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PROBABILISTIC UNIFORM CONVERGENCE SPACES REDEFINED

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Abstract. We develop a theory of probabilistic uniform convergence spaces based on Tardiff's neighbourhood systems for probabilistic metric spaces. We show that the resulting category is topological and Cartesian closed. A subcategory is identified that is isomorphic to the category of probabilistic metric spaces.

1. Introduction

Probabilistic uniform convergence spaces were first defined by Nusser [6]. They form a generalization of both uniform convergence spaces as defined by Cook and Fischer [2] (and improved by Wyler [13]) and probabilistic uniform spaces as defined by Florescu [3]. A probabilistic uniform convergence structure can loosely be described as a "tower" of uniform convergence structures, indexed by the unit interval $[0,1]$. In this paper, we generalize Nusser's definition by replacing the "index set" $[0,1]$ by the set Δ^+ of distance distribution functions [10]. A similar idea was used by Tardiff [12] to generate a family of neighbourhood structures, and thus a family of topologies, for a probabilistic metric space. Although the use of a fixed distance distribution function, a so-called *profile function* $\varphi \in \Delta^*$, allows a probabilistic interpretation (see e.g. [4,10]), Tardiff does not attach such an interpretation to his neighbourhood systems. Likewise, we also do not see such an interpretation in our case but use the new index set solely as a technical tool and keep the name "probabilistic uniform convergence space" because of the similarity of our spaces with the ones of Nusser and the relation to probabilistic metric spaces.

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Our generalization solves two problems. Firstly, just like metric spaces carry in a natural way a uniform structure (and hence also a uniform convergence structure), we can show, using an idea of Tardiff [12], that each probabilistic metric space carries a natural probabilistic uniform structure. Secondly, we can identify a subcategory of the category of probabilistic uniform convergence spaces that is isomorphic to the category of probabilistic metric spaces. In this sense, we can characterize probabilistic metric spaces entirely by their probabilistic uniform (convergence) structure.

2. Preliminaries

For an ordered set $(A, \leq)$ we denote, in case of existence, by $\bigwedge_{i\in I} \alpha_i$ the infimum and by $\bigvee_{i\in I} \alpha_i$ the supremum of $\{\alpha_i : i \in I\} \subseteqq A$. In case of a two-point set $\{\alpha, \beta\}$ we write $\alpha \wedge \beta$ and $\alpha \vee \beta$, respectively.

For a set S, we denote its power set by $P(S)$ and the set of all filters $\mathbb{F}$, $\mathbb{G}$, … on S by $\mathbb{F}(S)$. The set $\mathbb{F}(S)$ is ordered by set inclusion and maximal elements of $\mathbb{F}(S)$ in this order are called *ultrafilters*. In particular, for each $p \in S$, the point filter $[p] = \{A \subseteqq S : p \in A\} \in \mathbb{F}(S)$ is an ultrafilter. For $\mathbb{F} \in \mathbb{F}(S)$ and $\mathbb{G} \in \mathbb{F}(T)$ we denote $\mathbb{F} \times \mathbb{G}$ the filter on $S \times T$ generated by the sets $F \times G$ where $F \in \mathbb{F}$ and $G \in \mathbb{G}$. For $\Phi, \Psi \in \mathbb{F}(S \times S)$ we define Φ^{-1} to be the filter generated by the sets $\phi^{-1} = \{(p, q) \in S \times S : (q, p) \in \phi\}$ and if for all $\phi \in \Phi$ and $\psi \in \Psi$ the sets $\phi \circ \psi = \{(p, q) \in S \times S : \exists r \in S$ such that $(p, r) \in \phi, (r, q) \in \psi\}$ are non-empty, we denote the filter generated by these sets by $\Phi \circ \Psi$.

We assume some familiarity with category theory and refer to the textbooks [1] and [7] for more details. A *construct* is a category $\mathcal{C}$ whose objects are structured sets (S, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i \colon S \longrightarrow (S_i, \xi_i))_{i\in I}$ there is a unique structure ξ on S, such that a mapping $g \colon (T, \eta) \longrightarrow (S, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g \colon (T, \eta) \longrightarrow (S_i, \xi_i)$ is a morphism.

A topological construct is called *Cartesian closed* if for each pair of objects (S, ξ), (T, η) there is a function space structure on the set $C(S, T)$ of morphisms from S to T such that the *evaluation mapping* $\mathrm{ev} \colon C(S, T) \times S \longrightarrow T$, $(f, s) \longmapsto f(s)$ is a morphism and that for each object (Z, ζ) and each morphism $f \colon S \times Z \longrightarrow T$ the mapping $f^* \colon Z \longrightarrow C(S, T)$ defined by $f^*(z)(x) = f(x, z)$ is a morphism [7].

A function $\varphi \colon [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$ is called a *distance distribution function* [10]. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq a < \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases} \qquad \text{and} \qquad \varepsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$

are in Δ^+. The set Δ^+ is ordered pointwise with smallest element ε_∞ and largest element ε_0.

LEMMA 2.1 [10]. (i) *If* $\varphi, \psi \in \Delta^+$, *then also* $\varphi \wedge \psi \in \Delta^+$.
(ii) *If* $\varphi_i \in \Delta^+$ *for all* $i \in I$, *then also* $\bigvee_{i\in I} \varphi_i \in \Delta^+$.

Here, $\varphi \wedge \psi$ denotes the pointwise minimum of φ and ψ and $\bigvee_{i\in I} \varphi_i$ denotes the pointwise supremum of the family $\{\varphi_i : i \in I\}$.

On the set Δ^+, we can define a metric such that weak convergence of distribution functions is convergence in the metric [11]. We follow the exposition in Tardiff [12] and define the *modified Lévy metric* on Δ^+ by

$$d_L(\varphi, \psi) = \bigwedge \big\{ \varepsilon > 0 : A(\varphi, \psi, \varepsilon) \text{ and } B(\varphi, \psi, \varepsilon) \big\}$$

where

$$A(\varphi, \psi, \varepsilon) \iff \varphi(x - \varepsilon) - \varepsilon \le \psi(x), \quad \text{for } x \in [0, 1/\varepsilon + \varepsilon)$$
$$B(\varphi, \psi, \varepsilon) \iff \varphi(x + \varepsilon) + \varepsilon \ge \psi(x), \quad \text{for } x \in [0, 1/\varepsilon).$$

A binary operation, $\tau\colon \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and that satisfies the boundary condition $\tau(\varphi, \varepsilon_0) = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function* [10]. The largest triangle function is the pointwise minimum $\mu(\varphi, \psi) = \varphi \wedge \psi$. It is not difficult to show that a triangle function that is idempotent, i.e. for which $\tau(\varphi, \varphi) = \varphi$ for all $\varphi \in \Delta^+$, must be the largest triangle function. For a good survey on triangle functions see, e.g., [8,9]. A triangle function is called *continuous* [10,12] if it is a continuous function with respect to the topology and product topology induced by the modified Lévy metric. A triangle function is called *sup-continuous* [10,12], if $\tau(\bigvee_{i\in I} \varphi_i, \psi) = \bigvee_{i\in I} \tau(\varphi_i, \psi)$ for all $\varphi_i, \psi \in \Delta^+$, $(i \in I)$.

