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COMPLETIONS OF UNIFORM PARTIAL FRAMES

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Abstract. We address classical questions concerning the existence and properties of completions in a new context, namely, that of uniform partial frames.

A partial frame is a meet-semilattice in which certain joins exist and finite meets distribute over these joins. We specify these joins by means of a so-called selection function, which must satisfy certain axioms to produce a useful theory. The axioms we use here were introduced in [9] and are sufficiently general to encompass in the resulting theory frames, κ-frames and σ-frames.

Using covers to describe uniform structures on partial frames, we develop the notion of completeness for a uniform partial frame, using the frame-theoretic version of the well-known fact that a complete uniform space is isomorphic to any uniform space in which it is densely embedded.

In constructing a completion, we make substantial use of the functor which takes $\mathcal{S}$-ideals and the functor which takes $\mathcal{S}$-cozero elements, as well as the category equivalence that these functors induce. Our strategy involves the transfer of important properties concerning the completion from the category of uniform frames to that of uniform partial frames.

As a final application, we provide two constructions of the Samuel compactification. One involves the completion of the totally bounded coreflection; the other uses the functors mentioned above to transfer the corresponding compactification from uniform frames.

1. Introduction

Completeness and completions are of central importance in the study of a wide variety of uniform structures. Here we address classical questions concerning the existence and properties of completions in a new context,

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namely, that of uniform partial frames. We have found that partial frames have served to throw into relief some features of pointfree topology that are specific to certain cases and some that obtain more generally. In [9] we investigated some aspects of the rôle of countability and the importance of σ-frames in frame theory. On the other hand, we see in the current paper that the theory of completeness and completions is quite general and well-behaved and does not depend heavily on countability conditions. This is of interest, since the construction of a completion for a uniform σ-frame in [23] depends on a crucial countability argument of Ginsburg and Isbell ([10]).

A partial frame is a meet-semilattice in which certain joins exist and finite meets distribute over these joins. We specify these joins by means of a so-called selection function (see Definition 2.1). The following are well-known special cases:

1. In the case that all joins are specified, we have the notion of a frame.
2. In the case that countable joins are specified, we have the notion of a σ-frame.
3. In the case that joins of subsets with cardinality less than some regular cardinal κ are specified, we have the notion of a κ-frame.

Any selection function must satisfy certain axioms to produce a workable theory; the axioms we use here were all introduced in [9]. While the choice of these is not obvious, it is the mark of a good axiom, we feel, that it appears inevitable in retrospect (with apologies to Robert Louis Stevenson).

In [8] we use covers to describe uniform structures on partial frames; uniform frames and uniform σ-frames arise as special cases. In this paper we develop the notion of completeness for a uniform partial frame, using the frame-theoretic version of the well-known fact that a complete uniform space is isomorphic to any uniform space in which it is densely embedded.

In constructing a completion, we make substantial use of the functor which takes $\mathcal{S}$-ideals and the functor which takes $\mathcal{S}$-cozero elements, as well as the category equivalence that these functors induce. (See [9] for details.) Our strategy allows the transfer of important properties concerning the completion from the category of uniform frames to that of uniform partial frames.

As expected, a uniform partial frame is compact if and only if it is complete and totally bounded. We conclude by providing two constructions of the Samuel compactification. One involves the completion of the totally bounded coreflection; the other uses the functors mentioned above to transfer the corresponding compactification from uniform frames.

2. Background

See [19] and [12] as references for frame theory; see [3], [16], [20] and [22] for σ-frames; see [15] for κ-frames; see [14] and [1] for general category theory. We make substantial use of completions of uniform frames; see for instance [5], [2], [19], as well as [11], [13].

The specific topic of partial frames is relatively new. In [8] we introduce the axiom system that we use for $\mathcal{S}$-frames (our technical term for partial frames), and discuss regularity, normality and compactness as well as provide a method for constructing certain coreflections in the category of uniform $\mathcal{S}$-frames. In [9] we discuss the functors $\mathcal{H}_{\mathcal{S}}$ and $\mathrm{Coz}_{\mathcal{S}}$ for partial frames which will appear again in this paper, but with uniform structure added. Other approaches to partial frames can be found in [26], [25], [18] and [24]. For a discussion of how our work relates to these, see [8].

A *meet-semilattice* A is a partially ordered set in which all finite subsets have a meet. In particular, we regard the empty set as finite, so a meet-semilattice comes equipped with a top element, which we denote by 1. We also insist that a meet-semilattice should have a bottom element, which we denote by 0. (Technically, one might wish to refer to these as *bounded* meet-semilattices.) A function $f : A \to B$ is a *meet-semilattice map* if it preserves finite meets, as well as the top and the bottom element. D is a *downset* of A if $x \leqq y \in D$ implies that $x \in D$. We use the notation $\downarrow a = \{x \in A : x \leqq a\}$ for principal downsets and $\downarrow C = \bigcup\{\downarrow c : c \in C\}$ for any subset C of A. Similarly $\uparrow a = \{x \in A : x \geqq a\}$.

Definition 2.1. A *selection function* is a rule, which we usually denote by $\mathcal{S}$, which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A, such that the following conditions hold (for all meet-semilattices A and B):

(S1) For all $x \in A$, $\{x\} \in \mathcal{S}A$.

(S2) If $G, H \in \mathcal{S}A$ then $G \wedge H = \{x \wedge y : x \in G,\ y \in H\} \in \mathcal{S}A$.

(S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then $\bigcup_{x \in G} H_x \in \mathcal{S}A$.

(S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}\big(f[A]\big) = \big\{ f[G] : G \in \mathcal{S}A\big\} \subseteqq \mathcal{S}B.$$

Once a selection function, $\mathcal{S}$, has been fixed, we speak informally of the members of $\mathcal{S}A$ as the *designated* subsets of A.

Definition 2.2. Let $\mathcal{S}$ be a selection function.

1. An *$\mathcal{S}$-frame, L*, is a meet-semilattice that satisfies the following two conditions:

   (a) For all $G \in \mathcal{S}L$, G has a join in L (i.e. $\bigvee G$ exists).

   (b) For all $x \in L$, for all $G \in \mathcal{S}L$, $x \wedge \bigvee G = \bigvee_{y \in G} x \wedge y$.

2. Let L and M be $\mathcal{S}$-frames. An *$\mathcal{S}$-frame map* $f : L \to M$ is a meet-semilattice map such that, for all $G \in \mathcal{S}L$, $f\big(\bigvee G = big) = \bigvee_{y \in G} f(y)$.

3. $\mathcal{S}$**Frm** is the category with $\mathcal{S}$-frames as objects and $\mathcal{S}$-frame maps as morphisms. In the case where $\mathcal{S}$ selects all subsets, we get the category **Frm** of frames and frame maps.

4. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame. We shall call a subset M of L a *sub $\mathcal{S}$-frame of L* if it satisfies M is an $\mathcal{S}$-frame and the identical embedding $i : M \to L$ is an $\mathcal{S}$-frame map.

The following axiom, which we impose on all our selection functions, makes the behaviour of subobjects tractable:

(S5) For any $\mathcal{S}$-frame L, if M is a sub $\mathcal{S}$-frame of L, $G \subseteqq M$ and $G \in \mathcal{S}L$, then $G \in \mathcal{S}M$.

DEFINITION 2.3. Let $\mathcal{S}$ be a selection function and A an arbitrary meet-semilattice. We now further require that $\mathcal{S}$ satisfies the following axioms:

(SCount) Every countable subset of A is designated.

(SCov) Every subset of a designated cover of A is designated. (By a *cover* of A we mean, as usual, a subset C of A such that $\bigvee C = 1$.)

(SRef) Let $X, Y \subseteqq A$. If $X \leqq Y$ with X designated in A there is a designated $C \subseteqq A$ such that $X \leqq C \subseteqq Y$. (By $X \leqq Y$ we mean, as usual, that for each $x \in X$ there exists $y \in Y$ such that $x \leqq y$.)

Below we give the standard definitions of the rather below and completely below relations; they apply in an $\mathcal{S}$-frame as they do in any lattice.

DEFINITION 2.4. Let L be an $\mathcal{S}$-frame.

(a) For $a, b \in L$, we say a is *rather below* b, written $a \prec b$, if there exists $t \in L$ satisfying $a \wedge t = 0$ and $t \vee b = 1$.

