}} - ES_{k^{(2)}_{l(n)}}}{\left(k^{(2)}_{l(n)}\right)^\alpha} \cdot \frac{\left(k^{(2)}_{l(n)}\right)^\alpha}{n^\alpha} - \frac{ES_n}{n^\alpha} + \frac{ES_{k^{(2)}_{l(n)}}}{\left(k^{(2)}_{l(n)}\right)^\alpha} + \frac{ES_{k^{(2)}_{l(n)}}}{\left(k^{(2)}_{l(n)}\right)^\alpha} \cdot \left(\frac{\left(k^{(2)}_{l(n)}\right)^\alpha}{n^\alpha} - 1 \right). \tag{12}$$

By (12), (8), (10) and (11), we get

$$\frac{S_n - ES_n}{n^\alpha} \leqq \frac{S_{k^{(2)}_{l(n)}} - ES_{k^{(2)}_{l(n)}}}{\left(k^{(2)}_{l(n)}\right)^\alpha} \cdot (b_2)^\alpha + \varepsilon + C\left(1 + \frac{\varepsilon}{C} - 1\right) = \frac{S_{k^{(2)}_{l(n)}} - ES_{k^{(2)}_{l(n)}}}{\left(k^{(2)}_{l(n)}\right)^\alpha} \cdot (b_2)^\alpha + 2\varepsilon. \tag{13}$$

Similarly, for sufficiently large n, we get the lower estimation

$$\frac{S_n - ES_n}{n^\alpha} \geqq \frac{S_{k^{(1)}_{l(n)}} - ES_{k^{(1)}_{l(n)}}}{\left(k^{(1)}_{l(n)}\right)^\alpha} \cdot \frac{\left(k^{(1)}_{l(n)}\right)^\alpha}{n^\alpha} - \frac{ES_n}{n^\alpha} + \frac{ES_{k^{(1)}_{l(n)}}}{\left(k^{(1)}_{l(n)}\right)^\alpha} + \frac{ES_{k^{(1)}_{l(n)}}}{\left(k^{(1)}_{l(n)}\right)^\alpha} \cdot \left(\frac{\left(k^{(1)}_{l(n)}\right)^\alpha}{n^\alpha} - 1 \right) \tag{14}$$

and by (14), (9), (10) and (11), we obtain

$$\frac{S_n - ES_n}{n^\alpha} \geqq \frac{S_{k^{(1)}_{l(n)}} - ES_{k^{(1)}_{l(n)}}}{\left(k^{(1)}_{l(n)}\right)^\alpha} \cdot \frac{1}{(b_1)^\alpha} - \varepsilon + C\left(\frac{1}{1 + \frac{\varepsilon}{C-\varepsilon}} - 1 \right) = \frac{S_{k^{(1)}_{l(n)}} - ES_{k^{(1)}_{l(n)}}}{\left(k^{(1)}_{l(n)}\right)^\alpha} \cdot \frac{1}{(b_1)^\alpha} - 2\varepsilon. \tag{15}$$

From (13) and (15), we have

$$\left|\frac{S_n - ES_n}{n^\alpha}\right|$$

$$\leqq \max\left\{\left|\frac{S_{k^{(1)}_{l(n)}} - ES_{k^{(1)}_{l(n)}}}{\big(k^{(1)}_{l(n)}\big)^\alpha}\right| \cdot \frac{1}{(b_1)^\alpha} + 2\varepsilon,\ \left|\frac{S_{k^{(2)}_{l(n)}} - ES_{k^{(2)}_{l(n)}}}{\big(k^{(2)}_{l(n)}\big)^\alpha}\right| \cdot (b_2)^\alpha + 2\varepsilon\right\}$$

and

$$\big\{|S_n - ES_n| \geqq 3\varepsilon \cdot n^\alpha\big\} \subset \Big\{\big|S_{k^{(1)}_{l(n)}} - ES_{k^{(1)}_{l(n)}}\big| \geqq \varepsilon \cdot \big(k^{(1)}_{l(n)}\big)^\alpha \cdot (b_1)^\alpha\Big\}$$

$$\cup \Big\{\big|S_{k^{(2)}_{l(n)}} - ES_{k^{(2)}_{l(n)}}\big| \geqq \varepsilon \cdot \big(k^{(2)}_{l(n)}\big)^\alpha \cdot \frac{1}{(b_2)^\alpha}\Big\}.$$

