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ON THE CLONES OF NILPOTENT GROUPS WITH A VERBAL SUBGROUP OF PRIME ORDER

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Abstract. We show that if G is a finite nilpotent group with a verbal subgroup of prime order then the clone of G is not determined by the subgroups of G^2.

1. Introduction

For a group G it is of interest to determine when a function $f : G^n \to G$ can be represented as the interpretation of a group word w in G. Such operations on a group G are called the *term operations* of the group G, and the collection of these operations is called the *clone* of G, commonly notated as $\mathrm{Clo}(G)$. For a positive integer k, we say that a function f *preserves* a subgroup H of G^k if H is closed under the function f operating coordinatewise on the elements of H. The following fact from clone theory will aid us in determining when an operation is in $\mathrm{Clo}(G)$.

THEOREM 1.1 ([7, Corollary 1.4]). *Let* $\mathbf{A}$ *be a finite algebra. An operation* f *on* A *is in* $\mathrm{Clo}(\mathbf{A})$ *if and only if* f *preserves the subalgebras of* $\mathbf{A}^k$ *for all integers* k.

Hence, if G is a finite group, the above states that an operation f is in $\mathrm{Clo}(G)$ if and only if f preserves the subgroups G^k for all finite integers k. Alternatively, it is also said that the subgroups of G^k (for all integers k) *determine* $\mathrm{Clo}(G)$.

It was shown in [1] that for all finite groups G there exists a finite k such that if f preserves the subgroups of G^k then f preserves the subgroups of G^m for all positive integers m, and hence f is in $\mathrm{Clo}(G)$. However, no

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estimate for k is given for a particular group G. Hence, given a group G it is natural to try to find the smallest integer k such that the subgroups of G^k determine $\mathrm{Clo}(G)$. Much work has been done along these lines (see [2], [3] and [6]).

It is useful to know how certain properties that a group G may possess affect the smallest integer k such that the subgroups of G^k determine $\mathrm{Clo}(G)$. For instance, in [3] Kearnes and Szendrei showed that if G is an abelian group with a verbal subgroup of prime order, then the clone of G is not determined by the subgroups of G^2. This result lead Kearnes and Szendrei to pose the following conjecture:

CONJECTURE 1.2 ([3, Problem 3.6]). If G is a finite group with a verbal subgroup of prime order, then the clone of G is not determined by the subgroups of G^2.

The main result of this paper shows that Conjecture 1.2 is true when G is a finite nilpotent group (see Theorem 3.1).

2. Preliminaries

In this section we summarize the ideas and previous results needed for the proof of Theorem 3.1. First, we standardize some of the notation used throughout this paper. By the *commutator* $[x, y]$ of two elements x, y in a group we mean $[x, y] = x^{-1}y^{-1}xy$. Also, given a group G and a group word t, we will use the notation t^G for the term operation that is the interpretation of t in G.

Next, we need a lemma to help understand the form that group words take in terms of products of commutators. But first we need a definition.

DEFINITION 2.1 ([6, Section 2.2]). If X is a set of variables, we define the set C of *commutator words* on X to be the smallest set C such that C contains 1, the variables in X and their inverses, and C is closed under formation of commutators. By this we mean that if $c_1, c_2 \in C$, then $[c_1, c_2] \in C$. Further, for each element $c \in C$ we define the set $\mathcal{V}(c)$ of variables occurring in c by $\mathcal{V}(c) = \emptyset$ if $c = 1$, $\mathcal{V}(c) = \{x\}$ if $c = x$ or $c = x^{-1}$, and $\mathcal{V}(c) = \mathcal{V}(c_1) \cup \mathcal{V}(c_2)$ if $c = [c_1, c_2]$ for some $c_1, c_2 \in C$.

Now we are ready to state the lemma.

LEMMA 2.2 ([6, Lemma 2.2]). *Let $X = \{x_1, \dots, x_k\}$ be a set of variables, and let $U_0, U_1, \dots, U_m$ be an enumeration of the subsets of X such that $U_0 = \emptyset$ and for each i all proper subsets of U_i are among $U_0, \dots, U_{i-1}$.*

If $t = t(x_1, \ldots, x_k)$ *is a group word in the variables* X, *then* t *may be factored as*

$$t = \tau_{U_0}\tau_{U_1}\cdots\tau_{U_m}$$

where each τ_{U_i} *is a product of commutator words* $c \in C$, *with* $\mathcal{V}(c) = U_i$.