(b) For $a, b \in L$, we say a is *completely below* b, written $a \prec\!\!\prec b$, if there exists a set $\{ x_r \in L : r \in [0,1] \cap \mathbb{Q} \}$ for which $r < t$ implies $x_r \prec x_t$.

(c) We call an $\mathcal{S}$-frame L *completely regular* if each $a \in L$ can be written $a = \bigvee T$ for some designated subset T of L such that $t \prec\!\!\prec a$ for all $t \in T$. The full subcategory of $\mathcal{S}$**Frm** with completely regular $\mathcal{S}$-frames as objects, will be denoted by **Creg**$\mathcal{S}$**Frm**.

DEFINITION 2.5. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame.

1. We call C an *$\mathcal{S}$-cover* of L if $C \in \mathcal{S}L$ and $\bigvee C = 1$. If C and D are $\mathcal{S}$-covers of L, then $C \wedge D = \{c \wedge d : c \in C,\, d \in D\}$ is an $\mathcal{S}$-cover of L.

2. If C and D are $\mathcal{S}$-covers of L, we say that C *refines* D and write $C \leqq D$ if, for all $c \in C$, there exists $d \in D$ such that $c \leqq d$.

3. If $a \in L$ and C is an $\mathcal{S}$-cover of L, we set $C_a = \{c \in C : c \wedge a \neq 0\}$. If $a, b \in L$ and C is an $\mathcal{S}$-cover of L, we write $a \lhd_C b$ if $C_a \subseteqq \downarrow b$. We say that a is *uniformly below b with respect to C*. In the presence of (SCov), $C_a \subseteqq \downarrow b$ amounts to $Ca = \bigvee C_a \leqq b$.

4. If C and D are $\mathcal{S}$-covers of L, we say that C *star-refines* D, and write $C <^* D$, if, for all $c \in C$, there is $d \in D$ such that $c \lhd_C d$.

5. A non-empty collection of $\mathcal{S}$-covers, $\mathcal{U}L$, of L is an *$\mathcal{S}$-pre-uniformity* on L if it is filtered by meet and star-refinement. $\mathcal{U}L$ is an *$\mathcal{S}$-uniformity* on L if it further satisfies the following *compatibility condition:* For any $x \in L$, there exists a designated subset T of L such that $x = \bigvee T$ and $t \in T$ implies

that $t \lhd_C x$ for some $C \in \mathcal{U}L$. The pair $(L,\mathcal{U}L)$ is called a *uniform $\mathcal{S}$-frame.* We frequently use the notation $a \lhd b$ whenever there is some $C \in \mathcal{U}L$ such that $a \lhd_C b$ and say that a is *uniformly below b in* $(L,\mathcal{U}L)$. In the case that $\mathcal{S}$ selects all subsets we recover the usual definitions of pre-uniformities and uniformities on a frame.

6. Let $(L,\mathcal{U}L)$ and $(M,\mathcal{U}M)$ be uniform $\mathcal{S}$-frames. Then $f : (L,\mathcal{U}L) \to (M,\mathcal{U}M)$ is said to be *uniform* if $f : L \to M$ is an $\mathcal{S}$-frame map and for each $C \in \mathcal{U}L$, $f[C] \in \mathcal{U}M$. We denote the category of uniform $\mathcal{S}$-frames and uniform maps by **Uni$\mathcal{S}$Frm**. In the case that $\mathcal{S}$ selects all subsets, we get the category **UniFrm**.

7. If $(L,\mathcal{U}L)$ is a uniform $\mathcal{S}$-frame, then L is a completely regular $\mathcal{S}$-frame: If $a \lhd b$ in $(L,\mathcal{U}L)$ then $a \prec b$ in L. Since $\lhd$ interpolates by the existence of star-refinements, this yields that $a \lhd b$ implies $a \prec\!\!\prec b$ which is sufficient. (Details can be found in [8].)

3. $\mathcal{H}_\mathcal{S}$ and $\mathrm{Coz}_\mathcal{S}$ for uniform $\mathcal{S}$-frames

In this section, we extend the functors $\mathcal{H}_\mathcal{S}$ (taking $\mathcal{S}$-ideals) and $\mathrm{Coz}_\mathcal{S}$ (taking $\mathcal{S}$-cozero elements) to the uniform setting. The aim will be to provide a category equivalence between the uniform $\mathcal{S}$-frames and the $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frames.

Definition 3.1. Let N be a frame.

1. We call an element a of N an *$\mathcal{S}$-Lindelöf element* of N if it satisfies: If $a = \bigvee B$ for some $B \subseteqq N$, then $a = \bigvee C$ for some designated subset C of N such that $C \subseteqq B$. (There are other equivalent definitions; see Lemma 4.1 of [9].)

2. We call N an *$\mathcal{S}$-Lindelöf frame* if its top element is an $\mathcal{S}$-Lindelöf element.

Definition 3.2. We call a uniform frame $(N,\mathcal{V}N)$ *$\mathcal{S}$-separable* if, for each $A \in \mathcal{V}N$ there exists $B \in \mathcal{V}N$ such that B is an $\mathcal{S}$-cover of N and $B \leqq A$; that is, every uniform cover is refined by a uniform $\mathcal{S}$-cover.

We now extend the functor $\mathcal{H}_\mathcal{S}$ to the uniform setting making use of the following definitions and results, which appear in [9].

Definition 3.3. (a) A subset J of an $\mathcal{S}$-frame L is an *$\mathcal{S}$-ideal of L* if J is a non-empty downset closed under designated joins (the latter meaning that if $X \subseteqq J$, for X a designated subset of L, then $\bigvee X \in J$).

(b) The collection of all $\mathcal{S}$-ideals of an $\mathcal{S}$-frame L will be denoted $\mathcal{H}_\mathcal{S}L$, and called the *$\mathcal{S}$-ideal frame of L.*

(c) Taking $\mathcal{S}$-ideals provides a functor $\mathcal{H}_\mathcal{S} : \mathcal{S}\mathbf{Frm} \to \mathbf{Frm}$. Its action on a morphism h is defined by: For $J \in \mathcal{H}_\mathcal{S}L$, let $\mathcal{H}_\mathcal{S}h(J) = \langle h[J] \rangle$, the $\mathcal{S}$-ideal generated by the image $h[J]$.

We now define a uniformity on the frame $\mathcal{H}_{\mathcal{S}}L$.

Definition 3.4. Let $(L,\mathcal{U}L)$ be a uniform $\mathcal{S}$-frame. For $C \in \mathcal{U}L$, define $\hat{C} = \{\downarrow c : c \in C\}$. Let $\mathcal{V}\mathcal{H}_{\mathcal{S}}L = \{A : A$ is a cover of $\mathcal{H}_{\mathcal{S}}L$ and $\hat{C} \leqq A$ for some $C \in \mathcal{U}L\}$. Set $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) = (\mathcal{H}_{\mathcal{S}}L, \mathcal{V}\mathcal{H}_{\mathcal{S}}L)$.

Lemma 3.5. *$\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ is an $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame.*

Proof. We first note that, since $\downarrow: L \to \mathcal{H}_{\mathcal{S}}L$ is an $\mathcal{S}$-frame map (see Proposition 3.8 of [9]), it is, in particular, a meet-semilattice map, and so $\hat{C}$ is a designated subset of $\mathcal{H}_{\mathcal{S}}L$, whenever C is a designated subset of L. Moreover, $\bigvee \hat{C} = \bigvee\{\downarrow c : c \in C\} = \downarrow 1$, by the construction of joins in $\mathcal{H}_{\mathcal{S}}L$. Thus $\hat{C} \in \mathcal{V}\mathcal{H}_{\mathcal{S}}L$ for every $C \in \mathcal{U}L$.

To show that $\mathcal{V}\mathcal{H}_{\mathcal{S}}L$ is closed under binary meets, we note that $\hat{C} \wedge \hat{D} = \widehat{C \wedge D}$ for any $C, D \in \mathcal{U}L$.

For compatibility, we must show that, for each $I \in \mathcal{H}_{\mathcal{S}}L$, we have $I = \bigvee \big\{ J \in \mathcal{H}_{\mathcal{S}}L : J \lhd I$ in $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)\big\}$; it is sufficient to do so for the principal ideals. So take $a \in L$, and write $a = \bigvee T$ for some designated subset T of L satisfying $t \in T \Rightarrow t \lhd a$ in $(L,\mathcal{U}L)$. For each such t, there exists $C \in \mathcal{U}L$ with $Ct \leqq a$. It follows easily that $\hat{C}(\downarrow t) \subseteqq \downarrow a$, so $\downarrow t \lhd \downarrow a$ in $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$. Further, using the fact that T is a designated subset of L, we obtain $\downarrow a = \bigvee\{\downarrow t : t \in T\}$, so $\downarrow a$ has been exhibited as required.