Hence, using the estimations (8), (9), Markov's inequality, the assumption (5), the estimation (7) and Lemma 2, we get

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\big[|S_n - ES_n| \geqq 3\varepsilon \cdot n^\alpha\big]$$

$$\leqq \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big[\big|S_{k^{(1)}_{l(n)}} - ES_{k^{(1)}_{l(n)}}\big| \geqq \varepsilon \cdot \big(k^{(1)}_{l(n)}\big)^\alpha \cdot (b_1)^\alpha\Big]$$

$$+ \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big[\big|S_{k^{(2)}_{l(n)}} - ES_{k^{(2)}_{l(n)}}\big| \geqq \varepsilon \cdot \big(k^{(2)}_{l(n)}\big)^\alpha \cdot \frac{1}{(b_2)^\alpha}\Big]$$

$$\leqq \sum_{n=1}^{\infty} \left(\frac{n}{k^{(1)}_{l(n)}}\right)^{\alpha p} \cdot \left(\frac{k^{(1)}_{l(n)}}{n}\right)^2 \cdot \big(k^{(1)}_{l(n)}\big)^{\alpha p-2} P\Big[\big|S_{k^{(1)}_{l(n)}} - ES_{k^{(1)}_{l(n)}}\big|$$

$$\geqq \varepsilon \cdot \big(k^{(1)}_{l(n)}\big)^\alpha \cdot (b_1)^\alpha\Big]$$

$$+ \sum_{n=1}^{\infty} \left(\frac{n}{k^{(2)}_{l(n)}}\right)^{\alpha p} \cdot \left(\frac{k^{(2)}_{l(n)}}{n}\right)^2 \cdot \big(k^{(2)}_{l(n)}\big)^{\alpha p-2} P\Big[\big|S_{k^{(2)}_{l(n)}} - ES_{k^{(2)}_{l(n)}}\big|$$

$$\geqq \varepsilon \cdot \big(k^{(2)}_{l(n)}\big)^\alpha \cdot \frac{1}{(b_2)^\alpha}\Big]$$

$$\leqq \sum_{n=1}^{\infty} (b_1)^{\alpha p+2} \cdot \left(k_{l(n)}^{(1)}\right)^{\alpha p-2} \frac{E\left|S_{k_{l(n)}^{(1)}} - ES_{k_{l(n)}^{(1)}}\right|^p}{\varepsilon^p \cdot \left(k_{l(n)}^{(1)}\right)^{\alpha p} \cdot (b_1)^{\alpha p}}$$

$$+ \sum_{n=1}^{\infty} (b_2)^{\alpha p+2} \cdot \left(k_{l(n)}^{(2)}\right)^{\alpha p-2} \frac{E\left|S_{k_{l(n)}^{(2)}} - ES_{k_{l(n)}^{(2)}}\right|^p}{\varepsilon^p \cdot \left(k_{l(n)}^{(2)}\right)^{\alpha p}} \cdot (b_2)^{\alpha p}$$

$$\leqq \frac{(b_1)^2}{\varepsilon^p} \sum_{n=1}^{\infty} \left(k_{l(n)}^{(1)}\right)^{-2} E\left|S_{k_{l(n)}^{(1)}} - ES_{k_{l(n)}^{(1)}}\right|^p$$

$$+ \frac{(b_2)^{2\alpha p+2}}{\varepsilon^p} \sum_{n=1}^{\infty} \left(k_{l(n)}^{(2)}\right)^{-2} E\left|S_{k_{l(n)}^{(2)}} - ES_{k_{l(n)}^{(2)}}\right|^p$$

$$\leqq \frac{(b_1)^2}{\varepsilon^p} \sum_{l=1}^{\infty} \left(k_l^{(1)}\right)^{-2} E\left|S_{k_l^{(1)}} - ES_{k_l^{(1)}}\right|^p \left((b_1)^{m_l+1} - (b_1)^{m_l}\right)$$

$$+ \frac{(b_2)^{2\alpha p+2}}{\varepsilon^p} \sum_{l=1}^{\infty} \left(k_l^{(2)}\right)^{-2} E\left|S_{k_l}^{(2)} - ES_{k_l}^{(2)}\right|^p \left((b_2)^{m_l+1} - (b_2)^{m_l}\right)$$

