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DUALITY THEOREMS FOR NONCOMMUTATIVE QUASI-MARTINGALE SPACES*

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Abstract. Let $\widetilde{L}_p(\mathcal{M})$ be the space of all bounded $L_p(\mathcal{M})$-quasi-martingales and $\widetilde{\mathcal{H}}_p(\mathcal{M})$ the Hardy space of noncommutative quasi-martingales. Then

$$\widetilde{L}_p(\mathcal{M})^* = L_q(\mathcal{M}) \oplus BD_q(\mathcal{M}), \quad \widetilde{\mathcal{H}}_p(\mathcal{M})^* = \mathcal{S}_q(\mathcal{M})$$

with equivalent norms for $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, where $BD_q(\mathcal{M})$ is a subspace of $l_\infty(L_q(\mathcal{M}))$ and $\mathcal{S}_q(\mathcal{M})$ is a kind of space which is like but bigger than $\widetilde{\mathcal{H}}_q(\mathcal{M})$. The results for the case of $p = 1$ are also obtained.

1. Introduction

The theory of noncommutative martingale inequalities has been rapidly developed since the establishment of the noncommutative Burkholder–Gundy inequalities in [4]. Many of the classical martingale inequalities (see e.g. [2], [3] or [6]) have been transferred to the noncommutative setting. We refer the reader to a survey by Xu [7] for an exposition of this topic.

In this paper we focus on duality theorems for non-commutative quasi-martingales. Before describing our main results, we recall some duality results for noncommutative martingale spaces. Let $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Since the space of all bounded $L_p(\mathcal{M})$-martingales is isometric to the noncommutative L_p-space $L_p(\mathcal{M})$ of operators, the dual space of all bounded $L_p(\mathcal{M})$-martingales is the space of all bounded $L_q(\mathcal{M})$-martingales. Moreover, because of Burkholder–Gundy inequalities, the noncommutative

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martingale Hardy space $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ with equivalent norms. Thus we have the duality between $\mathcal{H}_p(\mathcal{M})$ and $\mathcal{H}_q(\mathcal{M})$. For the case of $p = 1$, it is known that $\mathcal{H}_1(\mathcal{M})^* = \mathcal{BMO}(\mathcal{M})$ (see [4]).

However, the case of noncommutative quasi-martingale is quite different. In particular, the space $\widetilde{L}_p(\mathcal{M})$ of bounded $L_p(\mathcal{M})$-quasi-martingales and the Hardy space $\widetilde{\mathcal{H}}_p(\mathcal{M})$ of quasi-martingales are not isomorphic to the noncommutative $L_p(\mathcal{M})$. Hence their dual spaces can not come from that of $L_p(\mathcal{M})$. Let $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. We prove that the dual space of $\widetilde{L}_p(\mathcal{M})$ is $L_q(\mathcal{M}) \oplus BD_q(\mathcal{M})$, where $BD_q(\mathcal{M})$ is the space of all predicable sequences $x = (x_n)_{n\geqq 1}$ such that $dx = (dx_n)_{n\geqq 1} \in l_\infty\big(L_q(\mathcal{M})\big)$ and $x_1 = 0$. Moreover, we prove that the dual space of $\widetilde{\mathcal{H}}_p(\mathcal{M})$ is $\mathcal{S}_q(\mathcal{M})$, where $\mathcal{S}_q(\mathcal{M})$ is a kind of space which is like but bigger than $\widetilde{\mathcal{H}}_q(\mathcal{M})$. For the case of $p = 1$, we also obtain the dual results.

2. Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H and τ a normal faithful trace on $\mathcal{M}$ with $\tau(1) = 1$. We call $(\mathcal{M}, \tau)$ a noncommutative probability space. For $1 \leqq p \leqq \infty$, let $L_p(\mathcal{M})$ be the associated noncommutative L_p-space. Recall that for $1 \leqq p < \infty$, the norm on $L_p(\mathcal{M})$ is defined by

$$\|x\|_p = \tau\big(|x|^p\big)^{\frac{1}{p}}, \quad x \in L_p(\mathcal{M}),$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the usual modulus of x. Note that if $p = \infty$, $L_\infty(\mathcal{M})$ is just $\mathcal{M}$ with the usual operator norm.

The noncommutative column spaces $L_p(\mathcal{M}; l_2^c)$ and the row spaces $L_p(\mathcal{M}; l_2^r)$ were introduced in [4]. For $1 \leqq p < \infty$, define $L_p(\mathcal{M}; l_2^c)$ (resp. $L_p(\mathcal{M}; l_2^r)$) as the completion of the family of all finite sequences $x = (x_n)_{n\geqq 1}$ in $L_p(\mathcal{M})$ under the norm

$$\|x\|_{L_p(\mathcal{M};l_2^c)} = \bigg\|\Big(\sum_n |x_n|^2\Big)^{\frac{1}{2}}\bigg\|_p \quad \bigg(\text{resp. } \|x\|_{L_p(\mathcal{M};l_2^r)} = \bigg\|\Big(\sum_n |x_n^*|^2\Big)^{\frac{1}{2}}\bigg\|_p\bigg).$$

For $p = \infty$, define $L_\infty(\mathcal{M}; l_2^c)$ (resp. $L_\infty(\mathcal{M}; l_2^r)$) as the Banach space of (possible infinite) sequences $x = (x_n)_{n\geqq 1}$ in $L_\infty(\mathcal{M})$ such that $\sum_n x_n^* x_n$ (resp. $\sum_n x_n x_n^*$) converges in w*-topology.

Let $x = (x_n)_{n \geqq 1}$ be a sequence in $L_p(\mathcal{M})$. Set $dx_n = x_n - x_{n-1}$ for $n \geqq 1$ (with $x_0 = 0$) and $dx = (dx_n)_{n \geqq 1}$. Set

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2 \Big)^{\frac{1}{2}} \quad \text{and} \quad S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2 \Big)^{\frac{1}{2}}.$$

Then dx belongs to $L_p(\mathcal{M}; l_2^c)$ iff $\big(S_{c,n}(x)\big)$ is bounded in $L_p(\mathcal{M})$. In this case, we define $S_c(x) = \lim_{n\to\infty} S_{c,n}(x) = \big(\sum_{n=1}^{\infty} |dx_n|^2 \big)^{\frac{1}{2}}$. Similarly, if dx belongs to $L_p(\mathcal{M}; l_2^r)$ we define $S_r(x) = \big(\sum_{n=1}^{\infty} |dx_n^*|^2 \big)^{\frac{1}{2}}$.