For the star-refinement condition, it will suffice to show that, for $C, D \in \mathcal{U}L$, $C <^* D \Rightarrow \hat{C} <^* \hat{D}$. Now $\hat{C}(\downarrow c) = \bigvee\{\downarrow b : b \in C, \downarrow b \wedge \downarrow c \neq \downarrow 0\}$. For such b, $b \wedge c \neq 0$, so $b \leqq Cc$. But $Cc \leqq d$ for some $d \in D$, so $b \leqq d$, giving $\downarrow b \subseteqq \downarrow d$. Then $\hat{C}(\downarrow c) \subseteqq \downarrow d$, showing that $\hat{C} <^* \hat{D}$.

Thus far we have shown that $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ is a uniform frame. That $\mathcal{H}_{\mathcal{S}}L$ is $\mathcal{S}$-Lindelöf was shown in Corollary 4.4 of [9]. That $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ is $\mathcal{S}$-separable is clear, because $\mathcal{V}\mathcal{H}_{\mathcal{S}}L$ is generated by $\{\hat{C} : C \in \mathcal{U}L\}$ and each such $\hat{C}$ is an $\mathcal{S}$-cover of $\mathcal{H}_{\mathcal{S}}L$, as was mentioned in the first paragraph of this proof. □

Lemma 3.6. *Taking $\mathcal{S}$-ideals provides a functor*

$$\mathcal{H}_{\mathcal{S}} : \mathbf{Uni\mathcal{S}Frm} \to \mathbf{UniFrm}.$$

Proof. Let $h : (L,\mathcal{U}L) \to (M,\mathcal{U}M)$ be a morphism in **Uni$\mathcal{S}$Frm**. From Remark 3.7 of [9] we know that, for any $J \in \mathcal{H}_{\mathcal{S}}L$, $\mathcal{H}_{\mathcal{S}}h(J)$ is the $\mathcal{S}$-ideal generated by the image $h[J]$, which is $\downarrow h[J]$. We note that, for $a \in L$, $\mathcal{H}_{\mathcal{S}}h(\downarrow a) = \downarrow h(a)$. So, for $C \in \mathcal{U}L$, $\mathcal{H}_{\mathcal{S}}h[\hat{C}] = \big\{ \downarrow h(c) : c \in C\big\} = \widehat{h[C]}$ and $h[C] \in \mathcal{U}M$. This shows that, for any $A \in \mathcal{V}\mathcal{H}_{\mathcal{S}}L$, we have $\mathcal{H}_{\mathcal{S}}h[A] \in \mathcal{V}\mathcal{H}_{\mathcal{S}}M$, making $\mathcal{H}_{\mathcal{S}}h : \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \to \mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M)$ a uniform map. □

Clearly from the previous two results, taking $\mathcal{S}$-ideals actually provides a functor

$$\mathcal{H}_{\mathcal{S}} : \mathbf{Uni}\mathcal{S}\mathbf{Frm} \to \mathcal{S}\mathbf{Lind}\,\mathcal{S}\mathbf{sep\ UniFrm}$$

where $\mathcal{S}\mathbf{Lind}\,\mathcal{S}\mathbf{sep\ UniFrm}$ is the full subcategory consisting of the $\mathcal{S}$-Lindelöf $\mathcal{S}$-separable uniform frames.

We now provide a functor in the opposite direction, which will show eventually that $\mathcal{H}_{\mathcal{S}}$ actually gives a category equivalence. In order to do this, we need the following definitions and results which appear in [9].

Definition 3.7. Let N be a frame and $a \in N$.

(a) We call a an *$\mathcal{S}$-cozero element* of N if $a = \bigvee T$ for some designated subset T of N such that $t \prec\!\!\prec a$ for all $t \in T$.

(b) We denote the $\mathcal{S}$-frame consisting of all $\mathcal{S}$-cozero elements of a frame N by $\mathrm{Coz}_{\mathcal{S}}\, N$.

(c) Taking $\mathcal{S}$-cozero elements provides a functor $\mathrm{Coz}_{\mathcal{S}} : \mathbf{Frm} \to \mathcal{S}\mathbf{Frm}$. It acts on morphisms by restriction.

We now lift $\mathrm{Coz}_{\mathcal{S}}$ to a functor from $\mathcal{S}\mathbf{Lind}\,\mathcal{S}\mathbf{sep\ UniFrm}$ to $\mathbf{Uni}\mathcal{S}\mathbf{Frm}$ by constructing an $\mathcal{S}$-uniformity on $\mathrm{Coz}_{\mathcal{S}}\, N$ as follows:

Lemma 3.8. *Let $(N, \mathcal{V}N)$ be an $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame. Define $\mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N = \{A \subseteqq \mathrm{Coz}_{\mathcal{S}}\, N : A \in \mathcal{V}N$ and A designated in $\mathrm{Coz}_{\mathcal{S}}\, N\}$.*

Then $\mathrm{Coz}_{\mathcal{S}}(N, \mathcal{V}N) = (\mathrm{Coz}_{\mathcal{S}}\, N, \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N)$ is a uniform $\mathcal{S}$-frame.

Proof. The members of $\mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$ are clearly chosen to be $\mathcal{S}$-covers of $\mathrm{Coz}_{\mathcal{S}}\, N$. Since the members of $\mathcal{V}N$ and the designated subsets of $\mathrm{Coz}_{\mathcal{S}}\, N$ are closed under binary meets, so is $\mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$. If $C \in \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$ and D is an $\mathcal{S}$-cover of $\mathrm{Coz}_{\mathcal{S}}\, N$ with $C \leqq D$, then $D \in \mathcal{V}N$ and so $D \in \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$.

Next we show that, for $A \in \mathcal{V}N$, there exists $D \in \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$ with $D <^* A$. This is sufficient to show that every member of $\mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$ has a star-refinement in $\mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$. To this end, begin with $A \in \mathcal{V}N$. Since $(N, \mathcal{V}N)$ is $\mathcal{S}$-separable there exists an $\mathcal{S}$-cover $B \in \mathcal{V}N$ with $B <^* A$. Repeat this to get an $\mathcal{S}$-cover $C \in \mathcal{V}N$ with $C <^* B$. For $c \in C$, there exists $b_c \in B$ such that $c \lhd b_c$ (using $Cc \leqq b_c$). So $c \prec\!\!\prec b_c$ in N (see Definition 2.5). By Corollary 5.9 of [9], there exists $z_c \in \mathrm{Coz}_{\mathcal{S}}\, N$ such that $c \prec\!\!\prec z_c \prec\!\!\prec b_c$ in N. By (SRef), there exists D a designated subset of N, $D \subseteqq \{z_c : c \in C\}$ and $C \leqq D$. By (S5), D is a designated subset of $\mathrm{Coz}_{\mathcal{S}}\, N$. So $D \in \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$ and $D \leqq B <^* A$, so $D <^* A$.

The fact that every member of $\mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$ has a star-refinement in $\mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\, N$ follows directly from the result above.

For the compatibility condition, begin with $x \in \mathrm{Coz}_{\mathcal{S}}\, N$. Since $(N, \mathcal{V}N)$ is a uniform frame, $x = \bigvee \big\{ y \in N : y \lhd x \text{ in } (N, \mathcal{V}N) \big\}$. By Proposition 5.13 of [9], x, being a $\mathcal{S}$-cozero element of N is also an $\mathcal{S}$-Lindelöf element of N. So there exists $T \subseteqq \big\{ y \in N : y \lhd x \text{ in } (N, \mathcal{V}N) \big\}$, T a designated subset of N with $x = \bigvee T$.

For each $t \in T$ there exists $s_t \in N$ with $t \lhd s_t \lhd x$ in $(N, \mathcal{V}N)$, because $\lhd$ interpolates in $(N, \mathcal{V}N)$. Use Corollary 5.9 of [9] again to obtain $z_t \in \operatorname{Coz}_{\mathcal{S}} N$ with $t \leqq z_t \leqq s_t$. Then $T \leqq \{z_t : t \in T\}$, so by (SRef) there exists $Z \subseteqq \{z_t : t \in T\}$, Z a designated subset of N and $T \leqq Z$. Note that $\bigvee Z = x$ and that Z is a designated subset of $\operatorname{Coz}_{\mathcal{S}} N$, by (S5).