A *t-norm* $*\colon [0,1] \times [0,1] \longrightarrow [0,1]$ is a binary operation on $[0,1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. A t-norm is called continuous if it is continuous as a mapping from $[0,1] \times [0,1] \longrightarrow [0,1]$. It is shown e.g. in [10] that for a t-norm $*$, the mapping τ_* defined by $\tau_*(\varphi, \psi)(x) = \bigvee_{u+v=x} \varphi(u) * \psi(v)$ for $\varphi, \psi \in \Delta^+$ is a triangle function. For $\varphi \in \Delta^+$, we denote the right-hand limit $\varphi(0+) = \lim_{x\to 0+} \varphi(x)$.

LEMMA 2.2. *For a continuous t-norm* $*$ *the triangle funtion* $\tau_*(\varphi, \psi)(x) = \bigvee_{u+v=x} \varphi(u) * \psi(v)$ *satisfies* $\tau_*(\varphi, \psi)(0+) = \varphi(0+) * \psi(0+)$.

PROOF. Let $x_n \to 0$ for $n \to \infty$ and $x_n > 0$ for all n. Then $\tau_*(\varphi, \psi)(x_n) = \bigvee_{u+v=x_n} \varphi(u) * \psi(v) \to \tau_*(\varphi, \psi)(0+)$. Hence there are sequences u_k^n, v_k^n such that $u_k^n + v_k^n = x_n$ for all k and $\varphi(u_k^n) * \psi(v_k^n) \to \bigvee_{u+v=x_n} \varphi(u) * \psi(v)$.

Clearly, because $0 \le u_k^n, v_k^n \le x_n$ then $u_k^n, v_k^n \to 0$ for $n \to \infty$ for all k. Because $\varphi(0) = \psi(0) = 0$ we may assume that $u_n^n, v_n^n > 0$ and hence $\varphi(u_n^n) \to \varphi(0+)$ and $\psi(u_n^n) \to \psi(0+)$. From the continuity of $*$ we conclude $\varphi(u_n^n) * \psi(v_n^n) \to \varphi(0+) * \psi(0+)$. As subsequences of u_k^n and v_k^n also $\varphi(u_n^n) * \psi(v_n^n) \to \tau_*(\varphi, \psi)(0+)$ and the proof is complete. $\square$

3. Probabilistic uniform convergence spaces

Definition 3.1. A pair $(S, \overline{\Lambda})$ with a set S and $\overline{\Lambda} = (\Lambda_\varphi)_{\varphi \in \Delta^+}$, where $\Lambda_\varphi \subseteqq \mathbb{F}(S \times S)$, is called a *probabilistic uniform convergence space* if

(PUC1) $[(p,p)] \in \Lambda_\varphi$ for all $p \in S, \varphi \in \Delta^+$;
(PUC2) $\Phi \in \Lambda_\varphi$, $\Psi \ge \Phi$ implies $\Psi \in \Lambda_\varphi$;
(PUC3) $\Phi, \Psi \in \Lambda_\varphi$ implies $\Phi \wedge \Psi \in \Lambda_\varphi$;
(PUC4) $\Phi \in \Lambda_\varphi$ implies $\Phi^{-1} \in \Lambda_\varphi$;
(PUC5) $\Phi \in \Lambda_\varphi$ and $\Psi \in \Lambda_\psi$ and $\Phi \circ \Psi \in \mathbb{F}(S \times S)$ implies $\Phi \circ \Psi \in \Lambda_{\tau(\varphi,\psi)}$;
(PUC6) $\psi \le \varphi$ implies $\Lambda_\varphi \subseteqq \Lambda_\psi$;
(PUC7) $\Lambda_{\varepsilon_\infty} = \mathbb{F}(S \times S)$.

A mapping $f\colon (S, \overline{\Lambda}) \longrightarrow (S', \overline{\Lambda'})$ is called *uniformly continuous* if for all $\varphi \in \Delta^+$, $\Phi \in \Lambda_\varphi$ implies $(f \times f)(\Phi) \in \Lambda'_\varphi$. The category of probabilistic uniform convergence spaces and uniformly continuous mappings is denoted by *PUCS*.

Example 3.2. A *Nusser-probabilistic uniform convergence space under the t-norm* $*$ [6] is a pair $(S, \overline{\mathcal{J}})$ with a set S and $\overline{\mathcal{J}} = (\mathcal{J}_\alpha)_{\alpha \in [0,1]}$ such that

(NPUC1) $[(p,p)] \in \mathcal{J}_\alpha$ for all $p \in S, \alpha \in [0,1]$;
(NPUC2) $\Phi \in \mathcal{J}_\alpha$, $\Psi \ge \Phi$ implies $\Psi \in \mathcal{J}_\alpha$;
(NPUC3) $\Phi, \Psi \in \mathcal{J}_\alpha$ implies $\Phi \wedge \Psi \in \mathcal{J}_\alpha$;
(NPUC4) $\Phi \in \mathcal{J}_\alpha$ implies $\Phi^{-1} \in \mathcal{J}_\alpha$;
(NPUC5) $\Phi \in \mathcal{J}_\alpha$ and $\Psi \in \mathcal{J}_\beta$ and $\Phi \circ \Psi \in \mathbb{F}(S \times S)$ implies $\Phi \circ \Psi \in \mathcal{J}_{\alpha * \beta}$;
(NPUC6) $\beta \le \alpha$ implies $\mathcal{J}_\alpha \subseteqq \mathcal{J}_\beta$;
(NPUC7) $\mathcal{J}_0 = \mathbb{F}(S \times S)$.

A mapping $f\colon (S, \overline{\mathcal{J}}) \longrightarrow (S', \overline{\mathcal{J}'})$ is called *uniformly continuous* if for all $\alpha \in [0,1]$, $\Phi \in \mathcal{J}_\alpha$ implies $(f \times f)(\Phi) \in \mathcal{J}'_\alpha$. The category of Nusser-probabilistic uniform convergence spaces and uniformly continuous mappings is denoted by *N-PUCS*.

Let now $*$ be a continuous t-norm and consider the triangle function τ_*. For $(S, \overline{\mathcal{J}}) \in |N\text{-}PUCS|$, we define $\Phi \in \Lambda_\varphi^{\mathcal{J}}$ if $\Phi \in \mathcal{J}_{\varphi(0+)}$. With Lemma 2.2 we see that $(S, \overline{\Lambda^{\mathcal{J}}}) \in |PUCS|$. Also a uniformly continuous mapping $f\colon (S, \overline{\mathcal{J}}) \longrightarrow (S', \overline{\mathcal{J}'})$ is uniformly continuous as a mapping $f\colon (S, \overline{\Lambda^{\mathcal{J}}}) \longrightarrow (S', \overline{\Lambda^{\mathcal{J}'}})$. Hence we have an embedding functor $A\colon N\text{-}PUCS \longrightarrow PUCS$ and *N-PUCS* is isomorphic to a subcategory of *PUCS*. Given now $(S, \overline{\Lambda})$

we define $\Phi \in J_\alpha^\Lambda$ if there is $\varphi \in \Delta^+$ with $\varphi(0+) = \alpha$ and $\Phi \in \Lambda_\varphi$. Again we can easily see that this defines a functor $B\colon PUCS \longrightarrow N\text{-}PUCS$. Moreover $A \circ B \geq \mathrm{id}_{PUCS}$ and $B \circ A = \mathrm{id}_{N\text{-}PUCS}$. Hence N-$PUCS$ is a reflective subcategory of $PUCS$.