Let us recall the general setup for noncommutative martingales. Let $(\mathcal{M}_n)_{n \geqq 1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$ and $\mathcal{E}_n$ (with $\mathcal{E}_0 = 0$) the conditional expectation with respect to $\mathcal{M}_n$. A sequence $x = (x_n)_{n \geqq 1}$ is said to be adapted if $x_n \in L_1(\mathcal{M}_n)$ for all $n \geqq 1$, and predictable if $x_n \in L_1(\mathcal{M}_{n-1})$. A noncommutative martingale with respect to the filtration $(\mathcal{M}_n)_{n \geqq 1}$ is a sequence $x = (x_n)_{n \geqq 1}$ in $L_1(\mathcal{M})$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all} \quad n \geqq 1.$$

If additionally, $x = (x_n)_{n \geqq 1} \subset L_p(\mathcal{M})$ for some $1 \leqq p \leqq \infty$, we call x an $L_p(\mathcal{M})$-martingale. In this case, we set $\|x\|_p = \sup_n \|x_n\|_p$. If $\|x\|_p < \infty$, then x is called a bounded $L_p(\mathcal{M})$-martingale. We refer to [5] for more information on noncommutative martingales.

In this paper, we focus on noncommutative quasi-martingales, which are generalizations of noncommutative martingales and the noncommutative analogue of classical quasi-martingales.

Definition 2.1. Let $1 \leqq p \leqq \infty$. An adapted sequence $x = (x_n)_{n \geqq 1}$ in $L_1(\mathcal{M})$ is called a p-quasi-martingale with respect to $(\mathcal{M}_n)_{n \geqq 1}$ (or simply a quasi-martingale for $p = 1$) if

$$\sum_{n=1}^{\infty} \big\| \mathcal{E}_{n-1}(dx_n) \big\|_p < \infty.$$

If in addition $x = (x_n)_{n \geqq 1} \subset L_p(\mathcal{M})$ for some $1 \leqq p \leqq \infty$, we call x an $L_p(\mathcal{M})$-quasi-martingale. In this case, we set

$$\|x\|_p := \sup_n \|x_n\|_p + \sum_{n=1}^{\infty} \big\| \mathcal{E}_{n-1}(dx_n) \big\|_p.$$

If $\|x\|_p < \infty$, x is called a bounded $L_p(\mathcal{M})$-quasi-martingale. The noncommutative quasi-martingale space $\widetilde{L}_p(\mathcal{M})$ is defined as the space of all bounded $L_p(\mathcal{M})$-quasi-martingales, and is equipped with the norm $\|\cdot\|_p$.

A basic fact with respect to quasi-martingales is that each p-quasi-martingale can be decomposed as a sum of a martingale and a predicable quasi-martingale which we call Doob's decomposition. Doob's decomposition plays an important role in this paper.

LEMMA 2.2 (Doob's decomposition). *Let $1 \leqq p \leqq \infty$. Each p-quasi-martingale $x = (x_n)_{n \geqq 1}$ can be uniquely decomposed as a sum of two sequences $y = (y_n)_{n \geqq 1}$ and $z = (z_n)_{n \geqq 1}$, where $y = (y_n)_{n \geqq 1}$ is a martingale and $z = (z_n)_{n \geqq 1}$ is a predicable p-quasi-martingale with $z_1 = 0$. Moreover, when $x = (x_n)_{n \geqq 1}$ is $L_p(\mathcal{M})$-bounded, $y = (y_n)_{n \geqq 1}$ and $z = (z_n)_{n \geqq 1}$ are also $L_p(\mathcal{M})$-bounded.*

PROOF. We define two sequences $y = (y_n)_{n \geqq 1}$ and $z = (z_n)_{n \geqq 1}$ by

$$y_n = \sum_{k=1}^{n} \big(dx_k - \mathcal{E}_{k-1}(dx_k) \big) \quad \text{and} \quad z_n = \sum_{k=1}^{n} \big(\mathcal{E}_{k-1}(dx_k) \big). \tag{2.1}$$

Then $x_n = y_n + z_n$ holds for every $n \geqq 1$. It is clear that $y = (y_n)_{n \geqq 1}$ is a martingale and $z = (z_n)_{n \geqq 1}$ is predicable with $z_1 = 0$. Observe that $\sum_{n=1}^{\infty} \big\| \mathcal{E}_{n-1}(dz_n) \big\|_p = \sum_{n=1}^{\infty} \big\| \mathcal{E}_{n-1}(dx_n) \big\|_p < \infty$, thus $z = (z_n)_{n \geqq 1}$ is a p-quasi-martingale. To prove the uniqueness of the decomposition, assume that $x_n = y_n + z_n$ and $x_n = y'_n + z'_n$ are two decompositions of x. It comes from $y_n - y'_n = z'_n - z_n (n \geqq 1)$ that $(z'_n - z_n)_{n \geqq 1}$ is a predicable martingale. We get that $z'_n - z_n = z'_1 - z_1 = 0$ for all $n \geqq 1$. Hence $z_n = z'_n$ and $y_n = y'_n$ for all $n \geqq 1$.

Moreover, if $x = (x_n)_{n \geqq 1}$ is $L_p(\mathcal{M})$-bounded, then

$$\sup_n \|y_n\|_p \leqq \sup_n \|x_n\|_p + \sup_n \|z_n\|_p \leqq \sup_n \|x_n\|_p + \sum_{n=1}^{\infty} \|dz_n\|_p < \infty.$$

This shows that $y = (y_n)_{n \geqq 1}$ and $z = (z_n)_{n \geqq 1}$ are $L_p(\mathcal{M})$-bounded. □

REMARK 2.3. One can see from the proof above that for an adaptable sequence (not necessarily a quasi-martingale) $x = (x_n)_{n \geqq 1}$ in $L_p(\mathcal{M})$, if it can be decomposed as $x_n = y_n + z_n (n \geqq 1)$, where $(y_n)_{n \geqq 1}$ is a martingale and $z = (z_n)_{n \geqq 1}$ is predicable with $z_1 = 0$, then the decomposition is unique. We will use this fact later.