$$\leqq \frac{(b_1)^2(b_1-1)}{\varepsilon^p} \sum_{l=1}^{\infty} \left(k_l^{(1)}\right)^{-2} \cdot \frac{\left(k_l^{(1)}\right)^p}{\psi\left(k_l^{(1)}\right)} \cdot (b_1)^{m_l}$$

$$+ \frac{(b_2)^{2\alpha p+2}(b_2-1)}{\varepsilon^p} \sum_{l=1}^{\infty} \left(k_l^{(2)}\right)^{-2} \cdot \frac{\left(k_l^{(2)}\right)^p}{\psi\left(k_l^{(2)}\right)} \cdot (b_2)^{m_l}$$

$$\leqq \frac{(b_1)^2(b_1-1)}{\varepsilon^p} \sum_{l=1}^{\infty} \left((b_1)^{m_l}\right)^{-2} \cdot \frac{\left((b_1)^{m_l} \cdot b_1\right)^p}{\psi\left((b_1)^{m_l}\right)} \cdot (b_1)^{m_l}$$

$$+ \frac{(b_2)^{2\alpha p+2}(b_2-1)}{\varepsilon^p} \sum_{l=1}^{\infty} \left((b_2)^{m_l}\right)^{-2} \cdot \frac{\left((b_2)^{m_l} \cdot b_2\right)^p}{\psi\left((b_2)^{m_l}\right)} \cdot (b_2)^{m_l}$$

$$\leqq \frac{(b_1)^{2+p}(b_1-1)}{\varepsilon^p} \sum_{l=1}^{\infty} \frac{\left((b_1)^{m_l}\right)^{p-1}}{\psi\left((b_1)^{m_l}\right)}$$

$$+ \frac{(b_2)^{2\alpha p+2+p}(b_2-1)}{\varepsilon^p} \sum_{l=1}^{\infty} \frac{\left((b_2)^{m_l}\right)^{p-1}}{\psi\left((b_2)^{m_l}\right)} < \infty$$

and the proof is now complete. $\square$

Theorem 5 gives sufficient conditions under which the sequence of non-negative random variables without any assumptions of independence satisfies one of the conditions of complete convergence of Baum–Katz type in the strong law of large numbers.

This theorem also answers the question about the subset of $\mathbf{\Psi_c}$ for which Petrov's condition (5) is sufficient for complete convergence in SLLN.

The following example shows that the condition $\sum_{n=1}^{\infty} n^{p-2}/\psi(n) < \infty$ that defines the set $\mathbf{\Psi_c}^{(p)}$, the class of functions we are looking for, is essential for complete convergence.

EXAMPLE 1. Let $\Omega = (0,1)$ be an interval on the real line and P be the Lebesgue measure. We define the sequence $\{X_n,\ n \geqq 1\}$ in the following way

$$X_n = \begin{cases} 1 & \text{for } 0 < \omega < \dfrac{1}{n} \\ 0 & \text{otherwise,} \end{cases} \qquad n \geqq 1.$$

It is easy to see that

$$S_1(\omega) = 1,\ \omega \in (0,1), \quad S_n(\omega) = \begin{cases} n & \text{for } \omega \in \left(0, \dfrac{1}{n}\right) \\ n-1 & \text{for } \omega \in \left\langle \dfrac{1}{n}, \dfrac{1}{n-1}\right) \\ \dots & \\ 1 & \text{for } \omega \in \left\langle \dfrac{1}{2}, 1\right), \end{cases} \qquad n \geqq 2.$$

Thus we have

$$ES_n = O(n) \quad \text{and} \quad E(S_n - ES_n)^2 \leqq ES_n^2 \leqq 2n.$$

This proves that $\{X_n,\ n \geqq 1\}$ satisfies the assumptions of Theorem 3 for the function $\psi(n) = n$ and therefore we can state that (3) holds.