For each $z \in Z$, $z \lhd x$ in $(N, \mathcal{V}N)$. So there exists $A \in \mathcal{V}N$ with $Az \leqq x$. Applying the second paragraph of this proof gives $D \in \mathcal{U} \operatorname{Coz}_{\mathcal{S}} N$ with $D <^* A$. Then, in particular, $Dz \leqq x$. This shows that $z \lhd x$ in $(\operatorname{Coz}_{\mathcal{S}} N, \mathcal{U} \operatorname{Coz}_{\mathcal{S}} N)$ which concludes the proof of the compatibility condition. $\square$

LEMMA 3.9. *Taking $\mathcal{S}$-cozero elements provides a functor*

$$\operatorname{Coz}_{\mathcal{S}} : \mathcal{S}\mathbf{Lind}\,\mathcal{S}\mathbf{sep\ UniFrm} \to \mathbf{Uni}\mathcal{S}\mathbf{Frm}.$$

PROOF. Let $f : (K, \mathcal{V}K) \to (N, \mathcal{V}N)$ be a morphism in $\mathcal{S}\mathbf{Lind}\,\mathcal{S}\mathbf{sep}$ **UniFrm**. By Corollary 5.4 of [9], $\operatorname{Coz}_{\mathcal{S}} f = f : \operatorname{Coz}_{\mathcal{S}} K \to \operatorname{Coz}_{\mathcal{S}} N$ is an $\mathcal{S}$-frame map. To see that the map is uniform, begin with $C \in \mathcal{U} \operatorname{Coz}_{\mathcal{S}} K$. Then $C \in \mathcal{V}K$, $C \subseteqq \operatorname{Coz}_{\mathcal{S}} K$ and C is a designated subset of $\operatorname{Coz}_{\mathcal{S}} K$. So $f[C] \in \mathcal{V}N$, $f[C] \subseteqq \operatorname{Coz}_{\mathcal{S}} N$ and $f[C]$ is a designated subset of $\operatorname{Coz}_{\mathcal{S}} N$; so $f[C] \in \mathcal{U} \operatorname{Coz}_{\mathcal{S}} N$, as required. $\square$

In Theorem 5.14 of [9], it was shown that the category of completely regular $\mathcal{S}$-frames is equivalent to the category of completely regular $\mathcal{S}$-Lindelöf frames. In the process these results were proved:

- For any completely regular $\mathcal{S}$-frame L, the map $\downarrow_L : L \to \operatorname{Coz}_{\mathcal{S}} \mathcal{H}_{\mathcal{S}} L$ is an isomorphism.
- For any $\mathcal{S}$-frame map $h : L \to M$ between completely regular $\mathcal{S}$-frames, $(\operatorname{Coz}_{\mathcal{S}} \mathcal{H}_{\mathcal{S}} h) \circ \downarrow_L = \downarrow_M \circ h$.
- For any completely regular $\mathcal{S}$-Lindelöf frame K, the map

$$j_K : \mathcal{H}_{\mathcal{S}} \operatorname{Coz}_{\mathcal{S}} K \to K$$

given by join, is an isomorphism.

- For any frame map $f : N \to P$ between completely regular $\mathcal{S}$-Lindelöf frames, $f \circ j_N = j_P \circ (\mathcal{H}_{\mathcal{S}} \operatorname{Coz}_{\mathcal{S}} f)$.

These results will be used in the proof of the next proposition in which we lift the category equivalence mentioned above to the uniform level.

PROPOSITION 3.10. *The category of uniform $\mathcal{S}$-frames is equivalent to the category of $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frames.*

PROOF. We show that, for any uniform $\mathcal{S}$-frame $(L, \mathcal{U}L)$, the map $\downarrow_L : (L, \mathcal{U}L) \to \operatorname{Coz}_{\mathcal{S}} \mathcal{H}_{\mathcal{S}}(L, \mathcal{U}L)$ is an isomorphism. We already have that the underlying $\mathcal{S}$-frame map is an isomorphism. So take $C \in \mathcal{U}L$. Then $\downarrow_L[C] = \hat{C} \in \mathcal{V}\mathcal{H}_{\mathcal{S}} L$ (see Lemma 3.5). Further, $\hat{C}$ is a designated subset of $\operatorname{Coz}_{\mathcal{S}} \mathcal{H}_{\mathcal{S}} L$, so $\hat{C} \in \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\,\mathcal{H}_{\mathcal{S}}L$. Moreover, the $\mathcal{S}$-uniformity on $\mathrm{Coz}_{\mathcal{S}}\,\mathcal{H}_{\mathcal{S}}L$ consists precisely of $\{\hat{C} : C \in \mathcal{U}L\}$ which establishes the desired isomorphism.

For any uniform $\mathcal{S}$-frame map $h : (L,\mathcal{U}L) \to (M,\mathcal{U}M)$, the following diagram commutes, since the underlying $\mathcal{S}$-frame maps are already known to commute:

$$\begin{array}{ccc} (L,\mathcal{U}L) & \xrightarrow{\ \downarrow_L\ } & \mathrm{Coz}_{\mathcal{S}}\,\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathrm{Coz}_{\mathcal{S}}\,\mathcal{H}_{\mathcal{S}}h} \\ (M,\mathcal{U}M) & \xrightarrow{\ \downarrow_M\ } & \mathrm{Coz}_{\mathcal{S}}\,\mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M) \end{array}$$

Next we show that, for any $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame $(K,\mathcal{V}K)$, the join map $j_K : \mathcal{H}_{\mathcal{S}}\,\mathrm{Coz}_{\mathcal{S}}(K,\mathcal{V}K) \to (K,\mathcal{V}K)$ is an isomorphism. Again, the underlying frame map is already known to be an isomorphism. To show that j_K is uniform, begin with $\hat{C}$, for some $C \in \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\,K$. Then $C \in \mathcal{V}K$, $C \subseteqq \mathrm{Coz}_{\mathcal{S}}\,K$ and C is a designated subset of $\mathrm{Coz}_{\mathcal{S}}\,K$. Now $j_K[\hat{C}] = \{ j_K(\downarrow c) : c \in C \} = \{ \bigvee \downarrow c : c \in C \} = C$ which is a member of $\mathcal{V}K$ as required. Finally use the second paragraph of the proof of Lemma 3.8 to obtain, for any $A \in \mathcal{V}K$, $D \in \mathcal{U}\,\mathrm{Coz}_{\mathcal{S}}\,K$ with $D <^* A$; in particular $D \leqq A$. Then $j_K[\hat{D}] = D \leqq A$ which concludes the proof that j_K is an isomorphism.

For any uniform frame map $f : (N,\mathcal{V}N) \to (P,\mathcal{V}P)$ between $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frames, the following diagram commutes, since the underlying frame maps are already known to commute:

$$\begin{array}{ccc} \mathcal{H}_{\mathcal{S}}\,\mathrm{Coz}_{\mathcal{S}}(N,\mathcal{V}N) & \xrightarrow{\ j_N\ } & (N,\mathcal{V}N) \\ {\scriptstyle \mathcal{H}_{\mathcal{S}}\,\mathrm{Coz}_{\mathcal{S}}\,f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{H}_{\mathcal{S}}\,\mathrm{Coz}_{\mathcal{S}}(P,\mathcal{V}P) & \xrightarrow{\ j_P\ } & (P,\mathcal{V}P) \end{array} \qquad \square$$

4. The strategy

The main aim of this paper is to construct a completion for uniform $\mathcal{S}$-frames; we use the category equivalence between the uniform $\mathcal{S}$-frames and the $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frames for this purpose.

Starting with a uniform $\mathcal{S}$-frame, we apply the functor $\mathcal{H}_{\mathcal{S}}$ to obtain an $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame; this has a completion. We then apply the functor $\mathrm{Coz}_{\mathcal{S}}$ to this completion, thus providing the required completion of the original uniform $\mathcal{S}$-frame.

In order for this to work, we need to show that the completion of an $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame is again $\mathcal{S}$-Lindelöf and $\mathcal{S}$-separable. We do this in Section 5.