3. The dual spaces of $\widetilde{L}_p(\mathcal{M})$ and $\widetilde{\mathcal{H}}_p(\mathcal{M})$

In this section, we first focus on the dual space of $\widetilde{L}_p(\mathcal{M})$ for $1 < p < \infty$. The basic ideal is to use Doob's decomposition. In fact, each $x = (x_n)_{n \geqq 1}$ in $\widetilde{L}_p(\mathcal{M})$ could be decomposed as $x_n = y_n + z_n$ for every $n \geqq 1$, where $(y_n)_{n \geqq 1}$ is a bounded $L_p(\mathcal{M})$-martingale and $z = (z_n)_{n \geqq 1}$ is a predicable bounded $L_p(\mathcal{M})$-quasi-martingale with $z_1 = 0$. The first part of the decomposition is the "good part" and it is easy to deal with, since the space of all bounded $L_p(\mathcal{M})$-martingales equipped with $\|\cdot\|_p$ is isometric to $L_p(\mathcal{M})$. The second part of the decomposition is the "main part" and for it we should focus on to deal with. We recall that $l_1\big(L_p(\mathcal{M})\big)$ is defined as the space of all sequences $x = (x_n)_{n \geqq 1}$ in $L_p(\mathcal{M})$ such that

$$\|x\|_{l_1(L_p(\mathcal{M}))} = \sum_{n=1}^{\infty} \|x_n\|_p < \infty$$

and $l_\infty\big(L_p(\mathcal{M})\big)$ is defined as the space of all sequences $x = (x_n)_{n \geqq 1}$ in $L_p(\mathcal{M})$ such that

$$\|x\|_{l_\infty(L_p(\mathcal{M}))} = \sup_n \|x_n\|_p < \infty.$$

Noting that the space of all predicable p-quasi-martingale difference sequences $dx = (dx_n)_{n \geqq 1}$ is a subspace of $l_1\big(L_p(\mathcal{M})\big)$ and $\big(l_1\big(L_p(\mathcal{M})\big)\big)^* = l_\infty\big(L_q(\mathcal{M})\big)$, this suggests us to consider the space $BD_p(\mathcal{M})$ defined in the following.

DEFINITION 3.1. Let $1 \leqq p \leqq \infty$. We define $BD_p(\mathcal{M})$ as the space of all predicable sequences $x = (x_n)_{n \geqq 1}$ such that $dx = (dx_n)_{n \geqq 1} \in l_\infty\big(L_p(\mathcal{M})\big)$ and $x_1 = 0$, equipped with the norm

$$\|x\|_{BD_p(\mathcal{M})} = \|dx\|_{l_\infty(L_p(\mathcal{M}))} = \sup_n \|dx_n\|_p.$$

DEFINITION 3.2. Let $1 < p \leqq \infty$. We define $L_p(\mathcal{M}) \oplus BD_p(\mathcal{M})$ as the space of all adaptable sequences $x = (x_n)_{n \geqq 1}$ in $L_p(\mathcal{M})$ which can be decomposed as

$$x_n = y_n + z_n \quad (n \geqq 1),$$

where $(y_n)_{n \geqq 1}$ is a bounded $L_p(\mathcal{M})$-martingale and $z = (z_n)_{n \geqq 1} \in BD_p(\mathcal{M})$. Given $x = (x_n)_{n \geqq 1} \in L_p(\mathcal{M}) \oplus BD_p(\mathcal{M})$, define

$$\|x\|_{L_p(\mathcal{M}) \oplus BD_p(\mathcal{M})} = \sup_n \|y_n\|_p + \|z\|_{BD_p(\mathcal{M})}.$$

Note that by Remark 2.3, the decomposition in Definition 3.2 is unique, thus the norm $\|\cdot\|_{L_p(\mathcal{M})\oplus BD_p(\mathcal{M})}$ is well defined. We can now state the first result of this section.

THEOREM 3.3. *Let* $1<p<\infty$ *and* $p^{-1}+q^{-1}=1$. *Then* $\widetilde{L}_p(\mathcal{M})^*=L_q(\mathcal{M})\oplus BD_q(\mathcal{M})$ *with equivalent norms.*

PROOF. Let $u=(u_n)_{n\geqq 1}\in L_q(\mathcal{M})\oplus BD_q(\mathcal{M})$ and $u_n=v_n+w_n$ $(n\geqq 1)$ be the decomposition such that $(v_n)_{n\geqq 1}$ is a bounded $L_q(\mathcal{M})$-martingale and $(w_n)_{n\geqq 1}\in BD_q(\mathcal{M})$. Let $x=(x_n)_{n\geqq 1}\in\widetilde{L}_p(\mathcal{M})$ and $x_n=y_n+z_n$ $(n\geqq 1)$ be its Doob's decomposition. Then $y=(y_n)_{n\geqq 1}$ is a bounded $L_p(\mathcal{M})$-martingale and $\sup_n\|y_n\|_p\leqq\|x\|_p$.

Now we define a linear functional on $\widetilde{L}_p(\mathcal{M})$ by

$$l_u(x)=\tau(v_\infty y_\infty)+\sum_{n=1}^{\infty}\tau(dw_n dz_n),$$

where v_∞ is the limit of $(v_n)_{n\geqq 1}$ in $L_q(\mathcal{M})$ and y_∞ is the limit of $(y_n)_{n\geqq 1}$ in $L_p(\mathcal{M})$. Then by Hölder's inequality,

$$\begin{aligned}\bigl|l_u(x)\bigr| &\leqq \|v_\infty\|_q\|y_\infty\|_p+\sup_n\|dw_n\|_q\sum_{n=1}^{\infty}\|dz_n\|_p\\ &\leqq\Bigl(\sup_n\|v_n\|_q+\sup_n\|dw_n\|_q\Bigr)\Bigl(\sup_n\|y_n\|_p+\sum_{n=1}^{\infty}\|dz_n\|_p\Bigr)\\ &\leqq 2\|u\|_{L_q(\mathcal{M})\oplus BD_q(\mathcal{M})}\|x\|_p.\end{aligned}$$

Thus $l_u(x)$ is continuous on $\widetilde{L}_p(\mathcal{M})$ and $\|l_u\|\leqq 2\|u\|_{L_q(\mathcal{M})\bigoplus BD_q(\mathcal{M})}$.