EXAMPLE 3.3. Let (S, F) be a *probabilistic metric space* [10], i.e. $F\colon S \times S \longrightarrow \Delta^+$ with the properties that for all $p, q \in S$

(PM1) $F(p, q) = \varepsilon_0 \iff p = q$,
(PM2) $F(p, q) = F(q, p)$,
(PM3) $\tau(F(p, q), F(q, r)) \leq F(p, r)$,

with a triangle function τ. We usually use the index notation $F_{pq} = F(p, q)$.

A mapping $f\colon (S, F) \longrightarrow (S', F')$ is *non-expansive* if $F_{pq} \leq F'_{f(p)f(q)}$ for all $p, q \in S$. The category $PMET$ has as objects the probabilistic metric spaces and as morphisms the non-expansive mappings.

Let $\varphi \in \Delta^+$ and $\varepsilon > 0$. Define (cf. Tardiff [12])

$$N_\varphi^\varepsilon = \left\{ (p, q) \in S \times S : F_{pq}(x + \varepsilon) + \varepsilon \geq \varphi(x) \ \forall\, x \in [0, \tfrac{1}{\varepsilon}) \right\}.$$

Then, clearly $D = \{(p, p) : p \in S\} \subseteqq N_\varphi^\varepsilon$.

Further, let $(p, q) \in N_{\varphi, \varepsilon_1 \wedge \varepsilon_2}$. Then $F_{pq}(x + \varepsilon_1 \wedge \varepsilon_2) + \varepsilon_1 \wedge \varepsilon_2 \geq \varphi(x)$ for all $x \in [0, \frac{1}{\varepsilon_1 \wedge \varepsilon_2})$. Because $F_{pq} \in \Delta^+$ we conclude that for all $x \in [0, \frac{1}{\varepsilon_1})$, $[0, \frac{1}{\varepsilon_2}) \subseteqq [0, \frac{1}{\varepsilon_1 \wedge \varepsilon_2})$ we have

$$F_{pq}(x + \varepsilon_1) + \varepsilon_1 \geq \varphi(x), F_{pq}(x + \varepsilon_2) + \varepsilon_2 \geq \varphi(x)$$

and hence $(p, q) \in N_\varphi^{\varepsilon_1} \cap N_\varphi^{\varepsilon_2}$.

From the symmetry, $F_{pq} = F_{qp}$, we further obtain $(N_\varphi^\varepsilon)^{-1} = N_\varphi^\varepsilon$.

Let now τ be a continuous triangle function. By a result of Tardiff [12, Lemma 2.4], for $F, G \in \Delta^+$ and $\varepsilon > 0$ there is $\delta > 0$ such that

$$\tau(F, G)(x + \varepsilon) + \varepsilon \geq \tau(F^\delta, G^\delta)(x) \quad \forall\, x \in [0, \tfrac{1}{\varepsilon}).$$

Here,

$$F^\delta(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ F(x + \delta) + \delta \wedge 1 & \text{if } x \in (0, \frac{1}{\delta}] \\ 1 & \text{if } x > \frac{1}{\delta}. \end{cases}$$

Furthermore, by Tardiff [12, Lemma 2.5], $(p, q) \in N_\varphi^\delta$ iff $F_{pq}^\delta \geq \varphi$. Let now $(p, q) \in N_\varphi^\delta \circ N_\psi^\delta$. Then there is $r \in S$ such that $(p, r) \in N_\varphi^\delta$ and $(r, q) \in N_\psi^\delta$. Hence $F_{pr}^\delta \geq \varphi$ and $F_{rq}^\delta \geq \psi$. It follows that for all $x \in [0, \frac{1}{\varepsilon})$ we have

$$\tau(\varphi, \psi)(x) \leq \tau(F_{pr}^\delta, F_{rq}^\delta)(x) \leq \tau(F_{pr}, F_{rq})(x + \varepsilon) + \varepsilon \leq F_{pq}(x + \varepsilon) + \varepsilon.$$

Hence $(p,q) \in N^{\varepsilon}_{\tau(\varphi,\psi)}$.

We can therefore define the *φ-entourage filter* $\mathbb{N}^F_\varphi$, which is the filter on $S \times S$ generated by the family $\{N^{\varepsilon}_{\varphi} : \varepsilon > 0\}$.

LEMMA 3.4. *Let $(S,F) \in |PMET|$ with a continuous triangle function τ. Then*
(1) $\mathbb{N}^F_\varphi \in \mathbb{F}(S \times S)$.
(2) $\mathbb{N}^F_\varphi \leq [D]$.
(3) $\mathbb{N}^F_\varphi \leq (\mathbb{N}^F_\varphi)^{-1}$.
(4) $\mathbb{N}^F_{\tau(\varphi,\psi)} \leq \mathbb{N}^F_\varphi \circ \mathbb{N}^F_\psi$.
(5) $\varphi \leq \psi$ *implies* $\mathbb{N}^F_\varphi \leq \mathbb{N}^F_\psi$.
(6) $\mathbb{N}^F_{\varepsilon_\infty} = [S \times S]$.

LEMMA 3.5. *Let $f \colon (S,F) \longrightarrow (S',F')$ be non-expansive. Then $\mathbb{N}^{F'}_\varphi \leq (f \times f)(\mathbb{N}^F_\varphi)$.*

PROOF. Let $(p,q) \in N^{\varepsilon}_{\varphi}$. Then for all $x \in [0, \frac{1}{\varepsilon})$ we have $F'_{f(p)f(q)}(x+\varepsilon) +\varepsilon \geq F_{pq}(x+\varepsilon) + \varepsilon \geq \varphi(x)$ and hence $(f \times f)(p,q) = (f(p), f(q)) \in N'^{\varepsilon}_{\varphi}$. $\square$

We define now $\Phi \in \Lambda^F_\varphi$ if $\Phi \geq \mathbb{N}^F_\varphi$. It is then clear that $(S, \overline{\Lambda^F}) \in |PUCS|$ In this sense, probabilistic metric spaces carry a natural probabilistic uniform convergence structure.

After these examples, we look at the categorical properties of $PUCS$.

