Acta Math. Hungar., **144** (**2**) (2014), 425–432
DOI: 10.1007/s10474-014-0457-6
First published online October 19, 2014

LOW MOMENTS OF DIRICHLET SERIES

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(Received January 7, 2014; revised May 28, 2014; accepted June 30, 2014)

Abstract. We determine the maximum possible size of the q^{th} moment of a Dirichlet series, for $1 \leqq q \leqq 2$.

1. Introduction

In order to bound the mean value of multiplicative functions, Halász [5] introduced a majorant principle which (after a little refining) asserts that if $\lambda_1, \lambda_2, \dots$ are real numbers, if $|a_n| \leqq A_n$ for all n, and $\sum_{n\geqq 1} A_n < \infty$, then

$$\int_{-T}^{T} \Big| \sum_{n\geqq 1} a_n \mathrm{e}^{i\lambda_n t} \Big|^2 \,\mathrm{d}t \leqq 3 \int_{-T}^{T} \Big| \sum_{n\geqq 1} A_n \mathrm{e}^{i\lambda_n t} \Big|^2 \,\mathrm{d}t. \tag{1.1}$$

For a proof of the principle in this form, see Montgomery [8, §7.3]. In Halász's theory, one needs bounds for integrals of the shape

$$I(q) = \int_{-1}^{1} \Big| \sum_{p} \frac{a_p \log p}{p^{\sigma+it}} \Big|^q \,\mathrm{d}t, \qquad J(q) = \int_{-1}^{1} \Big| \sum_{n\geqq 1} \frac{b_n}{n^{\sigma+it}} \Big|^q \,\mathrm{d}t, \tag{1.2}$$

[*] Corresponding author.
Key words and phrases: Dirichlet series, multiplicative function.
Mathematics Subject Classification: primary 30B50, secondary 11Q20, 11N37.

when $|a_p| \leqq 1$ for all p, and $|b_n| \leqq 1$ for all n. From (1.1) it is immediate that

$$I(2) \leqq 3\int_{-1}^{1}\Big|\sum_{n\geqq 2}\frac{\Lambda(n)}{n^{\sigma+it}}\Big|^2\,\mathrm{d}t = 3\int_{-1}^{1}\Big|\frac{\zeta'}{\zeta}(\sigma+it)\Big|^2\,\mathrm{d}t$$

$$\ll \int_{-1}^{1}\frac{\mathrm{d}t}{|\sigma+it-1|^2} \ll \frac{1}{\sigma-1}$$

uniformly for $1<\sigma\leqq 2$. Similarly, $J(2) \ll 1/(\sigma-1)$ for σ in this range. By applying the majorant principle to the squares of these Dirichlet series we find that $I(4) \ll (\sigma-1)^{-3}$ and $J(4) \ll (\sigma-1)^{-3}$. Hence by Hölder's inequality,

$$I(q) \ll (\sigma-1)^{1-q}, \quad J(q) \ll (\sigma-1)^{1-q} \quad (2\leqq q\leqq 4), \tag{1.3}$$

$$I(q) \ll (\sigma-1)^{-q/2}, \quad J(q) \ll (\sigma-1)^{-q/2} \quad (1\leqq q\leqq 2) \tag{1.4}$$

uniformly for $1<\sigma\leqq 2$. The estimate (1.3) is best possible, as we see by taking $a_p=1$ for all p and $b_n=1$ for all n. For purposes of Halász's theory, it would be helpful if (1.3) held also when $1<q\leqq 2$. However, we construct examples that show that the weaker estimate (1.4) is best possible.

THEOREM 1.1. *Let $I(q)$ and $J(q)$ be defined as in* (1.2). *There exist numbers a_p with $a_p=\pm 1$ for all p, and b_n with $b_n=\pm 1$ for all n, such that*

$$I(q) \asymp (\sigma-1)^{-q/2}, \qquad J(q) \asymp (\sigma-1)^{-q/2}$$

uniformly for $1<\sigma\leqq 2$, $1\leqq q\leqq 2$.