We pass to the converse inclusion. Let $l\in\widetilde{L}_p(\mathcal{M})^*$. Let l_1 be the restriction of l on $L_p(\mathcal{M})$. Then there exists a element $v\in L_q(\mathcal{M})$ and $\|v\|_q\leqq\|l\|$ such that

$$l_1(a)=\tau(av),\quad a\in L_p(\mathcal{M}). \tag{3.1}$$

On the other hand, let F_p be the subspace of $l_1\bigl(L_p(\mathcal{M})\bigr)$ of all sequences $db=(db_n)_{n\geqq 1}$ such that $b=(b_n)_{n\geqq 1}$ is a predictable quasi-martingale in $\widetilde{L}_p(\mathcal{M})$ with $b_1=0$. It is easy to see that $\|db\|_{l_1(L_p(\mathcal{M}))}\leqq\|b\|_{\widetilde{L}_p(\mathcal{M})}\leqq 2\|db\|_{l_1(L_p(\mathcal{M}))}$ for any $db=(db_n)_{n\geqq 1}\in F_p$. Define a functional on F_p by

$$l_2(db)=l(b),\quad db=(db_n)_{n\geqq 1}\in F_p.$$

Then l_2 is a continuous linear functional on F_p and $\|l_2\| \leqq 2\|l\|$. By the Hahn-Banach theorem, l_2 extends to a functional on $l_1\big(L_p(\mathcal{M})\big)$. Since $\big(l_1\big(L_p(\mathcal{M})\big)\big)^* = l_\infty\big(L_q(\mathcal{M})\big)$, the representation theorem allows us to find a sequence $w' = (w'_n)_{n\geqq 1} \in l_\infty\big(L_q(\mathcal{M})\big)$ such that

$$(3.2)\qquad l_2(s) = \sum_{n=1}^{\infty} \tau(w'_n s_n) \quad \big(s = (s_n)_{n\geqq 1} \in l_1\big(L_p(\mathcal{M})\big)\big)$$

and $\|w'\|_{l_\infty(L_q(\mathcal{M}))} \leqq \|l_2\|$. Set $w_1 = 0$ and $w_n = \sum_{k=1}^{n} \mathcal{E}_{k-1}(w'_k)$ $(n \geqq 2)$. For any $db = (db_n)_{n\geqq 1} \in F_p$, noting that $db = (db_n)_{n\geqq 1}$ is predicable, it follows from (3.2) that

(3.3)
$$l_2(db) = \sum_{n=1}^{\infty} \tau\big(\mathcal{E}_{n-1}(w'_n db_n)\big) = \sum_{n=1}^{\infty} \tau\big(db_n \mathcal{E}_{n-1}(w'_n)\big) = \sum_{n=1}^{\infty} \tau(dw_n db_n).$$

It remains to show that $w = (w_n)_{n\geqq 1} \in BD_q(\mathcal{M})$. This is true since $w = (w_n)_{n\geqq 1}$ is predicable with $w_1 = 0$ and

$$\|w\|_{BD_q(\mathcal{M})} = \sup_n \|dw_n\|_q \leqq \sup_n \|w'_n\|_q \leqq \|l_2\| \leqq 2\|l\|.$$

Set $u_n = v_n + w_n (n \geqq 1)$, where $v_n = \mathcal{E}_n(v)(n \geqq 1)$. Then $u = (u_n)_{n\geqq 1} \in L_q(\mathcal{M}) \oplus BD_q(\mathcal{M})$ and

$$\|u\|_{L_q(\mathcal{M})\oplus BD_q(\mathcal{M})} = \|v\|_q + \|w\|_{BD_q(\mathcal{M})} \leqq \|l\| + 2\|l\| = 3\|l\|.$$

For any $x = (x_n)_{n\geqq 1} \in \widetilde{L}_p(\mathcal{M})$, let $x_n = y_n + z_n (n \geqq 1)$ be its Doob's decomposition. Noting that $y = (y_n)_{n\geqq 1}$ is a bounded $L_p(\mathcal{M})$-martingale and $dz = (dz_n)_{n\geqq 1} \in F_p$, it follows from (3.1) and (3.3) that

$$l(x) = l(y) + l(z) = \tau(y_\infty v_\infty) + \sum_{n=1}^{\infty} \tau(dw_n dz_n). \quad \square$$

Remark 3.4. Let $\widetilde{\widetilde{L}}_1(\mathcal{M})$ be the space of all bounded $L_1(\mathcal{M})$-quasi-martingales $x = (x_n)_{n\geqq 1}$ such that $x = (x_n)_{n\geqq 1}$ can be decomposed as a sum of a uniformly integrable $L_1(\mathcal{M})$-martingale $y = (y_n)_{n\geqq 1}$ and a predicable $L_1(\mathcal{M})$-quasi-martingale $z = (z_n)_{n\geqq 1}$ with $z_1 = 0$. Recall that the space of all uniformly integrable $L_1(\mathcal{M})$-martingales is isometric to $L_1(\mathcal{M})$ (see [1]) and $L_1(\mathcal{M})^* = \mathcal{M}$. Then $\widetilde{\widetilde{L}}_1(\mathcal{M})^* = \mathcal{M} \oplus BD_\infty(\mathcal{M})$. This proof is similar to that of Theorem 3.3.

The second part of this section is devoted to the duality theorems for Hardy spaces of noncommutative quasi-martingales. Now we introduce the Hardy spaces of noncommutative quasi-martingales.

DEFINITION 3.5. Let $1 \leqq p < \infty$.

(i) The column Hardy space $\widetilde{\mathcal{H}}_p^c(\mathcal{M})$ of noncommutative quasi-martingales is defined as the space of all $L_p(\mathcal{M})$-quasi-martingales $x = (x_n)_{n \geqq 1}$ such that $(dx)_{n \geqq 1} \in L_p(\mathcal{M}; l_2^c)$, equipped with the norm

$$\|x\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{M})} = \|dx\|_{L_p(\mathcal{M}; l_2^c)} + \sum_{n=1}^{\infty} \big\| \mathcal{E}_{n-1}(dx_n) \big\|_p.$$

Similarly, the row space $\widetilde{\mathcal{H}}_p^r(\mathcal{M})$ is defined as the space of all $L_p(\mathcal{M})$-quasi-martingales $x = (x_n)_{n \geqq 1}$ such that $x^* \in \widetilde{\mathcal{H}}_p^c(\mathcal{M})$, equipped with the norm $\|x\|_{\widetilde{\mathcal{H}}_p^r(\mathcal{M})} = \|x^*\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{M})}$.