We call a k-ary term operation g an *absorptive* term operation if

$$g(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_k) = 1 \quad \text{for each} \quad i \quad (1 \leqq i \leqq k).$$

Lemma 2.2 yields an interesting corollary for absorptive term operations.

COROLLARY 2.3 ([6, Corollary 2.3]). *Let* t^G *be a* k*-ary absorptive term operation of a group* G *that is the interpretation of the group word* $t = t(x_1, \ldots, x_k)$. *If* t *has the form* $t = \tau_{U_0}\tau_{U_1}\cdots\tau_{U_m}$ *as described in Lemma* 2.2, *then* t^G *is also the interpretation of the group word* τ_{U_m} *which is a product of commutator words* c *with* $\mathcal{V}(c) = \{x_1, \ldots, x_k\}$.

This corollary will help us to prove the following fact.

LEMMA 2.4. *If* g *is an* n*-ary absorptive term operation of a nilpotent group* G *such that* n *is greater than the nilpotence class of* G, *then* g *is constant* 1.

PROOF. Let $t = t(x_1, \ldots, x_n)$ be a group word such that $g = t^G$. By Lemma 2.2, we know that we can factor t as $t = \tau_{U_0}\tau_{U_1}\cdots\tau_{U_m}$, where each τ_{U_i} is a product of commutator words $c \in C$. By Corollary 2.3 we know that $g = t^G = \tau_{U_m}^G$, where τ_{U_m} is a product of commutator words c such that $\mathcal{V}(c) = \{x_1, \ldots, x_n\}$. From group theory we know that $c^G = 1$, as c is a commutator word with more variables than the nilpotence class of a nilpotent group (see [4]). Hence, it follows that g is the constant 1 on G. $\square$

Next we need to define a verbal subgroup of a group G. We follow the definition from [5] but adapt the notation and terminology to the conventions used in this paper.

DEFINITION 2.5 ([5, Section 2.2]). A subgroup V of a group G is called a *verbal subgroup* of G if there exists a set W of group words in the variables $x_1, x_2, \ldots$ such that the union of the images of the term operations w^G, $w \in W$, generates V.

Now we prove a lemma regarding a verbal subgroup of prime order in a finite group.

LEMMA 2.6. *If* G *is a finite group with a verbal subgroup* V *of prime order*, *then there exists a group word* t *such that the term operation* t^G *is absorptive and the image of* t^G *generates the subgroup* V.

Proof. Since V is verbal, there exist group words $t = t(x_1, \ldots, x_k)$ such that t^G is not constant 1 and its image is contained in V. Choose such a word such that k is minimal. The $(k-1)$-ary term functions $t^G(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_k)$ for each i $(1 \leqq i \leqq k)$ also have their images contained in V (since t^G does); therefore, by the minimality of k they are all the constant 1. Hence, t^G is absorptive. Lastly, by the choice of t^G and the fact that V is of prime order, its image generates V. $\square$

Lastly, we need the following useful lemma, which follows from the proof of Theorem 3.5 in [3].

Lemma 2.7 ([3, Theorem 3.5]). *If G is a finite group with a verbal subgroup V of prime order such that V is generated by the range of a unary term operation of G, then the clone of G is not determined by the subgroups of G^2.*

3. Main theorem and proof

Now we state and prove the main theorem of this paper.

Theorem 3.1. *If G is a finite nilpotent group with a verbal subgroup of prime order, then* $\mathrm{Clo}(G)$ *is not determined by the subgroups of G^2.*

Proof. Suppose G is a finite nilpotent group of class $c \geqq 1$ with a verbal subgroup V of prime order p. By Lemma 2.6 there exists a group word t of arity k, such that t^G is absorptive, and V is generated by the image of t^G in G. Note that $k \leqq c$ by Lemma 2.4. If $k = 1$ then by Lemma 2.7 the result of the theorem holds. Hence, for the remainder of the proof assume that $k > 1$. Let $h = t^G$ for ease of notation.

Now, let m be the smallest positive integer such that

(i) $c < mk$,

(ii) $p \nmid m$.

Since we will be using mk-tuples of elements of G, for ease of notation we will write the mk-tuple as $(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1})$, where $\mathbf{x}_i = (x_{ik+1}, \ldots, x_{ik+k})$ for each i $(0 \leqq i \leqq m-1)$.

We now define the mk-ary operation f as follows.

$$f(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}) = \begin{cases} \displaystyle\prod_{i=0}^{m-1} h(\mathbf{x}_i), & \text{if } \ h(\mathbf{x}_i) \neq 1 \text{ for each } i \ (0 \leqq i \leqq m-1), \\ 1, & \text{else.} \end{cases}$$

Claim 3.2. $f \notin \mathrm{Clo}(G)$.