The function $\psi(n) = n$ belongs to $\mathbf{\Psi_c}$ but does not belong to $\mathbf{\Psi_c}^{(2)}$ (the series $\sum_{n=1}^{\infty} 1/\psi(n)$ diverges). This leads us to conclude that the sequence $\{X_n,\ n \geqq 1\}$ does not satisfy the strong law of large number of Hsu-Robbins type, i.e.,

$$\sum_{n=1}^{\infty} P\big[|S_n - ES_n| > \varepsilon n\big] = \infty \quad \text{for some} \quad \varepsilon > 0.$$

Indeed we have

$$\sum_{n=1}^{\infty} P\big[|S_n - ES_n| > \varepsilon n\big] \geqq \sum_{n=1}^{\infty} P[S_n - ES_n > \varepsilon n]$$

$$= \sum_{n=1}^{\infty} P[S_n > ES_n + \varepsilon n] \geqq \sum_{n=1}^{\infty} P\big[S_n > [\ln n] + 1 + \varepsilon n\big]$$

$$\geqq \sum_{n=n_0}^{\infty} P[S_n > n] = \sum_{n=n_0}^{\infty} \frac{1}{\psi(n)} = \infty$$

for some $\varepsilon > 0$ and some $n_0 \in \mathbb{N}$.

In the next section, we will consider some specified types of dependence and give sufficient conditions for convergence rate in the strong law of large numbers in terms of Petrov's condition.

3. Dependent random variables

In this section we consider three types of dependence: negatively associated random variables, ρ^*-mixing random variables and ϕ-mixing random variables. For each of these kinds of dependence, we have the maximal Rosenthal inequality.

Let us recall definitions of the above mentioned type of dependence.

Definition 1. The random variables $X_1, X_2, \dots, X_n$ are said to be negatively associated if for any disjoint subsets $A, B \subset \{1, 2, \dots, \}$ and any real coordinatewise non-decreasing functions f on $\mathbb{R}^A$ and g on $\mathbb{R}^B$,

$$\operatorname{cov}\big(f(X_k, k \in A), g(X_k, k \in B)\big) \leqq 0,$$

provided the covariance exists.

An infinite sequence $\{X_n,\ n \geqq 1\}$ of random variables is said to be negatively associated if every finite subset $\{X_{i_1}, X_{i_2}, \dots, X_{i_k}\}$ is a set of negatively associated random variables.

Definition 2. A sequence of random variables $\{X_n,\ n \geqq 1\}$ is said to be a ρ^*-mixing sequence if there exists $k \in \mathbb{N}$ such that

$$\rho^*(k) = \sup_{S,T} \left(\sup_{X \in L^2(\mathcal{F}_S),\, Y \in L^2(\mathcal{F}_T)} \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{Var}(X) \cdot \operatorname{Var}(Y)}} \right) < 1,$$

where S, T are the finite subsets of positive integers such that dist $(S, T) \geqq k$ and $\mathcal{F}_W$ is the σ-field generated by the random variable $\{X_i,\ i \in W \subset \mathbb{N}\}$.

DEFINITION 3. A sequence of random variables $\{X_n,\ n \geqq 1\}$ is called to be ϕ-mixing (or uniformly strong mixing) if

$$\phi(n) = \sup_{k\geqq 1,\ A\in\mathcal{F}_1^k,\ P(A)>0,\ B\in\mathcal{F}_{k+n}^{\infty}} \bigl|P(B|A) - P(B)\bigr| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

where $\mathcal{F}_n^m$ is the σ-field generated by random variables $X_n, X_{n+1}, \dots, X_m$.

For these three above mentioned types of dependence, Shao [8] (for negatively associated random variables), Peligrad and Gut [5] (for ρ^*-mixing random variables) and Wang et al. [10] (for ϕ-mixing random variables) proved the maximal Rosenthal-type inequalities.

Let $\{X_n,\ n \geqq 1\}$ be a sequence of random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for $n \geqq 1$ and $p \geqq 1$.

The maximal inequality

$$E \max_{1\leqq k\leqq n} \Bigl|\sum_{i=1}^{k} X_i\Bigr|^p \leqq C(p)\Biggl\{\Bigl(\sum_{i=1}^{n} EX_i^2\Bigr)^{p/2} + \sum_{i=1}^{n} E|X_i|^p\Biggr\} \quad \text{for} \quad p \geqq 2 \tag{16}$$

holds for:

(1) negatively associated random variables,

(2) ρ^*-mixing random variables

(3) ϕ-mixing random variables satisfying $\sum_{n=1}^{\infty} \phi^{1/2}(n) < \infty$.