(ii) The space $\widetilde{\mathcal{H}}_p(\mathcal{M})$ is defined as follows. For $1 \leqq p < 2$,

$$\widetilde{\mathcal{H}}_p(\mathcal{M}) = \widetilde{\mathcal{H}}_p^c(\mathcal{M}) + \widetilde{\mathcal{H}}_p^r(\mathcal{M})$$

equipped with the sum norm

$$\|x\|_{\widetilde{\mathcal{H}}_p(\mathcal{M})}$$

$$= \inf \left\{ \|y\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{M})} + \|z\|_{\widetilde{\mathcal{H}}_p^r(\mathcal{M})} : x = y + z,\ y \in \widetilde{\mathcal{H}}_p^c(\mathcal{M}),\ z \in \widetilde{\mathcal{H}}_p^r(\mathcal{M}) \right\}.$$

For $2 \leqq p < \infty$,

$$\widetilde{\mathcal{H}}_p(\mathcal{M}) = \widetilde{\mathcal{H}}_p^c(\mathcal{M}) \cap \widetilde{\mathcal{H}}_p^r(\mathcal{M}),$$

equipped with the intersection norm

$$\|x\|_{\widetilde{\mathcal{H}}_p(\mathcal{M})} = \max \left\{ \|x\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{M})}, \|x\|_{\widetilde{\mathcal{H}}_p^r(\mathcal{M})} \right\}.$$

REMARK 3.6. It is easy to see that $\widetilde{\mathcal{H}}_p^c(\mathcal{M})$ and $\widetilde{\mathcal{H}}_p^r(\mathcal{M})$ are Banach spaces, so is $\widetilde{\mathcal{H}}_p(\mathcal{M})$. Moreover, replacing noncommutative quasi-martingales by noncommutative martingales in Definition 3.5, we get the Hardy spaces $\mathcal{H}_p^c(\mathcal{M})$, $\mathcal{H}_p^r(\mathcal{M})$ and $\mathcal{H}_p(\mathcal{M})$ of noncommutative martingales.

Definition 3.7. Let $1 \leqq p < \infty$.

(i) We define $\mathcal{S}_p^c(\mathcal{M})$ as the space of all adaptable sequences $x = (x_n)_{n \geqq 1}$ in $L_p(\mathcal{M})$ which can be decomposed as

$$x_n = y_n + z_n \quad (n \geqq 1), \tag{3.4}$$

where $y = (y_n)_{n \geqq 1} \in \mathcal{H}_p^c(\mathcal{M})$ and $z = (z_n)_{n \geqq 1} \in BD_p(\mathcal{M})$. Given $x = (x_n)_{n \geqq 1} \in \mathcal{S}_p^c(\mathcal{M})$, define

$$\|x\|_{\mathcal{S}_p^c(\mathcal{M})} = \|y\|_{\mathcal{H}_p^c(\mathcal{M})} + \|z\|_{BD_p(\mathcal{M})}.$$

Similarly, we define $\mathcal{S}_p^r(\mathcal{M})$ as the space of all adaptable sequences $x = (x_n)_{n \geqq 1}$ in $L_p(\mathcal{M})$ such that $x^* \in \mathcal{S}_p^c(\mathcal{M})$, equipped with the norm $\|x\|_{\mathcal{S}_p^r(\mathcal{M})} = \|x^*\|_{\mathcal{S}_p^c(\mathcal{M})}$.

(ii) We define $\mathcal{S}_p(\mathcal{M})$ as the corresponding sum space for $1 \leqq p < 2$ and the corresponding intersection space for $2 \leqq p < \infty$.

Note that by Remark 2.3, the decomposition in (3.4) is unique, thus the norm $\| \cdot \|_{\mathcal{S}_p^c(\mathcal{M})}$ is well defined.

Now we are ready to state the following results.

Theorem 3.8. *Let $1 < p < \infty$ and q be the conjugate index of p. Then*

(i) $\widetilde{\mathcal{H}}_p^c(\mathcal{M})^* = \mathcal{S}_q^c(\mathcal{M})$ *and* $\widetilde{\mathcal{H}}_p^r(\mathcal{M})^* = \mathcal{S}_q^r(\mathcal{M})$ *with equivalent norms.*

(ii) $\widetilde{\mathcal{H}}_p(\mathcal{M})^* = \mathcal{S}_q(\mathcal{M})$ *with equivalent norms.*

Proof. (i) Let $u = (u_n)_{n \geqq 1} \in \mathcal{S}_q^c(\mathcal{M})$ and $u_n = v_n + w_n$ $(n \geqq 1)$ be the decomposition in (3.4). Define a linear functional on $\widetilde{\mathcal{H}}_p^c(\mathcal{M})$ by

$$l_u(x) = \sum_{n=1}^{\infty} \tau(dv_n^* dy_n) + \sum_{n=1}^{\infty} \tau(dw_n^* dz_n) \quad \big(x \in \widetilde{\mathcal{H}}_p^c(\mathcal{M})\big),$$

where $x_n = y_n + z_n (n \geqq 1)$ be its Doob's decomposition. To show l_u is continuous, we need the following inequality for $1 \leqq p < \infty$,

$$\Big\| \Big(\sum_{n=1}^{\infty} |dz_n|^2 \Big)^{\frac{1}{2}} \Big\|_p \leqq \sum_{n=1}^{\infty} \|dz_n\|_p. \tag{3.5}$$

It suffices to prove this for finite sequences. Let $2 \leqq p < \infty$. By the triangle inequality in $L_{p/2}(\mathcal{M})$ we have

$$\Big\| \Big(\sum_{k=1}^{n} |dz_n|^2 \Big)^{1/2} \Big\|_p = \Big(\Big\| \sum_{k=1}^{n} |dz_n|^2 \Big\|_{p/2} \Big)^{1/2}$$

$$\leqq \Big(\sum_{k=1}^{n} \big\| |dz_n|^2 \big\|_{p/2} \Big)^{1/2} = \Big(\sum_{k=1}^{n} \|dz_n\|_p^2 \Big)^{1/2} \leqq \sum_{n=1}^{\infty} \|dz_n\|_p.$$