This is analogous to the situation for Fourier series. For example, if $|b_n|\leqq 1$ for $-N\leqq n\leqq N$ and $\mathrm{e}(\vartheta)=\mathrm{e}^{2\pi i\vartheta}$, then

$$\int_0^1\Big|\sum_{|n|\leqq N} b_n\mathrm{e}(nx)\Big|^q\,\mathrm{d}x \ll N^{q-1} \tag{1.5}$$

uniformly for $2\leqq q\leqq 4$, and

$$\int_0^1\Big|\sum_{|n|\leqq N} b_n\mathrm{e}(nx)\Big|^q\,\mathrm{d}x \ll N^{q/2} \tag{1.6}$$

for $1 \leqq q \leqq 2$, but there exists a choice of the b_n with $b_n = \pm 1$ for all n such that

$$\int_0^1 \Big| \sum_{|n| \leqq N} b_n \mathrm{e}(nx) \Big|^q \, \mathrm{d}x \asymp N^{q/2} \tag{1.7}$$

uniformly for $1 \leqq q \leqq 2$. Indeed, we use such b_n in our construction.

Antecedents of Halász's majorant principle (1.1) are found in Wiener and Wintner [15] and in Erdős and Fuchs [3]. Logan [6] showed that the constant 3 in (1.1) is best-possible.

2. Lemmas

We begin with a generalization of a result of H. S. Shapiro [12].

Let the sequence $\{r_n\}_{n=0}^{\infty}$ be defined by the relations $r_0 = 1$, $r_{2n} = r_n$ and $r_{2n+1} = (-1)^n r_n$. The sequence $\{r_n\}_{n=0}^{\infty}$ is the classical Rudin–Shapiro sequence. Suppose that the binary expansion of n is $n = \sum_{j \geqq 0} e_j(n) 2^j$ where $e_j(n) = 0$ or 1. A well-known alternative definition is

$$r_n = (-1)^{H(n)}, \text{ where } H(n) := \sum_{j \geqq 0} e_j(n) e_{j+1}(n) \qquad (n \geqq 0).$$

Let $p_m(z)$, $q_m(z)$ denote polynomials defined recursively by the relations $p_0(z) = 1$, $q_0(z) = 1$ and

$$p_{m+1}(z) = p_m(z) + z^{2^m} q_m(z), \quad q_{m+1}(z) = p_m(z) - z^{2^m} q_m(z). \tag{2.1}$$

One can easily check that

$$p_m(z) = \sum_{0 \leqq n \leqq 2^m - 1} r_n z^n \qquad (m \geqq 0).$$

The Rudin–Shapiro sequence may be generalized by the so-called paper-folding twist. This amounts to introducing a sequence $\{\varepsilon_m\}_{m=0}^{\infty} \in \{\pm 1\}^{\mathbb{N}}$ and replacing (2.1) by

$$p_{m+1}(z) = p_m(z) + \varepsilon_m z^{2^m} q_m(z), \quad q_{m+1}(z) = p_m(z) - \varepsilon_m z^{2^m} q_m(z). \tag{2.2}$$

We then obtain

$$p_m(z) = \sum_{0 \leqq n \leqq 2^m - 1} c_n z^n \qquad (m = 0, 1, \ldots)$$

with

$$c_n = r_n \prod_{j\geqq 0} \varepsilon_j^{e_j(n)} \qquad (n \geqq 0), \tag{2.3}$$

a formula for which we did not find a reference in the literature and which B. Saffari kindly pointed out to us.

Lemma 2.1. *Let the sequence c_m be defined as above. Put*

$$P_M(\vartheta) = \sum_{0\leqq m<M} c_m \mathrm{e}(m\vartheta). \tag{2.4}$$

Then

$$\big|\, P_M(\vartheta)\big| \leqq \big(2+\sqrt{2}\,\big)\sqrt{M}$$

for all positive integers M and all real ϑ.