LEMMA 3.6. *$PUCS$ is a toppological category.*

PROOF. The proof is routine and we only state the initial structures. Let $(S_j, \overline{\Lambda^j}) \in |PUCS|$ and $f_j : S \longrightarrow S_j$ be mappings for all $j \in J$. We define, for $\Phi \in \mathbb{F}(S \times S)$ and $\varphi \in \Delta^+$, the initial structure $\overline{\Lambda}$ on S by $\Phi \in \Lambda_\varphi \iff (f_j \times f_j)(\Phi) \in \Lambda^j_\varphi \ \forall\, j \in J$. $\square$

As a consequence we have subspaces and product spaces in $PUCS$.

LEMMA 3.7. *$PUCS$ is Cartesian closed.*

PROOF. Also this proof is routine and we only state the function space structures. We define for $(S, \overline{\Lambda^S}), (T, \overline{\Lambda^T}) \in |PUCS|$ the set $UC(S,T) = \{f \colon S \longrightarrow T \text{ uniformly continuous}\}$. On $UC(S,T)$ we define the probabilistic uniform convergence structure $\overline{K}$ by $\Phi \in K_\varphi \iff \forall\, \psi \leq \varphi(\mathbb{F} \in \Lambda^S_\varphi \Rightarrow \Phi(\mathbb{F}) \in \Lambda^T_\varphi)$. Here $\Phi(\mathbb{F})$ is the filter generated by the sets $\phi(F) = \{(f(p), g(q)) : (f,g) \in \phi,\ (p,q) \in F\}$, where $\phi \in \Phi$ and $F \in \mathbb{F}$, and

$$\mathrm{ev} \colon UC(S,T) \times S \longrightarrow T, \quad (f,p) \longmapsto f(p)$$

is the evaluation mapping. $\square$

4. The underlying probabilistic convergence structure

DEFINITION 4.1 [5]. Let S be a set. A family of mappings $\overline{c} = (c_\varphi \colon \mathbb{F}(S) \longrightarrow P(S))_{\varphi \in \Delta^+}$ which satisfies the axioms

(PC1) $p \in c_\varphi([p])$ for all $p \in S, \varphi \in \Delta^+$,
(PC2) $c_\varphi(\mathbb{F}) \subseteqq c_\varphi(\mathbb{G})$ whenever $\mathbb{F} \le \mathbb{G}$,
(PC3) $c_\psi(\mathbb{F}) \subseteqq c_\varphi(\mathbb{F})$ whenever $\varphi \le \psi$,
(PC4) $p \in c_{\varepsilon_\infty}(\mathbb{F})$ for all $p \in S, \mathbb{F} \in \mathbb{F}(S)$,

is called a *probabilistic convergence structure on* S and the pair $(S, \overline{c})$ is called a *probabilistic convergence space.* A mapping $f \colon (S, \overline{c}) \longrightarrow (S', \overline{c'})$ is called *continuous* if $f(p) \in c'_\varphi(f(\mathbb{F}))$ whenever $p \in c_\varphi(\mathbb{F})$ for every $p \in S$ and for every $\mathbb{F} \in \mathbb{F}(S)$. The category of probabilistic convergence spaces with continuous mappings as morphisms is denoted by $PCONV$.

For $(S, F) \in |PMET|$, $N \in \mathbb{N}^F_\varphi$ and $p \in S$ we define $N(p) = \{q \in S : (p,q) \in N\}$ and $\mathbb{N}_\varphi(p)$ as the filter generated by the family $\{N(p) : N \in \mathbb{N}^F_\varphi\}$. Then $\mathbb{N}_\varphi(p)$ is the φ-neighbourhood filter of p, $\mathbb{N}^p_\varphi$ in Tardiff [12]. In fact, we have $(p,q) \in N^\varepsilon_\varphi$ iff $q \in N^\varepsilon_\varphi(p)$ iff $F_{pq}(x+\varepsilon) + \varepsilon \ge \varphi(x)\ \forall\, x \in [0, \frac{1}{\varepsilon})$ iff $q \in N^{p,\varepsilon}_\varphi$.

LEMMA 4.2. *Let* $(S,F) \in |PMET|$ *and* $p \in S$ *and* $\mathbb{F} \in \mathbb{F}(S)$. *Then* $\mathbb{F} \ge \mathbb{N}^p_\varphi$ *iff* $\mathbb{F} \times [p] \ge \mathbb{N}_\varphi$.

PROOF. We have $\mathbb{F} \times [p] \ge \mathbb{N}\varphi$ iff for all $F \in \mathbb{F}$ we have that $F \times \{p\} \in \mathbb{N}_\varphi$. This is equivalent to that for all $F \in \mathbb{F}$ there is $\varepsilon > 0$ such that $N^\varepsilon_\varphi \subseteqq F \times \{p\}$. The latter means that from $(p,q) \in N^\varepsilon_\varphi$ it follows that $q \in F$, i.e. that from $q \in N^\varepsilon_\varphi(p)$ it follows that $q \in F$. Hence $\mathbb{F} \times [p] \ge \mathbb{N}_\varphi$ is equivalent to that for all $F \in \mathbb{F}$ there is $\varepsilon > 0$ such that $N^\varepsilon_\varphi(p) \subseteqq F$, i.e. to $\mathbb{F} \ge \mathbb{N}^p_\varphi$. □

We therefore define, for $(S, \overline{\Lambda}) \in |PUCS|$, the probabilistic convergence structure $\overline{c^\Lambda}$ by $p \in c^\Lambda_\varphi(\mathbb{F}) \iff \mathbb{F} \times [p] \in \Lambda_\varphi$.

LEMMA 4.3. *Let* $(S, \overline{\Lambda}) \in |PUCS|$. *Then* $(S, \overline{c^\Lambda}) \in |PCONV|$ *and satisfies axiom*

(PLS5) *for all* $\varphi \in \Delta^+$ *and all* $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, $c^\Lambda_\varphi(\mathbb{F}) \cap c^\Lambda_\varphi(\mathbb{G}) \subseteqq c^\Lambda_\varphi(\mathbb{F} \wedge \mathbb{G})$.

LEMMA 4.4. *Let* $(S, \overline{\Lambda}), (T, \overline{M}) \in |PUCS|$ *and let* $f \colon (S, \overline{\Lambda}) \longrightarrow (T, \overline{M})$ *be uniformly continuous. Then* $f \colon (S, \overline{c^\Lambda}) \longrightarrow (T, \overline{c^M})$ *is continuous.*

PROOF. If $p \in c^\Lambda_\varphi(\mathbb{F})$, then $\mathbb{F} \times [p] \in \Lambda_\varphi$. The uniform continuity of f then implies $f(\mathbb{F}) \times [f(p)] = (f \times f)(\mathbb{F} \times [p]) \in M_\varphi$, i.e. $f(p) \in c^M_\varphi(f(\mathbb{F}))$. □

Hence we can define a functor

$$C : \begin{cases} PUCS \longrightarrow PCONV \\ (S, \overline{\Lambda}) \longmapsto (S, \overline{c^\Lambda}) \\ f \longmapsto f. \end{cases}$$