In Section 6 we define completeness for uniform $\mathcal{S}$-frames and investigate the properties of the functors $\mathcal{H}_\mathcal{S}$ and $\mathrm{Coz}_\mathcal{S}$ that are needed to enable the transfer of completions.

In Section 7 we show that this procedure does indeed give the required completion and a coreflection to complete uniform $\mathcal{S}$-frames.

5. Completions of $\mathcal{S}$-Lindelöf $\mathcal{S}$-separable uniform frames

The main result in this section is that the completion of an $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame is again $\mathcal{S}$-Lindelöf and $\mathcal{S}$-separable. The bulk of the work goes into proving that a complete, $\mathcal{S}$-separable uniform frame is $\mathcal{S}$-Lindelöf, which we do in Proposition 5.1. The special case of this result, where $\mathcal{S}$ selects countable sets, appeared in [6]; our proof is modelled on theirs.

The proof of Proposition 5.1 below requires the following technique: Given a preuniformity on a frame, one can define a uniformity on a subframe in such a way that the uniformity on the subframe generates the original preuniformity on the frame. This is the content of Lemma 2 of [4]:

Let K be a frame and $\mathcal{P}K$ a preuniformity on K. Define $i : K \to K$ by $i(x) = \bigvee \{ y \in K : y \lhd x \text{ in } (K, \mathcal{P}K) \}$ where $y \lhd x$ in $(K, \mathcal{P}K)$ means that there exists $C \in \mathcal{P}K$ with $Cy \leqq x$. Then i satisfies these conditions for all $x, y \in K$:

$$i(x) \leqq x, \quad i(x) = i^2(x), \quad i(x \wedge y) = i(x) \wedge i(y) \quad \text{and} \quad i(1) = 1.$$

Further $\mathrm{Fix}\, i = \{ x \in K : i(x) = x \}$ is a subframe of K. The set $\{ i[C] : C \in \mathcal{P}K \}$ generates a uniformity, denoted $\mathcal{V}\,\mathrm{Fix}\, i$, on $\mathrm{Fix}\, i$, and generates $\mathcal{P}K$ on K.

Proposition 5.1. *If $(N, \mathcal{V}N)$ is a complete, $\mathcal{S}$-separable uniform frame, then N is an $\mathcal{S}$-Lindelöf frame.*

Proof. Let $(N, \mathcal{V}N)$ be a complete, $\mathcal{S}$-separable uniform frame. Consider $\mathcal{H}_\mathcal{S}N$, the frame of all $\mathcal{S}$-ideals of N. (We note that earlier in this paper, we considered $\mathcal{S}$-ideals of an $\mathcal{S}$-frame, but since any frame is an $\mathcal{S}$-frame, the same construction applies here.)

For $C \in \mathcal{V}N$, define $\hat{C} = \{{\downarrow} c : c \in C\}$. Then $\{\hat{C} : C \in \mathcal{V}N$ and C is an $\mathcal{S}$-cover of $N\}$ is a basis for a preuniformity, denoted $P\mathcal{H}_\mathcal{S}N$, on $\mathcal{H}_\mathcal{S}N$; an argument like that of Lemma 3.5 applies, with the omission of the compatibility criterion.

Let $i : \mathcal{H_S}N \to \mathcal{H_S}N$ be defined by

$$i(J) = \bigvee \left\{ I \in \mathcal{H_S}N : I \lhd J \text{ in } (\mathcal{H_S}N, \mathcal{PH_S}N) \right\}.$$

By the method outlined above, $\operatorname{Fix} i$ is a subframe of $\mathcal{H_S}N$ and $\left\{ i[\hat{C}] : C \in \mathcal{V}N \text{ and } C \text{ is an } \mathcal{S}\text{-cover of } N \right\}$ generates a uniformity, denoted $\mathcal{V}\operatorname{Fix} i$ on $\operatorname{Fix} i$.

A routine calculation shows that the join map $\bigvee : \mathcal{H_S}N \to N$ is a frame map, so its restriction $j_N : \operatorname{Fix} i \to N$, is also a frame map, which is clearly dense.

For $a \in N$, we have $\downarrow\! a \in \mathcal{H_S}N$ and so $i(\downarrow\! a) \in \operatorname{Fix} i$. Now, if $x \in N$ with $x \lhd a$ in $(N, \mathcal{V}N)$, then there exists $C \in \mathcal{V}N$ with C an $\mathcal{S}$-cover of N and $Cx \leqq a$ by the $\mathcal{S}$-separability of N. Then $\hat{C}(\downarrow\! x) \subseteqq \downarrow\! a$, so $\downarrow\! x \lhd \downarrow\! a$ in $(\mathcal{H_S}N, \mathcal{PH_S}N)$. So $x \in i(\downarrow\! a)$. Since $a = \bigvee \left\{ x \in N : x \lhd a \text{ in } (N, \mathcal{V}N) \right\}$, we obtain $a = \bigvee i(\downarrow\! a)$. It now follows that the join map $j_N : (\operatorname{Fix} i, \mathcal{V}\operatorname{Fix} i) \to (N, \mathcal{V}N)$ is uniform, because $j_N i[\hat{C}] = \left\{ \bigvee i(\downarrow\! c) : c \in C \right\} = C$, for all $C \in \mathcal{V}N$, C an $\mathcal{S}$-cover of N.

The map $j_N : (\operatorname{Fix} i, \mathcal{V}\operatorname{Fix} i) \to (N, \mathcal{V}N)$ is onto, again because $a = \bigvee i(\downarrow\! a)$ for all $a \in N$. Further, j_N is a surjection, again using $\mathcal{S}$-separability of $(N, \mathcal{V}N)$ and $j_N i[\hat{C}] = C$ for all $C \in \mathcal{V}N$, C an $\mathcal{S}$-cover of N. This makes j_N a dense surjection onto a complete uniform frame, and hence an isomorphism.

Now $\mathcal{H_S}N$ is $\mathcal{S}$-Lindelöf, so $\operatorname{Fix} i$, since it is a subframe of $\mathcal{H_S}N$, is also $\mathcal{S}$-Lindelöf. But $\operatorname{Fix} i$ is isomorphic to N, and so N is $\mathcal{S}$-Lindelöf. □

Dense surjections preserve and reflect $\mathcal{S}$-separability:

Lemma 5.2. *Let $f : (N, \mathcal{V}N) \to (P, \mathcal{V}P)$ be a dense surjection between uniform frames. Then $(N, \mathcal{V}N)$ is $\mathcal{S}$-separable if and only if $(P, \mathcal{V}P)$ is $\mathcal{S}$-separable.*

Proof. ($\Rightarrow$) Given $A \in \mathcal{V}P$, there exists $B \in \mathcal{V}N$ such that $f[B] \leqq A$. Take $C \in \mathcal{V}N$ with C an $\mathcal{S}$-cover of N and $C \leqq B$, by $\mathcal{S}$-separability. Then $f[C] \leqq A$, $f[C] \in \mathcal{V}P$ and $f[C]$ is an $\mathcal{S}$-cover of P, as required.

($\Leftarrow$) We denote the right adjoint of f by r. Give $A \in \mathcal{V}N$, there exists $B \in \mathcal{V}P$ with $r[B] \leqq A$ and $r[B] \in \mathcal{V}N$. (See [2] for more details.) Now r is, in particular, a meet-semilattice map (it preserves 0 because f is dense) and so $r[B]$ is a designated subset of N, as required. □

We refer the reader to the preliminaries (Section 2) for references concerning completions of uniform frames.

Corollary 5.3. *Let $\gamma_K : C(K, \mathcal{V}K) \to (K, \mathcal{V}K)$ be the completion of the uniform frame $(K, \mathcal{V}K)$. Then $C(K, \mathcal{V}K)$ is $\mathcal{S}$-separable if and only if $(K, \mathcal{V}K)$ is $\mathcal{S}$-separable.*

Corollary 5.4. *If* $(K, \mathcal{V}K)$ *an* $\mathcal{S}$*-separable uniform frame, then* CK *is an* $\mathcal{S}$*-Lindelöf frame.*

Proof. Here $C(K, \mathcal{V}K) = (CK, \mathcal{V}CK)$ is again the completion of $(K, \mathcal{V}K)$. By Lemma 5.2, $C(K, \mathcal{V}K)$ is $\mathcal{S}$-separable, so by Proposition 5.1, CK is $\mathcal{S}$-Lindelöf. □

It is now clear that the completion of an $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame is again $\mathcal{S}$-Lindelöf and $\mathcal{S}$-separable.