For negatively associated random variables, we have additionally

$$E \max_{1\leqq k\leqq n} \Bigl|\sum_{i=1}^{k} X_i\Bigr|^p \leqq C(p) \sum_{i=1}^{n} E|X_i|^p \quad \text{for} \quad 1 \leqq p < 2 \tag{17}$$

THEOREM 6. *Let $\{X_n,\ n \geqq 1\}$ be a sequence of negatively associated random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for $n \geqq 1$ and some $p \geqq 1$. Let $\psi \in \mathbf{\Psi_c}^{(p)}$. If for some $\alpha > 1/2$ and $\alpha p \geqq 1$*

$$\sum_{i=1}^{n} E|X_i|^p = O\left(\frac{n^p}{\psi(n)}\right), \tag{18}$$

and additionally for $p > 2$

$$\sum_{i=1}^{n} E|X_i|^2 = O\left(\frac{n^2}{\bigl(\psi(n)\bigr)^{2/p}}\right), \tag{19}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Bigl[\max_{1\leqq k\leqq n} |S_k| > \varepsilon \cdot n^{\alpha}\Bigr] < \infty \qquad \textit{for all} \qquad \varepsilon > 0. \tag{20}$$

PROOF. Note that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big[\max_{1\leqq k\leqq n} |S_k| > \varepsilon\cdot n^{\alpha}\Big] = \sum_{l=1}^{\infty}\sum_{n=2^{l-1}}^{2^l-1} n^{\alpha p-2} P\Big[\max_{1\leqq k\leqq n} |S_k| > \varepsilon\cdot n^{\alpha}\Big]$$

$$\leqq \sum_{l=1}^{\infty}\sum_{n=2^{l-1}}^{2^l-1} n^{\alpha p-1} n^{-1} P\Big[\max_{1\leqq k\leqq 2^l} |S_k| > \varepsilon\cdot n^{\alpha}\Big]$$

$$\leqq \sum_{l=1}^{\infty}\sum_{n=2^{l-1}}^{2^l-1} \big(2^l\big)^{\alpha p-1}\big(2^{l-1}\big)^{-1} P\Big[\max_{1\leqq k\leqq 2^l} |S_k| > \varepsilon\cdot \big(2^{l-1}\big)^{\alpha}\Big]$$

$$\leqq 2^{-1}\sum_{l=1}^{\infty} 2^l\big(2^l\big)^{\alpha p-1}\big(2^{l-1}\big)^{-1} P\Big[\max_{1\leqq k\leqq 2^l} |S_k| > \varepsilon\cdot \big(2^{l-1}\big)^{\alpha}\Big]$$

$$\leqq \sum_{l=1}^{\infty}\big(2^l\big)^{\alpha p-1} P\Big[\max_{1\leqq k\leqq 2^l} |S_k| > \varepsilon\cdot \big(2^{l-1}\big)^{\alpha}\Big]$$

$$\leqq 2^{\alpha p}\cdot\varepsilon^{-p}\sum_{l=1}^{\infty}\big(2^l\big)^{\alpha p-1}\cdot\big(2^l\big)^{-\alpha p} E\max_{1\leqq k\leqq 2^l} |S_k|^p := I.$$

Let $p \geqq 2$. Then by (16), (18), (19) and Lemma 2, we have

$$I \leqq 2^{\alpha p}\cdot\varepsilon^{-p}\cdot C(p)\sum_{l=1}^{\infty}\big(2^l\big)^{-1}\left\{\sum_{j=1}^{2^l} E|X_j|^p + \Big(\sum_{j=1}^{2^l} E(X_j)^2\Big)^{p/2}\right\}$$

$$\leqq 2^{\alpha p}\cdot\varepsilon^{-p}\cdot C(p)\left\{\sum_{l=1}^{\infty}\big(2^l\big)^{-1}\frac{\big(2^l\big)^p}{\psi\big(2^l\big)} + \sum_{l=1}^{\infty}\big(2^l\big)^{-1}\left(\frac{\big(2^l\big)^2}{\big(\psi(2^l)\big)^{2/p}}\right)^{p/2}\right\}$$

$$\leqq 2^{\alpha p+1}\cdot\varepsilon^{-p}\cdot C(p)\sum_{l=1}^{\infty}\frac{\big(2^l\big)^{p-1}}{\psi\big(2^l\big)} < \infty$$

which proves Theorem 6 in the case $p \geqq 2$.