Shapiro proved this in the case $c_m = r_m$ $(m \geqq 0)$ but never published his work. The coefficients r_m were independently discovered by Golay [4]. Rudin [11] published an account of Shapiro's argument in the case $M = 2^k$, but obtained an inferior constant in the general case. The above lemma is proved in [7, théorème 2].

We note in passing that it follows from the proof of theorem 2 of [7] that, given an arbitrary sequence $\{\eta_j\}_{j=0}^{\infty} \in \{\pm 1\}^{\mathbb{N}}$, a generalized Rudin–Shapiro sequence may alternatively be written as

$$c_m = (-1)^{v_m} \tag{2.5}$$

where v_m equals 0 or 1 according to whether $\sum_{j\geqq 0} \eta_j \big| e_j(m) - e_{j+1}(m)\big|$ belongs to $\{0, 1\}$ or to $\{2, 3\}$ modulo 4. In this setting, we recover r_m by selecting $\eta_j = (-1)^j$ $(j \geqq 0)$. Also, this easily enables retrieving (2.3).

Lemma 2.2. *For $|z| < 1$, let $f(z) = \sum_{m\geq 0} c_m z^m$ where c_m is defined as in* (2.3). *Then*

$$\big| f\big(r\mathrm{e}(\vartheta)\big)\,\big| \leqq \frac{2+\sqrt{2}}{\sqrt{1-r}}.$$

Numerical studies suggest that, at least in the case $c_j = r_j$ $(j \geqq 0)$,

$$\max_{\vartheta} \big| f\big(r\mathrm{e}(\vartheta)\big)\,\big| = f(r). \tag{2.6}$$

Moreover, it is easy to show that, in the above circumstance, $f(r)\sqrt{1-r}$ does not tend to a limit as $r \to 1^-$. Indeed, it is proved in Brillhart, Erdős and Morton [1] that

$$\sum_{m<n} r_m = \sqrt{n}\, G\left(\frac{\log n}{\log 4}\right) \qquad (n \geqq 0) \tag{2.7}$$

where G is 1-periodic and continuous. Moreover, Dumont and Thomas [2] showed that G is nowhere differentiable and Tenenbaum [13] obtained the oscillation result

$$G(x+h) - G(x) = \Omega\big(\sqrt{h}\,\big) \qquad (h \geqq 0)$$

for any given real number x.

By partial summation it is readily derived from (2.7) that $f(r)\sqrt{1-r}$ oscillates as $r \to 1-$: indeed, as $y \to \infty$,

$$2^y f\big(\exp(-4^{-y})\big) \tag{2.8}$$

tends to a 1-periodic, nowhere differentiable function of y.

It is noteworthy that

$$f(z) = f\big(z^2\big) + f\big(-z^2\big), \quad f(-z) = f\big(z^2\big) - f\big(-z^2\big). \tag{2.9}$$

Kumiko Nishioka [10] showed that $f(z)$ and $f(-z)$ are algebraically independent, and then used these recurrences and Mahler's method to show that if α is algebraic with $0 < |\alpha| < 1$, then $f(\alpha)$ and $f(-\alpha)$ are algebraically independent.

Proof. Clearly

$$\frac{f\big(re(\vartheta)\big)}{1-r} = \sum_{m\geqq 0} P_{m+1}(\vartheta) r^m.$$

Hence by Lemma 2.1 and the triangle inequality it follows that

$$\left|\frac{f\big(re(\vartheta)\big)}{1-r}\right| \leqq \big(2+\sqrt{2}\,\big) \sum_{m\geqq 0} \sqrt{m+1}\, r^m. \tag{2.10}$$

But

$$\sqrt{m+1} \leqq \binom{m+1/2}{m} \tag{2.11}$$

for all non-negative integers m. Hence the right hand side of (2.10) is

$$\leqq \left(2+\sqrt{2}\right) \sum_{m\geqq 0} \binom{m+1/2}{m} r^m = \frac{2+\sqrt{2}}{(1-r)^{3/2}},$$

which gives the stated result.