The case of $1 \leqq p < 2$ is obtain by the inequality

$$\Big\| \Big(\sum_{k=1}^{n} |dz_n|^2 \Big)^{1/2} \Big\|_p = \Big(\Big\| \sum_{k=1}^{n} |dz_n|^2 \Big\|_{p/2}^{p/2} \Big)^{1/p}$$

$$\leqq \Big(\sum_{k=1}^{n} \big\| |dz_n|^2 \big\|_{p/2}^{p/2} \Big)^{1/p} \leqq \Big(\sum_{k=1}^{n} \|dz_n\|_p^p \Big)^{1/p} \leqq \sum_{n=1}^{\infty} \|dz_n\|_p.$$

This proves the inequality (3.5). It comes from (3.5) that

$$\Big\| \Big(\sum_{n=1}^{\infty} |dy_n|^2 \Big)^{\frac{1}{2}} \Big\|_p \leqq \Big\| \Big(\sum_{n=1}^{\infty} |dx_n|^2 \Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum_{n=1}^{\infty} |dz_n|^2 \Big)^{\frac{1}{2}} \Big\|_p \leqq \|x\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{M})}.$$

By Hölder's inequality (see [7]), the series $\sum_n dv_n^* dy_n$ converges in $L_1(\mathcal{M})$ and

$$\Big\| \sum_{n=1}^{\infty} dv_n^* dy_n \Big\|_1 \leqq \Big\| \Big(\sum_{n=1}^{\infty} |dv_n|^2 \Big)^{\frac{1}{2}} \Big\|_q \Big\| \Big(\sum_{n} |dy_n|^2 \Big)^{\frac{1}{2}} \Big\|_p.$$

It follows that the series $\sum_n \tau(dv_n^* dy_n)$ converges and

$$\Big| \sum_{n=1}^{\infty} \tau(dv_n^* dy_n) \Big| \leqq \Big\| \sum_{n=1}^{\infty} dv_n^* dy_n \Big\|_1 \leqq \Big\| \Big(\sum_{n=1}^{\infty} |dv_n|^2 \Big)^{\frac{1}{2}} \Big\|_q \Big\| \Big(\sum_{n=1}^{\infty} |dy_n|^2 \Big)^{\frac{1}{2}} \Big\|_p.$$

On the other hand, the series $\sum_n \tau(dw_n^* dz_n)$ converges and

$$\Big| \sum_{n=1}^{\infty} \tau(dw_n^* dz_n) \Big| \leqq \sup_n \|dw_n\|_q \sum_{n=1}^{\infty} \|dz_n\|_p.$$

Putting the preceding inequalities together, we deduce that

$$\big| l_u(x) \big| \leqq \Big\| \Big(\sum_{n=1}^{\infty} |dv_n|^2 \Big)^{\frac{1}{2}} \Big\|_q \Big\| \Big(\sum_{n=1}^{\infty} |dy_n|^2 \Big)^{\frac{1}{2}} \Big\|_p + \sup_n \|dw_n\|_q \sum_{n=1}^{\infty} \|dz_n\|_p$$

$$\leqq \Big(\Big\| \Big(\sum_{n=1}^{\infty} |dv_n|^2 \Big)^{\frac{1}{2}} \Big\|_q + \sup_n \|dw_n\|_q \Big) \Big(\Big\| \Big(\sum_{n=1}^{\infty} |dy_n|^2 \Big)^{\frac{1}{2}} \Big\|_p + \sum_{n=1}^{\infty} \|dz_n\|_p \Big)$$

$$\leqq 2\|u\|_{\mathcal{S}_q^c(\mathcal{M})}\|x\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{M})}.$$

Thus l_u is continuous on $\widetilde{\mathcal{H}}_p^c(\mathcal{M})$ and $\|l_u\| \leqq 2\|u\|_{\mathcal{S}_q^c(\mathcal{M})}$.

We pass to the converse inclusion. Let $l \in \widetilde{\mathcal{H}}_p^c(\mathcal{M})^*$. First we restrict l on the subspace $\mathcal{H}_p^c(\mathcal{M})$. If we identify a martingale $x = (x_n)_{n\geqq 1}$ with its difference sequence $dx = (dx_n)_{n\geqq 1}$, we may regard $\mathcal{H}_p^c(\mathcal{M})$ as a subspace of $L_p(\mathcal{M}; l_2^c)$. By the Hahn-Banach theorem, l extends to a functional on $L_p(\mathcal{M}; l_2^c)$. Since $L_p(\mathcal{M}; l_2^c)^* = L_q(\mathcal{M}; l_2^c)$, there exists a sequence $v' = (v'_n)_{n\geqq 1} \in L_q(\mathcal{M}; l_2^c)$ such that

$$l(s) = \sum_{n=1}^{\infty} \tau(v_n'^* s_n) \quad \big(s = (s_n)_{n\geqq 1} \in L_p(\mathcal{M}; l_2^c)\big)$$

and $\|v'\|_{L_q(\mathcal{M}; l_2^c)} \leqq \|l\|$. Then we have that

$$l(a) = \sum_{n=1}^{\infty} \tau(v_n'^* da_n) \quad \big(a = (a_n)_{n\geqq 1} \in \mathcal{H}_p^c(\mathcal{M})\big). \tag{3.6}$$

Set $dv_n = \mathcal{E}_n(v'_n) - \mathcal{E}_{n-1}(v'_n) (n \geqq 1)$. Then $v = (v_n)_{n\geqq 1}$ is a martingale. By Stein inequality, we have that

$$\Big\|\Big(\sum_{k=1}^{n} |dv_k|^2\Big)^{\frac{1}{2}}\Big\|_q = \Big\|\Big(\sum_{k=1}^{n} \big|\mathcal{E}_k(v'_k) - \mathcal{E}_{k-1}(v'_k)\big|^2\Big)^{\frac{1}{2}}\Big\|_q$$

$$\leqq 2C_q \Big\|\Big(\sum_{n=1}^{\infty} |v'_n|^2\Big)^{\frac{1}{2}}\Big\|_q = 2C_q\|v'\|_{L_q(\mathcal{M}; l_2^c)},$$

where C_q is the positive constant depending only on q. Thus $\|v\|_{\mathcal{H}_q^c(\mathcal{M})} \leqq 2C_q\|l\|$. For any $a = (a_n)_{n\geqq 1} \in \mathcal{H}_p^c(\mathcal{M})$, noting that $a = (a_n)_{n\geqq 1}$ is a martingale, we have that for any $n \geqq 1$

$$\tau(dv_n^* da_n) = \tau(dv_n^* a_n) - \tau\big(\mathcal{E}_{n-1}(dv_n^*) a_{n-1}\big) = \tau\Big(\big(\mathcal{E}_n(v_n'^*) - \mathcal{E}_{n-1}(v_n'^*)\big)\, a_n\Big)$$

$$= \tau\big(v_n'^* \mathcal{E}_n(a_n)\big) - \tau\big(v_n'^* \mathcal{E}_{n-1}(a_n)\big) = \tau(v_n'^* da_n).$$