6. Completeness for uniform $\mathcal{S}$-frames

We begin this section with the definition of completeness for uniform $\mathcal{S}$-frames; it is clearly a generalization of the corresponding concept for uniform frames.

Definition 6.1. (1) A uniform map $h : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ is called a *surjection* if, for each $D \in \mathcal{U}L$, there exists $C \in \mathcal{U}M$ such that $h[C] \leqq D$.

(2) An $\mathcal{S}$-frame map h is *dense* if $h(a) = 0$ implies $a = 0$; it is *codense* if $h(a) = 1$ implies $a = 1$.

(3) A uniform $\mathcal{S}$-frame $(L, \mathcal{U}L)$ is called *complete* if every dense surjection $h : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ from a uniform $\mathcal{S}$-frame $(M, \mathcal{U}M)$ is an isomorphism.

(4) A *completion* of a uniform $\mathcal{S}$-frame $(L, \mathcal{U}L)$ is a dense surjection $h : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ with $(M, \mathcal{U}M)$ a complete uniform $\mathcal{S}$-frame.

The next result shows that surjections are indeed onto:

Lemma 6.2. *Suppose that* $h : (L, \mathcal{U}L) \to (M, \mathcal{U}M)$ *is a uniform map between uniform* $\mathcal{S}$*-frames such that, for each* $D \in \mathcal{U}M$*, there exists* $C \in \mathcal{U}L$ *with* $h[C] \leqq D$*. Then* $h : L \to M$ *is onto.*

Proof. Let $b \in M$ and write $b = \bigvee T$ for some designated subset T of M with $t \in T \Rightarrow t \lhd b$ in $(M, \mathcal{U}M)$. Let $t \in T$. Then there exists $D \in \mathcal{U}M$ with $Dt \leqq b$. By assumption, there exists $C \in \mathcal{U}L$ with $h[C] \leqq D$. Since $t \leqq h[C]t \leqq b$, we obtain $b = \bigvee \{ h[C]t : t \in T \}$.

Since $C \in \mathcal{U}L$, C is a designated subset of L, so by (SCov), $K = \{ c \in C : h(c) \wedge t \neq 0 \}$ is also designated. Since h is an $\mathcal{S}$-frame map, $\bigvee h[K] = h\big(\bigvee K \big)$. But $\bigvee h[K] = h[C]t$, so we have shown that $h[C]t$ is in the image of h.

Since $T \leqq \{ h[C]t : t \in T \}$, by (SRef) there exists a designated subset S of $\{ h[C]t : t \in T \}$ such that $T \leqq S$. Then $b = \bigvee S$. Since S is a designated subset of M and is contained in the image of h, there exists a designated subset W of L such that $h[W] = S$. Then $b = \bigvee S = \bigvee h[W] = h\big(\bigvee W \big)$, showing that b is in the image of h. □

The remaining results in this section concern the preservation of relevant properties by the functors $\mathcal{H}_\mathcal{S}$ and $\mathrm{Coz}_\mathcal{S}$:

Lemma 6.3. *The functor* $\mathcal{H}_\mathcal{S} : \mathbf{Uni}\mathcal{S}\mathbf{Frm} \to \mathcal{S}\mathbf{Lind}\,\mathcal{S}\mathbf{sep\ UniFrm}$ *taking $\mathcal{S}$-ideals preserves dense surjections.*

Proof. Take $h : (L,\mathcal{U}L) \to (M,\mathcal{U}M)$ a dense surjection between uniform $\mathcal{S}$-frames; we show that $\mathcal{H}_\mathcal{S}h : (\mathcal{H}_\mathcal{S}L, \mathcal{V}\mathcal{H}_\mathcal{S}L) \to (\mathcal{H}_\mathcal{S}M, \mathcal{V}\mathcal{H}_\mathcal{S}M)$ is a dense surjection. Now $\mathcal{H}_\mathcal{S}h$ is dense, because $\mathcal{H}_\mathcal{S}h(J) = \downarrow 0$ for some $J \in \mathcal{H}_\mathcal{S}L$ implies that $h[J] = \{0\}$, giving $J = \downarrow 0$ by the density of h.

Next, take $D \in \mathcal{U}M$, and $C \in \mathcal{U}L$ with $h[C] \leqq D$. Then $\mathcal{H}_\mathcal{S}h[\hat{C}] = \widehat{h[C]}$ (see Lemma 3.6) so $\mathcal{H}_\mathcal{S}h[\hat{C}] \leqq \hat{D}$. The result follows, since $\{\hat{D} : D \in \mathcal{U}M\}$ generates the uniformity $\mathcal{V}\mathcal{H}_\mathcal{S}M$. $\square$

Lemma 6.4. *The functor* $\mathrm{Coz}_\mathcal{S} : \mathcal{S}\mathbf{Lind}\,\mathcal{S}\mathbf{sep\ UniFrm} \to \mathbf{Uni}\mathcal{S}\mathbf{Frm}$ *taking $\mathcal{S}$-cozero elements, preserves dense surjections.*

Proof. Take $f : (K, \mathcal{V}K) \to (N, \mathcal{V}N)$ a dense surjection between $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frames; we show that

$$\mathrm{Coz}_\mathcal{S} f : (\mathrm{Coz}_\mathcal{S} K, \mathcal{U}\,\mathrm{Coz}_\mathcal{S} K) \to (\mathrm{Coz}_\mathcal{S} N, \mathcal{U}\,\mathrm{Coz}_\mathcal{S} N)$$

is a dense surjection.

Certainly $\mathrm{Coz}_\mathcal{S} f$ is dense, since it acts by restriction of f. Now take $A \in \mathcal{U}\,\mathrm{Coz}_\mathcal{S} N$, that is, $A \in \mathcal{V}N$, $A \subseteqq \mathrm{Coz}_\mathcal{S} N$ and A is a designated subset of $\mathrm{Coz}_\mathcal{S} N$. There exists $B \in \mathcal{V}K$ such that $f[B] \leqq A$. By the proof of Lemma 3.8, there exists $D \in \mathcal{U}\,\mathrm{Coz}_\mathcal{S} K$ such that $D <^* B$; in particular $D \leqq B$. Then $f[D] \leqq A$. $\square$

Corollary 6.5. *The functor* $\mathrm{Coz}_\mathcal{S}$ *preserves completeness.*

Proof. Let $(K, \mathcal{V}K)$ be a complete $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frame, and $h : (L,\mathcal{U}L) \to \mathrm{Coz}_\mathcal{S}(K, \mathcal{V}K)$ a dense surjection from the uniform $\mathcal{S}$-frame $(L,\mathcal{U}L)$. Since $\mathcal{H}_\mathcal{S}$ preserves dense surjections (Lemma 6.3), $\mathcal{H}_\mathcal{S}h$ is a dense surjection. So the composite $\mathcal{H}_\mathcal{S}(L,\mathcal{U}L) \to \mathcal{H}_\mathcal{S}\,\mathrm{Coz}_\mathcal{S}(K, \mathcal{V}K) \sim (K, \mathcal{V}K)$ is a dense surjection to a complete uniform frame, so an isomorphism. Since $\mathcal{H}_\mathcal{S}$ provides a category equivalence, it reflects isomorphisms, so h is an isomorphism. $\square$

Corollary 6.6. *The functor* $\mathcal{H}_\mathcal{S}$ *preserves completeness.*

Proof. Let $(L,\mathcal{U}L)$ be a complete uniform $\mathcal{S}$-frame, and $f : (K, \mathcal{V}K) \to \mathcal{H}_\mathcal{S}(L,\mathcal{U}L)$ a dense surjection from the uniform frame $(K, \mathcal{V}K)$. Since $\mathcal{H}_\mathcal{S}(L,\mathcal{U}L)$ is $\mathcal{S}$-separable, so is $(K, \mathcal{V}K)$ by Lemma 5.2. So the completion