LEMMA 4.5. *The functor C preserves initial constructions: If $(f_j\colon S \longrightarrow (S_j, \overline{\Lambda^j}))_{j\in J}$ is a source in PUCS and $\overline{\Lambda}$ the initial structure on S, then $\overline{c^\Lambda}$ is the initial structure with respect to the source $(f_j\colon S \longrightarrow (S_j, \overline{c^{\Lambda^j}}))_{j\in J}$.*

PROOF. We have $p \in c^\Lambda_\varphi(\mathbb{F})$ iff $\mathbb{F} \times [p] \in \Lambda_\varphi$. This is equivalent to $f_j(\mathbb{F}) \times [f_j(p)] = (f_j \times f_j)(\mathbb{F} \times [p]) \in \Lambda^j_\varphi$ for all $j \in J$, i.e. to $f_j(p) \in c^{\Lambda^j}_\varphi(f_j(\mathbb{F}))$ for all $j \in J$. □

The probabibilistic convergence space underlying a probabilistic uniform convergence space has some strong properties. For a triangle function τ, a probabilistic convergence space $(S, \overline{c})$ is called *τ-transitive* [5] if for all $p, q, r \in S$, $p \in c_{\tau(\varphi,\psi)}([r])$ whenever $p \in c_\varphi([q])$ and $q \in c_\psi([r])$.

LEMMA 4.6. *If $(S, \overline{\Lambda}) \in |PUCS|$, then $(S, \overline{c^\Lambda})$ is τ-transitive.*

PROOF. If $p \in c^\Lambda_\varphi([q])$ and $q \in c^\Lambda_\psi([r])$, then $[q] \times [p] \in \Lambda_\varphi$ and $[r] \times [q] \in \Lambda_\psi$. Hence $[r] \times [p] = ([r] \times [q]) \circ ([q] \times [p]) \in \Lambda_{\tau(\varphi,\psi)}$, i.e. $p \in c^\Lambda_{\tau(\varphi,\psi)}([r])$. □

A probabilistic convergence space $(S, \overline{c})$ is called *symmetric* [5] if for all $p, q \in S$, $p \in c_\varphi([q])$ whenever $q \in c_\varphi([p])$.

LEMMA 4.7. *If $(S, \overline{\Lambda}) \in |PUCS|$, then $(S, \overline{c^\Lambda})$ is symmetric.*

PROOF. This follows from (PUC3) and $([p] \times [q])^{-1} = [q] \times [p]$. □

A probabilistic convergence space $(S, \overline{c})$ is called a T1-*space* [5] if $p = q$ whenever $\bigvee\{\varphi : p \in c_\varphi([q])\} = \varepsilon_0$. It is called a T2-*space* if $p = q$ whenever $\bigvee\{\varphi : p \in c_\varphi(\mathbb{F})\} = \varepsilon_0 = \bigvee\{\psi : q \in c_\psi(\mathbb{F})\}$. Clearly, every T2-space is always a T1-space.

LEMMA 4.8. *If $(S, \overline{\Lambda}) \in |PUCS|$ and the triangle function τ is sup-continuous, then $(S, \overline{c^\Lambda})$ is a* T1-*space if and only if it is a* T2-*space.*

PROOF. Let $\mathbb{F} \in \mathbb{F}(S)$ and $p, q \in S$. If $p \in c^\Lambda_\varphi(\mathbb{F})$ and $q \in c^\Lambda_\psi(\mathbb{F})$, then $\mathbb{F} \times [p] \in \Lambda_\varphi$ and $\mathbb{F} \times [q] \in \Lambda_\psi$. By symmetry then $[q] \times \mathbb{F} = (\mathbb{F} \times [q])^{-1} \in \Lambda_\psi$ and hence

$$[q] \times [p] = ([q] \times \mathbb{F}) \circ (\mathbb{F} \times [p]) \in \Lambda_{\tau(\varphi,\psi)},$$

i.e. $q \in c^\Lambda_{\tau(\varphi,\psi)}([p])$. So if $\bigvee_{p \in c^\Lambda_\varphi(\mathbb{F})} \varphi = \varepsilon_0 = \bigvee_{q \in c^\Lambda_\psi(\mathbb{F})} \psi$, then

$$\varepsilon_0 = \tau(\varepsilon_0, \varepsilon_0) = \bigvee_{p \in c^\Lambda_\varphi(\mathbb{F}), q \in c^\Lambda_\psi(\mathbb{F})} \tau(\varphi, \psi) \le \bigvee_{q \in c^\Lambda_{\tau(\varphi,\psi)}([p])} \tau(\varphi, \psi) \le \bigvee_{q \in c^\Lambda_\eta([p])} \eta,$$

and hence $p = q$, because $(S, \overline{c^\Lambda})$ is a T1-space. □

5. Probabilistic uniform spaces and principal probabilistic uniform convergence spaces

Motivated by Lemma 3.4 we give the following definition.

DEFINITION 5.1. A pair $(S,\overline{\mathbb{N}})$ with $\overline{\mathbb{N}} = (\mathbb{N}_\varphi)_{\varphi\in\Delta^+}$ is called a *probabilistic uniform space* (under the triangle function τ) if for all $\varphi,\psi \in \Delta^+$

(PU1) $\mathbb{N}_\varphi \in \mathbb{F}(S\times S)$.
(PU2) $\mathbb{N}_\varphi \le [D]$.
(PU3) $\mathbb{N}_\varphi \le (\mathbb{N}_\varphi)^{-1}$.
(PU4) $\mathbb{N}_{\tau(\varphi,\psi)} \le \mathbb{N}_\varphi \circ \mathbb{N}_\psi$.
(PU5) $\varphi \le \psi$ implies $\mathbb{N}_\varphi \le \mathbb{N}_\psi$.
(PU6) $\mathbb{N}_{\varepsilon_\infty} = [S\times S]$.

A mapping $f\colon (S,\overline{\mathbb{N}}) \longrightarrow (S',\overline{\mathbb{N}'})$ is called *uniformly continuous* if $\mathbb{N}'_\varphi \le (f\times f)(\mathbb{N}_\varphi)$ for all $\varphi\in\Delta^+$. The category with objects the probabilistic uniform spaces and uniformly continuous mappings as morphisms is denoted by *PUNIF*.

Let $(S,\overline{\mathbb{N}}) \in |PUNIF|$. If we define $\Phi \in \Lambda^{\mathbb{N}}_\varphi$ if $\Phi \ge \mathbb{N}_\varphi$, then $(S,\overline{\Lambda^{\mathbb{N}}}) \in |PUCS|$. It is also clear that uniformly continuous mappings between probabilistic uniform spaces are uniformly continuous as mappings between the corresponding probabilistic uniform convergence spaces. Hence *PUNIF* is isomorphic to a subcategory of *PUCS*.

EXAMPLE 5.2. Let $(S,F)\in |PMET|$. Then $(S,\overline{\mathbb{N}^F}) \in |PUNIF|$.