Proof. We begin by showing that f is absorptive. Suppose $(\mathbf{x}_0,\dots,\mathbf{x}_{m-1})$ is an mk-tuple of elements of G such that at least one of the coordinates is 1. Then for some i $(0 \leqq i \leqq m-1)$ at least one of the coordinates of $\mathbf{x}_i$ is 1. Since h is absorptive, $h(\mathbf{x}_i)=1$, which by the definition of f implies that $f(\mathbf{x}_1,\dots,\mathbf{x}_{m-1})=1$. Hence, f is an absorptive operation.

Now, since the image of h generates V, there exists $\mathbf{g}=(g_1,\dots,g_k)\in G^k$ such that $h(\mathbf{g})=\alpha$, where $V=\langle\alpha\rangle$. So

$$f(\mathbf{g},\dots,\mathbf{g})=\big(h(\mathbf{g})\big)^m=\alpha^m\neq 1$$

due to the fact that $p\nmid m$. Hence, we see that f is not the constant 1. Since the arity of f is $mk>c$, by Lemma 2.4 we see that $f\notin \mathrm{Clo}(G)$. $\square$

Claim 3.3. *f preserves the subgroups of G^2.*

Proof. We have to prove that every subgroup H of G^2 is closed under the function f operating coordinatewise on the elements of H. So, let H be a subgroup of G^2, and let $(\mathbf{g}_0,\dots,\mathbf{g}_{m-1})$ be an mk-tuple of elements of H where $\mathbf{g}_i=(g_{ik+1},\dots,g_{ik+k})$ for each i $(0\leqq i\leqq m-1)$, and $g_j=(g_j',g_j'')\in H$ for every j $(1\leqq j\leqq mk)$. Our task is to show that the pair $g:=f(\mathbf{g}_0,\dots,\mathbf{g}_{m-1})$ is in H. Since f is applied coordinatewise,

$$g=f(\mathbf{g}_0,\dots,\mathbf{g}_{m-1})=\big(f(\mathbf{g}_0',\dots,\mathbf{g}_{m-1}'),f(\mathbf{g}_0'',\dots,\mathbf{g}_{m-1}'')\big) \tag{3.1}$$

where $(\mathbf{g}_0',\dots,\mathbf{g}_{m-1}')$ and $(\mathbf{g}_0'',\dots,\mathbf{g}_{m-1}'')$ are the mk-tuples of the first and second coordinates of $(\mathbf{g}_0,\dots,\mathbf{g}_{m-1})$, respectively.

If $g=(1,1)$, then $g\in H$ is clear. If, when computing g by (3.1), it is the case that in both coordinates the first case of the definition of f applies, then

$$g=f(\mathbf{g}_0,\dots,\mathbf{g}_{m-1})=\prod_{i=0}^{m-1}h(\mathbf{g}_i).$$

Since multiplication and h are term operations of G and $(\mathbf{g}_0,\dots,\mathbf{g}_{m-1})$ is a tuple of pairs from H, we get that $g\in H$.

In the remaining case $g\neq(1,1)$, and when we compute $g=(g',g'')$ by (3.1), in one of the coordinates the first case of the definition of f applies, while in the other coordinate the second case of the definition of f applies. We may assume without loss of generality that the first case of the definition of f applies in the first coordinate, and the second case applies in the second coordinate. Then

$$g'=\prod_{i=0}^{m-1}h(\mathbf{g}_i')\quad\text{with}\quad h(\mathbf{g}_0'),\dots,h(\mathbf{g}_{m-1}')\neq 1,$$

and

$$g'' = 1 \quad \text{because} \quad h(\mathbf{g}_i'') = 1 \quad \text{for some} \quad i \ (0 \leqq i \leqq m-1).$$

Therefore, $(1,1) \neq (h(\mathbf{g}_i'), 1) = h(\mathbf{g}_i)$ is in H. Since the image of h generates V, $h(\mathbf{g}_i') \neq 1$, and V is of prime order, we can generate g' from $h(\mathbf{g}_i')$. It follows that $h(\mathbf{g}_i)$ generates $(g', 1) = g$, so $g \in H$. □

By Claims (3.2) and (3.3), we see that f is an operation on G that preserves the subgroups of G^2, but $f \notin \mathrm{Clo}(G)$. Hence, the result of our theorem follows. □

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