It follows from (3.6) that

$$l(a) = \sum_{n=1}^{\infty} \tau(dv_n^* da_n) \quad \big(a = (a_n)_{n\geqq 1} \in \mathcal{H}_p^c(\mathcal{M})\big). \tag{3.7}$$

On the other hand, let Q_p be the subspace of $l_1\big(L_p(\mathcal{M})\big)$ of all sequences $db=(db_n)_{n\geqq 1}$ such that $b=(b_n)_{n\geqq 1}$ is a predictable quasi-martingale in $\widetilde{\mathcal{H}}_p^c(\mathcal{M})$ with $b_1=0$. It follows from (3.5) that

$$\|db\|_{l_1(L_p(\mathcal{M}))}\leqq\|b\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{M})}\leqq 2\|db\|_{l_1(L_p(\mathcal{M}))}$$

for any $db=(db_n)_{n\geqq 1}\in Q_p$. Imitating the proof of Theorem 3.3, there exists a sequence $w=(w_n)_{n\geqq 1}\in BD_q(\mathcal{M})$ such that for any $db=(db_n)_{n\geqq 1}\in Q_p$,

$$l(b)=\sum_{n=1}^{\infty}\tau(dw_n^*db_n) \tag{3.8}$$

and $\|w\|_{BD_q(\mathcal{M})}\leqq 2\|l\|$. Set $u_n=v_n+w_n(n\geqq 1)$. Then $u=(u_n)_{n\geqq 1}\in\mathcal{S}_q^c(\mathcal{M})$ and

$$\|u\|_{\mathcal{S}_q^c(\mathcal{M})}=\|v\|_{\mathcal{H}_q^c(\mathcal{M})}+\|w\|_{BD_q(\mathcal{M})}\leqq 2C_q\|l\|+2\|l\|=2(C_q+1)\|l\|.$$

For any $x=(x_n)_{n\geqq 1}\in\widetilde{\mathcal{H}}_p^c(\mathcal{M})$, let $x_n=y_n+z_n(n\geqq 1)$ be its Doob's decomposition. Noting that $dz=(dz_n)_{n\geqq 1}\in Q_p$, it follows from (3.7) and (3.8) that

$$l(x)=l(y)+l(z)=\sum_{n=1}^{\infty}\tau(dv_n^*dy_n)+\sum_{n=1}^{\infty}\tau(dw_n^*dz_n).$$

Therefore, this proves that $\widetilde{\mathcal{H}}_p^c(\mathcal{M})^*=\mathcal{S}_q^c(\mathcal{M})$. Passing to adjoint, we obtain the identity $\widetilde{\mathcal{H}}_p^r(\mathcal{M})^*=\mathcal{S}_q^r(\mathcal{M})$.

(ii) The duality between $\widetilde{\mathcal{H}}_p(\mathcal{M})$ and $\mathcal{S}_q(\mathcal{M})$ is deduced from the standard duality between intersection and sum spaces. □

We turn to the case of $p=1$. Recall that the dual space of $\mathcal{H}_1(\mathcal{M})$ is $\mathcal{BMO}(\mathcal{M})$ which is defined in [4] as the intersection space $\mathcal{BMO}(\mathcal{M})=\mathcal{BMO}^c(\mathcal{M})\bigcap\mathcal{BMO}^r(\mathcal{M})$, where

$$\mathcal{BMO}^c(\mathcal{M})$$

$$=\Big\{x\in L_2(\mathcal{M}):\ \|x\|_{\mathcal{BMO}^c(\mathcal{M})}=\sup_n\big\|\mathcal{E}_n\big(\big|x-\mathcal{E}_{n-1}(x)\big|^2\big)\big\|_\infty^{1/2}\Big\},$$

$$\mathcal{BMO}^r(\mathcal{M})=\big\{x\in L_2(\mathcal{M}):\ \|x\|_{\mathcal{BMO}^r(\mathcal{M})}=\|x^*\|_{\mathcal{BMO}^c(\mathcal{M})}\big\}.$$

This suggests us to consider the spaces defined in the following.

DEFINITION 3.9. We define $E_\infty^c(\mathcal{M})$ as the space of all adaptable sequences $x = (x_n)_{n\geqq 1}$ in $L_2(\mathcal{M})$ which can be decomposed as

$$x_n = y_n + z_n \quad (n \geqq 1), \tag{3.9}$$

where $y = (y_n)_{n\geqq 1}$ is a martingale in $\mathcal{BMO}^c(\mathcal{M})$ and $dz = (dz_n)_{n\geqq 1} \in BD_\infty(\mathcal{M})$. Given $x \in E_\infty^c(\mathcal{M})$, define

$$\|x\|_{E_\infty^c(\mathcal{M})} = \|y\|_{\mathcal{BMO}^c(\mathcal{M})} + \|z\|_{BD_\infty(\mathcal{M})}.$$

Similarly, we define $E_\infty^r(\mathcal{M})$ as the space of all adaptable sequences $x = (x_n)_{n\geqq 1}$ in $L_2(\mathcal{M})$ such that $x^* = (x_n^*)_{n\geqq 1} \in E_\infty^c(\mathcal{M})$, equipped with the norm $\|x\|_{E_\infty^r(\mathcal{M})} = \|x^*\|_{E_\infty^c(\mathcal{M})}$. We define $E_\infty(\mathcal{M})$ as the corresponding sum space for $1 \leqq p < 2$ and the corresponding intersection space for $2 \leqq p < \infty$.