Now we consider the case $1 \leqq p < 2$. Using (17), (18) and Lemma 2, we have

$$I \leqq 2^{\alpha p}\cdot\varepsilon^{-p}\cdot C(p)\sum_{l=1}^{\infty}\big(2^l\big)^{-1}\sum_{j=1}^{2^l} E|X_j|^p$$

$$\leqq 2^{\alpha p}\cdot\varepsilon^{-p}\cdot C(p)\sum_{l=1}^{\infty}\left(2^l\right)^{-1}\frac{\left(2^l\right)^p}{\psi\left(2^l\right)}\leqq 2^{\alpha p}\cdot\varepsilon^{-p}\cdot C(p)\sum_{l=1}^{\infty}\frac{\left(2^l\right)^{p-1}}{\psi\left(2^l\right)}<\infty. \quad \square$$

Now we see that one can get a stronger result than (20).

THEOREM 7. *Under the assumption of Theorem* 6 *for some* $p \geqq 1$, $\alpha > 1/2$ *and* $\alpha p > 1$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big[\sup_{k\geqq n} k^{-\alpha}|S_k| > \varepsilon\Big] < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{21}$$

PROOF. Note that similarly as in the proof of Theorem 6, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big[\sup_{k\geqq n} k^{-\alpha}|S_k| > \varepsilon\Big] = \sum_{l=1}^{\infty}\sum_{n=2^{l-1}}^{2^l-1} n^{\alpha p-2} P\Big[\sup_{k\geqq n} k^{-\alpha}|S_k| > \varepsilon\Big]$$

$$\leqq \sum_{l=1}^{\infty}\left(2^l\right)^{\alpha p-1} P\Big[\sup_{k\geqq 2^{l-1}} k^{-\alpha}|S_k| > \varepsilon\Big]$$

$$= \sum_{l=1}^{\infty}\left(2^l\right)^{\alpha p-1}\sum_{m=l}^{\infty} P\Big[\max_{2^{m-1}\leqq k<2^m} k^{-\alpha}|S_k| > \varepsilon\Big]$$

$$= \sum_{m=1}^{\infty} P\Big[\max_{2^{m-1}\leqq k<2^m} k^{-\alpha}|S_k| > \varepsilon\Big]\sum_{l=1}^{m}\left(2^l\right)^{\alpha p-1}$$

$$\leqq C\sum_{m=1}^{\infty}(2^m)^{\alpha p-1} P\Big[\max_{2^{m-1}\leqq k<2^m} k^{-\alpha}|S_k| > \varepsilon\Big]$$

$$\leqq C\sum_{m=1}^{\infty}(2^m)^{\alpha p-1} P\Big[\max_{2^{m-1}\leqq k<2^m} |S_k| > \varepsilon\cdot\left(2^{m-1}\right)^{\alpha}\Big]$$

$$\leqq C\sum_{m=1}^{\infty}(2^m)^{\alpha p-1} P\Big[\max_{1\leqq k\leqq 2^m} |S_k| > \varepsilon\cdot\left(2^{m-1}\right)^{\alpha}\Big]$$

$$\leqq 2^{\alpha p}\cdot C\cdot\varepsilon^{-p}\sum_{m=1}^{\infty}(2^m)^{\alpha p-1}\cdot(2^m)^{-\alpha p} E\max_{1\leqq k\leqq 2^m}|S_k|^p := I.$$

Now following the considerations in the proof of Theorem 6, we can deduce (21). $\square$

Let us note that the following relationship is true

$$P\big[|S_n| > \varepsilon \cdot n^{\alpha}\big] \leqq P\Big[\max_{1\leqq k\leqq n} |S_k| > \varepsilon \cdot n^{\alpha}\Big].$$

Using this fact and following the above obtained results, we can formulate a result of Baum–Katz type.