EXAMPLE 5.3 (Florescu [3]). A *Florescu-probabilistic uniform space* $(S,\overline{\mathbb{U}})$ with $\overline{\mathbb{U}} = (\mathbb{U}_\alpha)_{\alpha\in[0,1]}$ (under the t-norm $*$) satisfies the axioms

(FPU1) $\mathbb{U}_\alpha \in \mathbb{F}(S\times S)$ for all $\alpha\in[0,1]$.
(FPU2) $\mathbb{U}_\alpha \le [D]$ for all $\alpha\in[0,1]$.
(FPU3) $\mathbb{U}_\alpha \le (\mathbb{U}_\alpha)^{-1}$.
(FPU4) $\mathbb{U}_{\alpha*\beta} \le \mathbb{U}_\alpha \circ \mathbb{U}_\beta$.
(FPU5) $\alpha\le\beta$ implies $\mathbb{U}_\alpha \le \mathbb{U}_\beta$.
(FPU6) $\mathbb{U}_0 = [S\times S]$.

Florescu [3] does not define morphisms, but we can define them by $f\colon (S,\overline{\mathbb{U}}) \longrightarrow (S',\overline{\mathbb{U}'})$ is uniformly continuous if $\mathbb{U}'_\alpha \le (f\times f)(\mathbb{U}_\alpha)$ for all $\alpha \in [0,1]$. We denote the resulting category then by *F-PUNIF*.

For a Florescu-probabilistic uniform space $(S,\overline{\mathbb{U}})$ we define a probabilistic uniform space by $\mathbb{N}^{\mathbb{U}}_\varphi = \mathbb{U}_\alpha$ if $\varphi(0+) = \alpha$. If the t-norm $*$ is continuous, it follows with Lemma 2.2 that $(S,\overline{\mathbb{N}^{\mathbb{U}}})$ is a probabilistic uniform space under the triangle function τ_*. Also, a uniformly continuous function $f\colon (S,\overline{\mathbb{U}}) \to (S',\overline{\mathbb{U}'})$ is uniformly continuous as mapping $f\colon (S,\overline{\mathbb{N}^{\mathbb{U}}}) \to (S',\overline{\mathbb{N}^{\mathbb{U}'}})$. Hence the category of Florescu-probabilistic uniform spaces *F-PUNIF* is a subcategory of *PUNIF*.

Example 5.4. If $(S,\mathcal{U})$ is a uniform space in the sense of Bourbaki and if we define for $\varphi\in\Delta^+$, $\mathbb{N}^{\mathcal{U}}_\varphi=\mathcal{U}$, then $(S,\overline{\mathbb{N}^{\mathcal{U}}})$ is a probabilistic uniform space. Clearly, uniform continuity is also preserved and hence $UNIF$ is a subcategory of $PUNIF$.

Definition 5.5. Let $(S,\overline{\Lambda})\in|PUCS|$. Then $(S,\overline{\Lambda})$ is called a *principal probabilistic uniform convergence space* if $\Phi\in\Lambda_\varphi$ whenever $\Phi\geq\mathbb{N}^\Lambda_\varphi$, where $\mathbb{N}^\Lambda_\varphi=\bigwedge_{\Psi\in\Lambda_\varphi}\Psi$. The subcategory of $PUCS$ consisting of the principal probabilistic uniform convergence spaces is denoted by $PPUCS$.

Clearly, for $(S,F)\in|PMET|$, $(S,\overline{\Lambda^F})$ is principal and we have $\mathbb{N}^{\Lambda^F}_\varphi=\bigwedge_{\Psi\in\Lambda^F_\varphi}\Psi=\bigwedge_{\Psi\geq\mathbb{N}^F_\varphi}\Psi=\mathbb{N}^F_\varphi$.

Lemma 5.6. *Let $(S,\overline{\Lambda})\in|PPUCS|$. Then $(S,\overline{\mathbb{N}^\Lambda})\in|PUNIF|$.*

Proof. Clearly (PU1) is true. For (PU2) we note that for all $p\in S$ we have $[(p,p)]\in\Lambda_\varphi$ and hence $[(p,p)]\geq\mathbb{N}^\Lambda_\varphi$. It follows that $[D]=\bigwedge_{p\in S}[(p,p)]\geq\mathbb{N}^\Lambda_\varphi$. (PU3), (PU4) and (PU5) follow from the fact that $\mathbb{N}^\Lambda_\varphi\in\Lambda_\varphi$. (PU6) finally follows from $[S\times S]\in\Lambda_{\varepsilon_\infty}$. $\square$

Lemma 5.7. *Let $(S,\overline{\Lambda}),(S',\overline{\Lambda'})\in|PPUCS|$ and let $f\colon(S,\overline{\Lambda})\longrightarrow(S',\overline{\Lambda'})$ be uniformly continuous. Then $f\colon(S,\overline{\mathbb{N}^\Lambda})\longrightarrow(S',\overline{\mathbb{N}^{\Lambda'}})$ is uniformly continuous.*

Proof. We have

$$\begin{aligned}(f\times f)(\mathbb{N}^\Lambda_\varphi)&=(f\times f)\Big(\bigwedge_{\Phi\in\Lambda_\varphi}\Phi\Big)=\bigwedge_{\Phi\in\Lambda_\varphi}(f\times f)(\Phi)\\&\geq\bigwedge_{(f\times f)(\Phi)\in\Lambda'_\varphi}(f\times f)(\Phi)\geq\mathbb{N}^{\Lambda'}_\varphi.\quad\square\end{aligned}$$

Hence we can define two functors:

$$A:\begin{cases}PPUCS\longrightarrow PUNIF\\(S,\overline{\Lambda})\longmapsto(S,\overline{\mathbb{N}^\Lambda})\\f\longmapsto f\end{cases}\quad\text{and}\quad B:\begin{cases}PUNIF\longrightarrow PPUCS\\(S,\overline{\mathbb{N}})\longmapsto(S,\overline{\Lambda^{\mathbb{N}}})\\f\longmapsto f.\end{cases}$$

Because $\mathbb{N}^{(\Lambda^{\mathbb{N}})}_\varphi=\bigwedge_{\Phi\in\Lambda^{\mathbb{N}}_\varphi}\Phi=\bigwedge_{\Phi\geq\mathbb{N}_\varphi}\Phi=\mathbb{N}_\varphi$ and $\Phi\in\Lambda^{(\mathbb{N}^\Lambda)}_\varphi\iff\Phi\geq\mathbb{N}^\Lambda_\varphi\iff\Phi\in\Lambda_\varphi$ (because (X,Λ) is principal), these functors are isomorphism functors.

Theorem 5.8. *The categories PUNIF and PPUCS are isomorphic.*

The embedding of *PUNIF* into *PUCS* is very nice if we consider the largest triangle function $\mu(\varphi, \psi) = \varphi \wedge \psi$.