THEOREM 3.10. (i) $\widetilde{\mathcal{H}}_1^c(\mathcal{M})^* = E_\infty^c(\mathcal{M})$ *and* $\widetilde{\mathcal{H}}_1^r(\mathcal{M})^* = E_\infty^r(\mathcal{M})$ *with equivalent norms.*

(ii) $\widetilde{\mathcal{H}}_1(\mathcal{M})^* = E_\infty(\mathcal{M})$ *with equivalent norms.*

PROOF. (i) Let $u = (u_n)_{n\geqq 1} \in E_\infty^c(\mathcal{M})$ and $u_n = v_n + w_n (n \geqq 1)$ be the decomposition as in (3.9). Define a linear functional on $\widetilde{\mathcal{H}}_1^c(\mathcal{M})$ by

$$l_u(x) = \sum_{n=1}^\infty \tau(dv_n^* dy_n) + \sum_{n=1}^\infty \tau(dw_n^* dz_n) \, \left(x \in \widetilde{\mathcal{H}}_1^c(\mathcal{M})\right),$$

where $x_n = y_n + z_n (n \geqq 1)$ is the Doob's decomposition of x. It follows from (3.5) that

$$\Big\| \Big(\sum_{n=1}^\infty |dy_n|^2 \Big)^{\frac{1}{2}} \Big\|_1 \leqq \Big\| \Big(\sum_{n=1}^\infty |dx_n|^2 \Big)^{\frac{1}{2}} \Big\|_1 + \Big\| \Big(\sum_{n=1}^\infty \big| \mathcal{E}_{n-1}(dx_n) \big|^2 \Big)^{\frac{1}{2}} \Big\|_1$$
$$\leqq \|x\|_{\widetilde{\mathcal{H}}_1^c(\mathcal{M})}.$$

Moreover,

$$\Big| \sum_{n=1}^\infty \tau(dv_n^* dy_n) \Big| \leqq \sqrt{2} \|y\|_{\mathcal{H}_1^c(\mathcal{M})} \|v\|_{\mathcal{BMO}^c(\mathcal{M})}$$

(see [4], Appendix). Putting the preceding inequalities together, we obtain that

$$\big| l_u(x) \big| \leqq \sqrt{2} \|y\|_{\mathcal{H}_1^c(\mathcal{M})} \|v\|_{\mathcal{BMO}^c(\mathcal{M})} + \sup_n \|dw_n\|_\infty \sum_{n=1}^\infty \|dz_n\|_1$$

$$\leqq \sqrt{2}\Big(\|y\|_{\mathcal{H}_1^c(\mathcal{M})} + \sum_{n=1}^{\infty}\|dz_n\|_1\Big)\Big(\|v\|_{\mathcal{BMO}^c(\mathcal{M})} + \sup_n \|dw_n\|_\infty\Big)$$

$$\leqq 2\sqrt{2}\|x\|_{\widetilde{\mathcal{H}}_1^c(\mathcal{M})}\|u\|_{E_\infty^c(\mathcal{M})}.$$

Thus $l_u \in \widetilde{\mathcal{H}}_1^c(\mathcal{M})^*$ and $\|l_u\| \leqq 2\sqrt{2}\|u\|_{E_\infty^c(\mathcal{M})}$.

We pass to the converse inclusion. Let $l \in \widetilde{\mathcal{H}}_1^c(\mathcal{M})^*$. First we restrict l on the subspace $\mathcal{H}_1^c(\mathcal{M})$. Since $\mathcal{H}_1^c(\mathcal{M})^* = \mathcal{BMO}^c(\mathcal{M})$, there exists a martingale $v = (v_n)_{n\geqq 1} \in \mathcal{BMO}^c(\mathcal{M})$ such that

$$l(s) = \sum_{n=1}^{\infty}\tau(dv_n^* ds_n) \quad \big(s = (s_n)_{n\geqq 1} \in \mathcal{H}_1^c(\mathcal{M})\big) \tag{3.10}$$

and $\|v\|_{\mathcal{BMO}^c(\mathcal{M})} \leqq \|l\|$.

On the other hand, let Q_1 be the subspace of $l_1\big(L_1(\mathcal{M})\big)$ of all sequences $db = (db_n)_{n\geqq 1}$ such that $b = (b_n)_{n\geqq 1}$ is a predictable quasi-martingale in $\widetilde{\mathcal{H}}_1^c(\mathcal{M})$ with $b_1 = 0$. It is easy to see that $\|db\|_{l_1(L_1(\mathcal{M}))} \leqq \|b\|_{\widetilde{\mathcal{H}}_1^c(\mathcal{M})} \leqq 2\|db\|_{l_1(L_1(\mathcal{M}))}$ for any $db = (db_n)_{n\geqq 1} \in Q_1$. Imitating the proof of Theorem 3.3, there exists a sequence $w = (w_n)_{n\geqq 1} \in BD_\infty(\mathcal{M})$ such that for any $db = (db_n)_{n\geqq 1} \in Q_1$,

$$l(b) = \sum_{n=1}^{\infty}\tau(dw_n^* db_n) \tag{3.11}$$

and $\|w\|_{BD_\infty(\mathcal{M})} \leqq 2\|l\|$. Set $u_n = v_n + w_n (n \geqq 1)$. Then $u = (u_n)_{n\geqq 1} \in E_\infty^c(\mathcal{M})$ and

$$\|u\|_{E_\infty^c(\mathcal{M})} = \|v\|_{\mathcal{BMO}^c(\mathcal{M})} + \|w\|_{BD_\infty(\mathcal{M})} \leqq \|l\| + 2\|l\| = 3\|l\|.$$

For any $x = (x_n)_{n\geqq 1} \in \widetilde{\mathcal{H}}_1^c(\mathcal{M})$, let $x_n = y_n + z_n (n \geqq 1)$ be its Doob's decomposition. Noting that $dz = (dz_n)_{n\geqq 1} \in Q_1$, it follows from (3.10) and (3.11) that

$$l(x) = l(y) + l(z) = \sum_{n=1}^{\infty}\tau(dv_n^* dy_n) + \sum_{n=1}^{\infty}\tau(dw_n^* dz_n).$$

Therefore, this proves that $\widetilde{\mathcal{H}}_1^c(\mathcal{M})^* = E_\infty^c(\mathcal{M})$. Passing to adjoint, we obtain the identity $\widetilde{\mathcal{H}}_1^r(\mathcal{M})^* = E_\infty^r(\mathcal{M})$.

(ii) The duality between $\widetilde{\mathcal{H}}_1(\mathcal{M})$ and $E_\infty(\mathcal{M})$ is deduced from the standard duality between intersection and sum spaces. □

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