$C(K,\mathcal{V}K)$ is $\mathcal{S}$-Lindelöf and $\mathcal{S}$-separable using Corollary 5.4 and Lemma 5.2. So the composite

$$C(K,\mathcal{V}K) \xrightarrow[\gamma_K]{} (K,\mathcal{V}K) \xrightarrow[f]{} \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$$

is a dense surjection between $\mathcal{S}$-Lindelöf, $\mathcal{S}$-separable uniform frames. Since $\mathrm{Coz}_{\mathcal{S}}$ preserves dense surjections, $\mathrm{Coz}_{\mathcal{S}}(f\circ\gamma_K)$ is a dense surjection. So the composite

$$\mathrm{Coz}_{\mathcal{S}}\, C(K,\mathcal{V}K) \xrightarrow[\mathrm{Coz}_{\mathcal{S}}(f\circ\gamma_K)]{} \mathrm{Coz}_{\mathcal{S}}\,\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \sim (L,\mathcal{U}L)$$

is a dense surjection to a complete uniform $\mathcal{S}$-frame and so is an isomorphism. Since $\mathrm{Coz}_{\mathcal{S}}$ provides a category equivalence, it relects isomorphisms, so $f\circ\gamma_K$ is an isomorphism. Since $C(K,\mathcal{V}K)$ is complete, so is $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$. □

7. The construction of the completion of a uniform $\mathcal{S}$-frame

Let $(L,\mathcal{U}L)$ be a uniform $\mathcal{S}$-frame. Let

$$\gamma_{\mathcal{H}_{\mathcal{S}}L} : C\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \to \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$$

be the uniform frame completion of $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$. Apply the functor $\mathrm{Coz}_{\mathcal{S}}$ to obtain:

$$\mathrm{Coz}_{\mathcal{S}}\, C\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \xrightarrow[\mathrm{Coz}_{\mathcal{S}}(\gamma_{\mathcal{H}_{\mathcal{S}}L})]{} \mathrm{Coz}_{\mathcal{S}}\,\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \sim (L,\mathcal{U}L).$$

We use the notation $C_{\mathcal{S}}(L,\mathcal{U}L) \xrightarrow[\tau_L]{(L,\mathcal{U}L)}$ for this composite, which is the desired completion of $(L,\mathcal{U}L)$.

We note that $C_{\mathcal{S}}(L,\mathcal{U}L)$ is complete by Corollary 6.5 and τ_L is a dense surjection by Lemma 6.4. The lemma below shows that the completion is unique.

LEMMA 7.1. *The completion of a uniform $\mathcal{S}$-frame is unique up to isomorphism.*

PROOF. Let $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ be a dense surjection from a complete uniform $\mathcal{S}$-frame $(M,\mathcal{U}M)$ to a uniform $\mathcal{S}$-frame $(L,\mathcal{U}L)$. Apply the functor $\mathcal{H}_\mathcal{S}$ and consider the following diagram:

$$\begin{CD} \mathcal{H}_\mathcal{S}(M,\mathcal{U}M) @>{\mathcal{H}_\mathcal{S}h}>> \mathcal{H}_\mathcal{S}(L,\mathcal{U}L)\\ @A{\gamma_{\mathcal{H}_\mathcal{S}M}}A{\sim}A @AA{\gamma_{\mathcal{H}_\mathcal{S}L}}A \\ C\mathcal{H}_\mathcal{S}(M,\mathcal{U}M) @>{\sim}>> C\mathcal{H}_\mathcal{S}(L,\mathcal{U}L)\end{CD}$$

For the isomorphism $C\mathcal{H}_\mathcal{S}(M,\mathcal{U}M) \to C\mathcal{H}_\mathcal{S}(L,\mathcal{U}L)$ see [5], Corollary, p. 69. The map $\gamma_{\mathcal{H}_\mathcal{S}M}$ is an isomorphism because $\mathcal{H}_\mathcal{S}(M,\mathcal{U}M)$ is complete, by Corollary 6.6. Then $\mathcal{H}_\mathcal{S}(M,\mathcal{U}M) \sim C\mathcal{H}_\mathcal{S}(L,\mathcal{U}L)$. Applying $\mathrm{Coz}_\mathcal{S}$ gives $(M,\mathcal{U}M) \sim \mathrm{Coz}_\mathcal{S}\,\mathcal{H}_\mathcal{S}(M,\mathcal{U}M) \sim C_\mathcal{S}(L,\mathcal{U}L)$. $\square$

Our method for constructing the completion provides a very straightforward proof of the coreflectivity of completeness for uniform $\mathcal{S}$-frames:

PROPOSITION 7.2. *The complete uniform $\mathcal{S}$-frames form a coreflective subcategory of the category of uniform $\mathcal{S}$-frames, with the coreflection map given by the completion map $\tau_L : C_\mathcal{S}(L,\mathcal{U}L) \to (L,\mathcal{U}L)$.*

PROOF. Let $(L,\mathcal{U}L)$ be a uniform $\mathcal{S}$-frame, and $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ be a uniform map from a complete uniform $\mathcal{S}$-frame $(M,\mathcal{U}M)$. We show that there exists a unique uniform map $g : (M,\mathcal{U}M) \to C_\mathcal{S}(L,\mathcal{U}L)$ such that $\tau_L \circ g = h$, i.e. making the following diagram commute:

$$\begin{array}{ccc} C_\mathcal{S}(L,\mathcal{U}L) & \xrightarrow{\ \tau_L\ } & (L,\mathcal{U}L) \\ {\scriptstyle \exists! g}\uparrow & \nearrow{\scriptstyle h} & \\ (M,\mathcal{U}M) & & \end{array}$$

Now $\mathcal{H}_\mathcal{S}C_\mathcal{S}(L,\mathcal{U}L) = \mathcal{H}_\mathcal{S}\,\mathrm{Coz}_\mathcal{S}\,C\mathcal{H}_\mathcal{S}(L,\mathcal{U}L) \sim C\mathcal{H}_\mathcal{S}(L,\mathcal{U}L)$, so applying $\mathcal{H}_\mathcal{S}$ to the diagram above gives the diagram below:

$$\begin{array}{ccc} C\mathcal{H}_\mathcal{S}(L,\mathcal{U}L) \sim \mathcal{H}_\mathcal{S}C_\mathcal{S}(L,\mathcal{U}L) & \xrightarrow{\ \mathcal{H}_\mathcal{S}\tau_L\ } & \mathcal{H}_\mathcal{S}(L,\mathcal{U}L) \\ & {\scriptstyle k}\nwarrow \quad \nearrow{\scriptstyle \mathcal{H}_\mathcal{S}h} & \\ & \mathcal{H}_\mathcal{S}(M,\mathcal{U}M) & \end{array}$$

Since $\mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M)$ is complete by Corollary 6.6, there exists a unique uniform map $k : \mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M) \to C\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ making the diagram above commute. Then $g = \mathrm{Coz}_{\mathcal{S}}\, k$ makes the first diagram above commute.

Uniqueness of this g follows because τ_L is dense, and so monic (see Proposition 8.10 of [8]). □

Remark 7.3. In the case where $\mathcal{S}$ selects all countable collections, we obtain a construction of the completion of uniform σ-frames. An alternative construction for this case was provided in [23], but the method we use has some substantially different features. In the case where $\mathcal{S}$ selects all collections with cardinality less than some fixed regular cardinal κ, we obtain a completion for uniform κ-frames.

8. The construction of the Samuel compactification of a uniform $\mathcal{S}$-frame

We view the Samuel compactification as the coreflection of uniform $\mathcal{S}$-frames to compact uniform $\mathcal{S}$-frames, which is the approach taken in, for instance, [5], [17], [7] and [23], as well as the original [21]. Our construction mirrors closely the construction of the completion in the previous section; we use the functors $\mathcal{H}_{\mathcal{S}}$ and $\mathrm{Coz}_{\mathcal{S}}$ to transfer the Samuel compactification of uniform frames to uniform $\mathcal{S}$-frames. Before doing this we relate the notions of compactness, total boundedness and completeness in the expected way:

Definition 8.1. (1) We call an $\mathcal{S}$-frame *compact* if any $\mathcal{S}$-cover has a finite sub $\mathcal{S}$-cover.

(2) We call a uniform $\mathcal{S}$-frame *totally bounded* if every uniform $\mathcal{S}$-cover is refined by a finite uniform $\mathcal{S}$-cover.