LEMMA 5.9. *If the triangle function is idempotent, then PPUCS is a reflective subcategory of PUCS.*

PROOF. For $(S, \Lambda) \in |PUCS|$ and for $\varphi \in \Delta^+$ we define the φ-entourage filter $\mathbb{N}_\varphi^\Lambda = \bigwedge_{\Phi \in \Lambda_\varphi} \Phi$. Then $\mathbb{N}_\varphi^\Lambda \le [D]$, $\mathbb{N}_\varphi^\Lambda \le (\mathbb{N}_\varphi^\Lambda)^{-1}$ and $\varphi \le \psi$ implies $\mathbb{N}_\varphi^\Lambda \le \mathbb{N}_\psi^\Lambda$. We define $\mathcal{J}_\varphi^\Lambda = \{\Phi \in \mathbb{F}(S \times S) : \Phi \le \mathbb{N}_\varphi^\Lambda,\ \Phi \le \Phi \circ \Phi\}$. Then $[S \times S] \in \mathcal{J}_\varphi^\Lambda$, i.e. $\mathcal{J}_\varphi^\Lambda \neq \emptyset$ and $\mathbb{N}_\varphi^* = \bigvee_{\Phi \in \mathcal{J}_\varphi^\Lambda} \Phi \in \mathbb{F}(S \times S)$. It is not difficult to show that $\mathbb{N}_\varphi^* \in \mathcal{J}_\varphi^\Lambda$ and hence $\mathbb{N}_\varphi^* \le \mathbb{N}_\varphi^* \circ \mathbb{N}_\varphi^*$. Furthermore, $\mathbb{N}_\varphi^* \le \mathbb{N}_\varphi^\Lambda \le [D]$ and also $\mathbb{N}_\varphi^* = (\mathbb{N}_\varphi^*)^{-1}$. We define now $\Phi \in \Lambda_\varphi^*$ if $\Phi \ge \mathbb{N}_\varphi^*$. Clearly then $(S, \overline{\Lambda^*}) \in |PPUCS|$ and for $\Phi \in \Lambda_\varphi$ we have $\Phi \ge \mathbb{N}_\varphi^\Lambda \ge \mathbb{N}_\varphi^*$, i.e. $\Phi \in \Lambda_\varphi^*$. Hence, the identity mapping $\mathrm{id}_S \colon (S, \overline{\Lambda}) \longrightarrow (S, \overline{\Lambda^*})$ is uniformly continuous. If $f \colon (S, \overline{\Lambda}) \longrightarrow (T, \overline{M})$ is uniformly continuous in *PUCS*, then $f \colon (S, \overline{\Lambda^*}) \longrightarrow (T, \overline{M^*})$ is also uniformly continuous as a mapping in *PPUCS*. In fact, let $\Phi \le \mathbb{N}_\varphi^M$ such that $\Phi \le \Phi \circ \Phi$. Then $\Phi \le (f \times f)(\mathbb{N}_\varphi^\Lambda)$ and hence $(f \times f)^{-1}(\Phi)$ exists and $(f \times f)^{-1}(\Phi) \le \mathbb{N}_\varphi^\Lambda$. As

$$(f \times f)^{-1}(\Phi) \le (f \times f)^{-1}(\Phi \circ \Phi) \le (f \times f)^{-1}(\Phi) \circ (f \times f)^{-1}(\Phi)$$

we see that $(f \times f)^{-1}(\Phi) \in \mathcal{J}_\varphi^\Lambda$ and therefore $(f \times f)^{-1}(\Phi) \le \mathbb{N}_\varphi^*$. We conclude from this that $\Phi \le (f \times f)(\mathbb{N}_\varphi^*)$ and because $\Phi \in \mathcal{J}_\varphi^M$ was arbitrary, we conclude $(f \times f)(\mathbb{N}_\varphi^*) \ge \mathbb{N}_\varphi^{\prime *}$. Hence we can define a functor

$$K : \begin{cases} PUCS \longrightarrow PPUCS \\ (S, \overline{\Lambda}) \longmapsto (S, \overline{\Lambda^*}) \\ \qquad f \longmapsto f. \end{cases}$$

If we denote the embedding functor $E \colon PPUCS \longrightarrow PUCS$, then for $(S, \overline{\Lambda}) \in |PPUCS|$ we have $\Lambda_\varphi = \Lambda_\varphi^*$ for all $\varphi \in \Delta^+$. This follows from the idempotency of τ as in this case $\mathbb{N}_\varphi^\Lambda \in \mathcal{J}_\varphi^\Lambda$. Hence $K \circ E = \mathrm{id}_{PPUCS}$. We have seen above that $E \circ K \ge \mathrm{id}_{PUCS}$ and hence the claim follows. □

We can state the last result in the following form:

THEOREM 5.10. *If the triangle function is the minimum on Δ^+ then PUNIF is isomorphic to a reflective subcategory of PUCS.*

6. A subcategory of *PUCS* which is isomorphic to *PMET*

In this section we identify a subcategory of *PUCS* which is isomorphic to *PMET*. To this end we note the following result.

LEMMA 6.1 (Tardiff [12]). *Let* $(S,F) \in |PMET|$ *and let* $\varphi \in \Delta^+$ *and* $p,q \in S$. *Then* $[(p,q)] \geq \mathbb{N}_\varphi^F$ *if and only if* $F_{pq} \geq \varphi$.

LEMMA 6.2. *Let* $(S,F) \in |PMET|$ *with a continuous triangle function* τ. *Then* $(S,\overline{\Lambda^F})$ *is principal and a* T1*-space.*

PROOF. By definition, $(S,\overline{\Lambda^F})$ is principal. In order to show that it is a T1-space, we apply the Lemma above. If

$$F_{pq} = \bigvee\{\varphi \in \Delta^+ : F_{pq} \geq \varphi\} = \bigvee\{\varphi \in \Delta^+ : [(p,q)] \geq \mathbb{N}_\varphi^F\}$$
$$= \bigvee\{\varphi \in \Delta^+ : p \in c_\varphi^{\Lambda^F}([q])\} = \varepsilon_0$$

then, by (PM1), we have $p = q$. $\square$

The following axiom will be central. We say that $(S,\overline{\Lambda}) \in |PUCS|$ satisfies the axiom (UPM) if for all ultrafilters $\Phi \in \mathbb{F}(S \times S)$ and all $\varphi \in \Delta^+$

$$\Phi \in \Lambda_\varphi \iff \forall\, \phi \in \Phi,\ \varepsilon > 0\ \exists\,(p,q) \in \phi$$

such that $\bigvee_{\psi:[(p,q)]\in\Lambda_\psi} \psi(x+\varepsilon) + \varepsilon \geq \varphi(x)\ \forall\, x \in [0,\frac{1}{\varepsilon})$.

LEMMA 6.3. *Let* $(S,F) \in |PMET|$ *and the triangle function* τ *be continuous. Then* (S,Λ^F) *satisfies* (UPM).