To prove (2.11), let $a_m = \binom{m+1/2}{m}/\sqrt{m+1}$. To show that $a_m \geqq 1$, it suffices to note that $a_0 = 1$, and to show that the a_m are increasing. As to this latter point, we observe that

$$\frac{a_m}{a_{m-1}} = \frac{m+1/2}{\sqrt{m(m+1)}} = \frac{2m+1}{\sqrt{(2m+1)^2-1}} > 1. \quad \square$$

Lemma 2.3. *Let f be defined as in Lemma 2.2. For each r, $0<r<1$, there is a measurable set $A_r \subseteqq \mathbb{T}$ with Lebesgue measure $\lambda(A_r) \geqq 1/50$ such that*

$$\left|f\left(re(\vartheta)\right)\right| \geqq \frac{1}{2\sqrt{1-r}} \tag{2.12}$$

for all $\vartheta \in A_r$.

Proof. Let $B_r = \mathbb{T} \setminus A_r$ be the complementary set of those ϑ on which $|f|$ is small; precisely $\left|f\left(re(\vartheta)\right)\right| < 1/\left(2\sqrt{1-r}\right)$. By Parseval's identity we know that

$$\int_0^1 \left|f\left(re(\vartheta)\right)\right|^2 \mathrm{d}\vartheta = \sum_{m\geqq 0} r^{2m} = \frac{1}{1-r^2} > \frac{1}{2(1-r)}. \tag{2.13}$$

By Lemma 2.2, the left hand side above equals

$$\int_{A_r} \left|f\left(re(\vartheta)\right)\right|^2 \mathrm{d}\vartheta + \int_{B_r} \left|f\left(re(\vartheta)\right)\right|^2 \mathrm{d}\vartheta \leqq \frac{\left(2+\sqrt{2}\right)^2}{1-r}\lambda(A_r) + \frac{1-\lambda(A_r)}{4(1-r)}.$$

On combining this with (2.13), we find that

$$\frac{1}{4} \leqq \left(\left(2+\sqrt{2}\right)^2 - \frac{1}{4}\right)\lambda(A_r),$$

which gives the stated result. $\square$

Lemma 2.4. *Write $s=\sigma+it$. Then*

$$\sum_{n\leqq x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{x^{1-\sigma}}{\exp\left(\sqrt{\log x}\right)}\right)$$

for $x \geqq 2$, $1 < \sigma \leqq 2$, $-1 \leqq t \leqq 1$.

This is included in equation (III.5.72) of [14] and is proved by Perron's summation formula (see Montgomery–Vaughan [9, Theorem 5.2] or Tenenbaum [14, Corollary II.2.4]) appealing to the classical zero-free region and estimates for $\zeta'(s)/\zeta(s)$ in the zero-free region.

Lemma 2.5. *For* $x \geqq 2$, $1 < \sigma \leqq 2$, *and* $-1 \leqq t \leqq 1$, *we have*

$$\sum_{n \leqq x} \frac{1}{n^s} = \zeta(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{1}{x^\sigma}\right).$$

This is immediate by partial summation; see Montgomery–Vaughan [9, Theorem 1.12] or Tenenbaum [14, Theorem II.3.5] for the details.