THEOREM 8. *Let* $p \geqq 1$, $\alpha > 1/2$ *and* $\alpha p \geqq 1$. *Suppose that* $\{X_n,\ n \geqq 1\}$ *are negatively associated random variables with* $EX_n = 0$, $E|X_n|^p < \infty$ *for each* $n \geqq 1$ *and* $\psi \in \mathbf{\Psi_c}^{(p)}$. *If*

$$\sum_{i=1}^{n} E|X_i|^p = O\left(\frac{n^p}{\psi(n)}\right),$$

and for $p \geqq 2$

$$\sum_{i=1}^{n} E|X_i|^2 = O\left(\frac{n^2}{\big(\psi(n)\big)^{2/p}}\right),$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\big[|S_n| > n^{\alpha}\varepsilon\big] < \infty, \quad \textit{for all} \quad \varepsilon > 0, \tag{22}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big[\max_{1\leqq k\leqq n} |S_k| > n^{\alpha}\varepsilon\Big] < \infty, \quad \textit{for all} \quad \varepsilon > 0.$$

If $\alpha p > 1$, *we also have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left[\sup_{k\geqq n}\left|\frac{S_k}{k^{\alpha}}\right| > \varepsilon\right] < \infty, \quad \textit{for all} \quad \varepsilon > 0.$$

Using the fact that the maximal Rosenthal-type inequality (16) is true for sequences $\{X_n,\ n \geqq 1\}$ of ρ^*-mixing and ϕ-mixing random variables, we can prove, in the same way, analogous theorems for sequences with ρ^*-mixing and ϕ-mixing types of dependence.

THEOREM 9. *Let* $p \geqq 2$, $\alpha > 1/2$. *Let* $\psi \in \mathbf{\Psi_c}^{(p)}$. *Suppose that* $\{X_n,\ n \geqq 1\}$ *are* ρ^**-mixing random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for* $n \geqq 1$. *If* (18) *and* (19) *are fulfilled, then* (22),(20) *and* (21) *hold.*

THEOREM 10. *Let* $p \geqq 2$, $\alpha > 1/2$. *Let* $\psi \in \mathbf{\Psi_c}^{(p)}$. *Suppose that* $\{X_n, n \geqq 1\}$ *are* ϕ*-mixing random variables with* $EX_n = 0$, $E|X_n|^p < \infty$ *for* $n \geqq 1$ *and* $\sum_{n=1}^{\infty} \phi^{1/2}(n) < \infty$. *If* (18) *and* (19) *are fulfilled, then* (22), (20) *and* (21) *hold.*

COROLLARY 1. *Let* $\{X_n,\ n \geqq 1\}$ *be negatively associated random variables or* ρ^**-mixing random variables or* ϕ*-mixing random variables with* $EX_n = 0$, $E|X_n|^2 < \infty$ *for each* $n \geqq 1$ *and* $\sum_{n=1}^{\infty} \phi^{1/2}(n) < \infty$ *in case of* ϕ*-mixing type dependence. If*

$$\sum_{i=1}^{n} E|X_i|^2 = O\left(\frac{n^2}{\psi(n)}\right) \qquad \text{for some} \quad \psi \in \mathbf{\Psi_c}^{(2)},$$

then

$$\sum_{n=1}^{\infty} P\big[|S_n| > n \cdot \varepsilon\big] < \infty, \quad \text{for all} \quad \varepsilon > 0, \tag{23}$$

$$\sum_{n=1}^{\infty} P\Big[\max_{1 \leqq k \leqq n} |S_n| > n \cdot \varepsilon\Big] < \infty, \quad \text{for all} \quad \varepsilon > 0, \tag{24}$$

and

$$\sum_{n=1}^{\infty} P\left[\sup_{k \geqq n}\left|\frac{S_k}{k}\right| > \varepsilon\right] < \infty, \quad \text{for all} \quad \varepsilon > 0. \tag{25}$$

PROOF. Put $\alpha = 1$ and $p = 2$. By Theorem 8, Theorem 9 and Theorem 10, we get (23), (24) and (25). $\square$

REMARKS. (1) Theorems 8, 9 and 10 give sufficient conditions for the rate convergence of Baum–Katz type in terms of Petrov's conditions for some non-decreasing function $\psi \in \mathbf{\Psi_c}^{(p)} \subset \mathbf{\Psi_c}$ without any identical distribution assumption.

(2) Corollary 1 shows that Petrov's condition (4) is sufficient not only for almost sure convergence in the strong law of large numbers but it is also sufficient for complete convergence for the three types of dependence considered above provided the function ψ belongs to the set $\mathbf{\Psi_c}^{(2)}$.

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