PROOF. Let $\Phi \in \mathbb{F}(S \times S)$ be an ultrafilter. Let first $\Phi \in \Lambda_\varphi^F$. Then $\Phi \geq \mathbb{N}_\varphi^F$ and hence for $\phi \in \Phi$ and $\varepsilon > 0$ we have $\phi \cap N_\varphi^\varepsilon \neq \emptyset$. Thus there is $(p,q) \in \phi$ such that

$$\bigvee_{\psi:[(p,q)]\in\Lambda_\psi^F} \psi(x+\varepsilon) + \varepsilon = F_{pq}(x+\varepsilon) + \varepsilon \geq \varphi(x) \quad \forall\, x \in [0,\tfrac{1}{\varepsilon}).$$

Conversely, let for all $\phi \in \Phi$ and all $\varepsilon > 0$ there exist $(p,q) \in \phi$ such that $\bigvee_{\psi:[(p,q)]\in\Lambda_\psi^F} \psi(x+\varepsilon)+\varepsilon \geq \varphi(x)$ for all $x \in [0,\frac{1}{\varepsilon})$. By Lemma 6.1 we conclude that $F_{pq}(x+\varepsilon)+\varepsilon \geq \varphi(x)$ for all $x \in [0,\frac{1}{\varepsilon})$. Hence $\Phi \vee \mathbb{N}_\varphi^F$ exists and because Φ is an ultrafilter we conclude $\Phi \geq \mathbb{F}_\varphi^F$. $\square$

We denote the subcategory of *PUCS* with objects the principal T1-spaces that satisfy the axiom (UPM) by *PM-PUCS*. From the results above and the results in Sections 4 and 5, we see that for a continuous triangle function we can define the following functor.

$$D : \begin{cases} PMET \longmapsto PM\text{-}PUCS \\ (S,F) \longmapsto (S,\overline{\Lambda^F}) \\ f \longmapsto f. \end{cases}$$

Let now $(S,\overline{\Lambda}) \in |PUCS|$. We define for $p,q \in S$ the following distance distribution function.

$$F^{\Lambda}_{pq} = \bigvee_{\varphi:[(p.q)]\in\Lambda_\varphi} \varphi.$$

LEMMA 6.4. *Let $(S,\overline{\Lambda}) \in |PUCS|$ be a* T1*-space and let the triangle function τ be sup-continuous. Then (S,F^{Λ}) is a probabilistic metric space.*

PROOF. (PM1) follows, because $[(p,q)] = [p] \times [q]$, directly from the T1-property, (PM2) follows from $[(p,q)]^{-1} = [(q,p)]$ and (PUC4). For (PM3) we use the sup-continuity and proceed as follows. We have

$$\tau(F^{\Lambda}_{pq},F^{\Lambda}_{qr}) = \bigvee_{\varphi:[(p,q)]\in\Lambda_\varphi,\psi:[(q,r)]\in\Lambda_\psi} \tau(\varphi,\psi) \le \bigvee_{\psi,\varphi:[(p,r)]\in\Lambda_{\tau(\varphi,\psi)}} \tau(\varphi,\psi) \le F^{\Lambda}_{pr}. \quad \square$$

LEMMA 6.5. *Let $(S,\overline{\Lambda}),(T,\overline{M}) \in |PUCS|$ and let $f\colon S \longrightarrow T$ be uniformly continuous. Then $f\colon (S,F^{\Lambda}) \longrightarrow (T,F^{M})$ is non-expansive.*

PROOF. Let $p,q \in S$. Then, because $(f\times f)([(p,q)]) = [(f(p),f(q))]$ we conclude, using the uniform continuity of f,

$$F^{\Lambda}_{pq} = \bigvee_{\varphi:[(p,q)]\in\Lambda_\varphi} \varphi \le \bigvee_{\varphi:[(f(p),f(q))]\in M_\varphi} \varphi = F^{M}_{f(p),f(q)}. \quad \square$$

Hence, if τ is a sup-continuous triangle function, we can define a functor

$$E : \begin{cases} PM\text{-}PUCS \longmapsto PMET \\ (S,\overline{\Lambda}) \longmapsto (S,F^{\Lambda}) \\ f \longmapsto f. \end{cases}$$

We will show that D and E are isomorphism functors.

LEMMA 6.6. *Let τ be a continuous and sup-continuous triangle function. Then $E \circ D = \mathrm{id}_{PMET}$.*

PROOF. Let $(S,F) \in |PMET|$ and $p,q \in S$. Then, using Lemma 6.1,

$$F^{(\Lambda^F)}_{pq} = \bigvee_{\varphi:[(p,q)]\in\Lambda^F_\varphi} \varphi = \bigvee_{\varphi:[(p,q)]\ge\mathbb{N}^F_\varphi} \varphi = \bigvee_{\varphi:F_{pq}\ge\varphi} \varphi = F_{pq}.$$

Hence $D(C((S,F))) = (S,F^{\Lambda^F}) = (S,F)$. $\square$

LEMMA 6.7. *Let τ be a continuous and sup-continuous triangle function. Then $D \circ E = \mathrm{id}_{PM\text{-}PUCS}$.*

PROOF. Let $(S,\overline{\Lambda}) \in |PM\text{-}PUCS|$ and let $\varphi \in \Delta^+$ and $\Phi \in \mathbb{F}(S \times S)$ be an ultrafilter. Let first $\Phi \in \Lambda_\varphi$. If $\phi \in \Phi$ and $\varepsilon > 0$, then, by the axiom (UPM), there is $(p,q) \in \phi$ such that

$$F^{\Lambda}_{pq}(x+\varepsilon)+\varepsilon = \bigvee_{\psi:[(p,q)]\in\Lambda_\psi} \psi(x+\varepsilon)+\varepsilon \geq \varphi(x) \quad \forall\, x \in [0, \tfrac{1}{\varepsilon}).$$

Hence $(p,q) \in N^{F^\Lambda}_\varphi$. Therefore $\Phi \vee \mathbb{N}^F_\varphi$ exists and because Φ is an ultrafilter, we conclude $\Phi \geq \mathbb{N}^{F^\Lambda}_\varphi$, i.e. $\Phi \in \Lambda^{F^\Lambda}_\varphi$.

Conversely, if $\Phi \in \Lambda^{F^\Lambda}_\varphi$, then $\Phi \geq \mathbb{N}^{F^\Lambda}_\varphi$. Hence for all $\phi \in \Phi$ and all $N^\varepsilon_\varphi \in \mathbb{N}^{F^\Lambda}_\varphi$ we have $\phi \cap N^\varepsilon_\varphi \neq \emptyset$. Hence there is $(p,q) \in \phi$ such that

$$\bigvee_{\psi:[(p,q)]\in\Lambda_\psi} \psi(x+\varepsilon)+\varepsilon = F^{\Lambda}_{pq}(x+\varepsilon)+\varepsilon \geq \varphi(x) \quad \forall\, x \in [0, \tfrac{1}{\varepsilon}).$$

From the axiom (UPM) we conclude that $\Phi \in \Lambda_\varphi$. Because both $(S,\overline{\Lambda})$ and $(S,\overline{\Lambda^{F^\Lambda}})$ are principal and every filter is the intersection of its finer ultrafilters, we obtain $(S,\overline{\Lambda}) = (S,\overline{\Lambda^{F^\Lambda}})$. $\square$

THEOREM 6.8. *Let τ be a continuous and sup-continuous triangle function. Then PMET and PM-PUCS are isomorphic categories.*

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