Proposition 8.2. *A uniform $\mathcal{S}$-frame is complete and totally bounded if and only if it is compact.*

Proof. (⇒) Suppose that $(L,\mathcal{U}L)$ is a complete, totally bounded uniform $\mathcal{S}$-frame. Then $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ is totally bounded, since its uniformity is generated by $\{\hat{C} : C \in \mathcal{U}L\}$, where $\hat{C} = \{\downarrow c : c \in C\}$. Then the completion $C\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ is a complete totally bounded uniform frame and so is compact. (See [2].) This makes $C_{\mathcal{S}}(L,\mathcal{U}L) = \mathrm{Coz}_{\mathcal{S}}\, C\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ a compact $\mathcal{S}$-frame.

(⇐) Let $(L,\mathcal{U}L)$ be a compact uniform $\mathcal{S}$-frame. That $(L,\mathcal{U}L)$ is totally bounded is clear. To show completeness, begin with a dense surjection $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ between uniform $\mathcal{S}$-frames. We show that h is codense which suffices to show that that h is $1-1$ (see Proposition 8.10 of [8]). So suppose that $h(a) = 1$ for some $a \in M$. Write $a = \bigvee T$ for some designated subset T of M with $t \in T \Rightarrow t \lhd a$ in $(M,\mathcal{U}M)$. Then $1 = \bigvee h[T]$, making $h[T]$ an $\mathcal{S}$-cover of L. Now L, being compact, has a unique $\mathcal{S}$-nearness (and hence $\mathcal{S}$-uniform) structure, consisting of all $\mathcal{S}$-covers of L (see Proposition 10.7 of [8]). So $h[T] \in \mathcal{U}L$. Then there exists $C \in \mathcal{U}M$ with $h[C] \leqq h[T]$. Let $c \in C$; there exists $t \in T$ with $h(c) \leqq h(t)$. Since $t \lhd a$ in $(M,\mathcal{U}M)$, $t \prec a$. So there exists $s \in M$ with $s \wedge t = 0$, $a \vee s = 1$. Then $h(c \wedge s) = h(c) \wedge h(s)$ $\leqq h(t) \wedge h(s) = h(t \wedge s) = 0$; by density of h, we obtain $c \wedge s = 0$. So $c = c \wedge (a \vee s) = c \wedge a$, so $c \leqq a$. So $\bigvee C \leqq a$, making $a = 1$. $\square$

The construction of the Samuel compactification of a uniform $\mathcal{S}$-frame. Let $(L,\mathcal{U}L)$ be a uniform $\mathcal{S}$-frame. Let $\rho_{\mathcal{H}_{\mathcal{S}}L} : \operatorname{Sam} \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \to \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$ be the Samuel compactification of the uniform frame $\mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$. Apply the functor $\operatorname{Coz}_{\mathcal{S}}$ to obtain:

$$\operatorname{Coz}_{\mathcal{S}} \rho_{\mathcal{H}_{\mathcal{S}}L} : \operatorname{Coz}_{\mathcal{S}} \operatorname{Sam} \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \to \operatorname{Coz}_{\mathcal{S}} \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \sim (L,\mathcal{U}L)$$

We use the notation $\xi_L : \operatorname{Sam}_{\mathcal{S}}(L,\mathcal{U}L) \to (L,\mathcal{U}L)$ for this composite, which is the desired Samuel compactification of $(L,\mathcal{U}L)$.

We note that $\operatorname{Sam}_{\mathcal{S}}(L,\mathcal{U}L)$ is a compact uniform $\mathcal{S}$-frame by Lemma 6.1 of [9]. Further, ξ_L is a dense onto uniform $\mathcal{S}$-frame map by Proposition 6.2 of [9].

We now prove the required coreflection property:

THEOREM 8.3. *For any uniform $\mathcal{S}$-frame $(L,\mathcal{U}L)$, the map*

$$\xi_L : \operatorname{Sam}_{\mathcal{S}}(L,\mathcal{U}L) \to (L,\mathcal{U}L)$$

is the Samuel compactification of $(L,\mathcal{U}L)$.

PROOF. Let $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ be a uniform map from a compact uniform $\mathcal{S}$-frame $(M,\mathcal{U}M)$ to a uniform $\mathcal{S}$-frame $(L,\mathcal{U}L)$. Since M is compact, so is $\mathcal{H}_{\mathcal{S}}M$ by Lemma 6.1 of [9] and so $\operatorname{Sam} \mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M)$ $\sim \mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M)$. This gives

$$\operatorname{Sam}_{\mathcal{S}}(M,\mathcal{U}M) = \operatorname{Coz}_{\mathcal{S}} \operatorname{Sam} \mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M) \sim \operatorname{Coz}_{\mathcal{S}} \mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M) \sim (M,\mathcal{U}M),$$

as illustrated in the diagram below:

$$\begin{array}{ccccc}
\operatorname{Coz}_{\mathcal{S}} \operatorname{Sam} \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) & \longrightarrow & & & \operatorname{Coz}_{\mathcal{S}} \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L) \sim (L,\mathcal{U}L) \\
\uparrow & & & \nearrow h & \\
\operatorname{Coz}_{\mathcal{S}} \operatorname{Sam} \mathcal{H}_{\mathcal{S}}(M,\mathcal{U}M) & \underset{\sim}{\xrightarrow{\xi_M}} & (M,\mathcal{U}M) & &
\end{array}$$

So, by functoriality, there exists a uniform map

$$g : (M,\mathcal{U}M) \to \operatorname{Coz}_{\mathcal{S}} \operatorname{Sam} \mathcal{H}_{\mathcal{S}}(L,\mathcal{U}L)$$

such that $\xi_L \circ g = h$. Uniqueness of g follows, since ξ_L is dense and hence monic. (See Proposition 8.10 of [8].) $\square$

We now provide an explicit construction of the totally bounded coreflection of a uniform $\mathcal{S}$-frame, which is preparatory to providing an alternative description of the Samuel compactification:

PROPOSITION 8.4. *Let* $(L, \mathcal{U}L)$ *be a uniform* $\mathcal{S}$*-frame. Define* $\mathrm{tb}\mathcal{U}L = \{C \in \mathcal{U}L : F \leqq C$ *for some finite* $F \in \mathcal{U}L\}$. *Then* $(L, \mathrm{tb}\mathcal{U}L)$ *is a totally bounded uniform* $\mathcal{S}$*-frame, and the identity map* $i : (L, \mathrm{tb}\mathcal{U}L) \to (L, \mathcal{U}L)$ *is the totally bounded coreflection of* $(L, \mathcal{U}L)$ *in the category of uniform* $\mathcal{S}$*-frames.*

PROOF. The proof of this result is in all essentials the same as the proof of the equivalent result for uniform frames appearing in [5]. We therefore only sketch the method.

The main ingredient is that $\mathrm{tb}\mathcal{U}L$ should have the star-refinement property. To see this, begin with a finite $A \in \mathcal{U}L$ and take $B \in \mathcal{U}L$ with $B <^* A$. Define an equivalence relation $\sim$ on B by $x \sim y$ iff $A \cap \uparrow x = A \cap \uparrow y$ and $A \cap \uparrow (Bx) = A \cap \uparrow (By)$. Since A is finite, this gives a finite partition of B.

For each $b \in B$, define $\overline{b} = \bigvee\{x \in B : x \sim b\}$ which exists by (SCov). Finally let $\overline{B} = \{\overline{b} : b \in B\}$. It is immediate that $\overline{B}$ is finite and since $B \leqq \overline{B}$, we have $\overline{B} \in \mathcal{U}L$. One then checks that if $Bb \leqq a$ then $\overline{B}\,\overline{b} \leqq a$, all required joins existing by application of (SCov).

For compatibility, one shows that $x \lhd y$ in $(L, \mathcal{U}L)$ iff $x \lhd y$ in $(L, \mathrm{tb}\mathcal{U}L)$. The coreflection property is then clear (since the image of a finite cover is finite). $\square$

REMARK 8.5. We note that the technique for constructing coreflections provided in [8] would suffice to provide the coreflection result of Proposition 8.4, but the explicit description of the totally bounded coreflection provided above would then not be available.

COROLLARY 8.6. *The Samuel compactification of a uniform* $\mathcal{S}$*-frame* $(L, \mathcal{U}L)$ *can equivalently be given by the composite*:

$$C_{\mathcal{S}}(L, \mathrm{tb}\mathcal{U}L) \xrightarrow[\tau]{} (L, \mathrm{tb}\mathcal{U}L) \xrightarrow[i]{} (L, \mathcal{U}L)$$

that is, by the completion of its totally bounded coreflection.

PROOF. This is a routine application of Propositions 7.2, 8.2 and 8.4. $\square$

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