3. Proof of the Theorem

In view of the upper bounds of (1.4), it suffices to establish lower bounds. For $I(q)$ we let c_m be defined as in (2.3), and take $a_p := c_m$ for $\mathrm{e}^{\pi m} < p < \mathrm{e}^{\pi(m+1)}$. Then, for $s = \sigma + it$, $1 < \sigma \leqq 2$, $|t| \leqq 1$,

$$(3.1) \qquad \begin{cases} \displaystyle\sum_p \frac{a_p \log p}{p^s} = \sum_{m \geqq 0} c_m \sum_{\mathrm{e}^{\pi m} < p < \mathrm{e}^{\pi(m+1)}} \frac{\log p}{p^s} \\ \displaystyle = \sum_{m \geqq 0} c_m \Bigg(\sum_{\mathrm{e}^{\pi m} < n < \mathrm{e}^{\pi(m+1)}} \frac{\Lambda(n)}{n^s} + O\big(\mathrm{e}^{\pi m(1-2\sigma)}\big)\Bigg). \end{cases}$$

By Lemma 2.4 this is

$$= \sum_{m \geqq 0} c_m \left(\frac{\mathrm{e}^{\pi(m+1)(1-s)}}{1-s} - \frac{\mathrm{e}^{\pi m(1-s)}}{1-s}\right) + O\left(\sum_{m \geqq 0} \frac{\mathrm{e}^{\pi m(1-\sigma)}}{\exp\left(c\sqrt{m}\right)}\right) + O(1).$$

Here the first error term is also $O(1)$, uniformly for $\sigma \geqq 1$. The main term is

$$(3.2) \qquad F(s) f\big(\mathrm{e}^{\pi(1-s)}\big)$$

where f is defined in Lemma 2.2, and

$$(3.3) \qquad F(s) = \frac{\mathrm{e}^{\pi(1-s)} - 1}{1-s}\cdot$$

The zeros of this entire function are the numbers $1 + 2im$ where m runs over non-zero integers. Thus $\big|F(s)\big|$ is bounded away from 0 uniformly on the rectangle $1 \leqq \sigma \leqq 2$, $-1 \leqq t \leqq 1$. With a little computation one can in fact show that the minimum of $\left|F(s)\right|$ in this rectangle is $\left|F(2 \pm i)\right| \approx 0.73766$. By Lemma 2.3 it follows that if σ is fixed, $1 < \sigma \leqq 2$, then

$$\left|\sum_p \frac{a_p \log p}{p^{\sigma+it}}\right| \gg \frac{1}{\sqrt{1-\mathrm{e}^{\pi(\sigma-1)}}} \asymp \frac{1}{\sqrt{\sigma-1}}$$

when $t/2 \in A_{\mathrm{e}^{\pi(\sigma-1)}}$, i.e. on a subset of $-1 \leqq t \leqq 1$ of measure $> 1/25$. Hence $I(q) \gg (\sigma-1)^{-q/2}$.

The proof for $J(q)$ is the same, except that now the passage from $\log p$ to $\Lambda(n)$ in (3.1) is unnecessary, and instead of Lemma 2.4 we use Lemma 2.5, in which the error term is smaller.

References

[1] J. Brillhart, P. Erdős and P. Morton, On sums of Rudin–Shapiro coefficients II, *Pacific J. Math.,* **107** (1983), 39–69.

[2] J.-M. Dumont and A. Thomas, Systèmes de numération et fonctions fractales relatifs aux substitutions, *Theoret. Comput. Sci.,* **65** (1989), 153–169.

[3] P. Erdős and W. H. J. Fuchs, On a problem in additive number theory, *J. London Math. Soc.,* **31** (1956), 67–73.

[4] M. J. E. Golay, Multi-slit spectroscopy, *J. Opt. Soc. Amer.,* **39** (1949), 437–444.

[5] G. Halász, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hungar.,* **19** (1968), 365–403.

[6] B. F. Logan, An interference problem for exponentials, *Michigan Math. J.,* **35** (1988), 369–393.

[7] M. Mendès France and G. Tenenbaum, Dimension des courbes planes, papiers pliés, et suites de Rudin–Shapiro, *Bull. Soc. Math. France,* **109** (1981), 207–215.

[8] H. L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis,* CBMS No. 84, Amer. Math. Soc. (Providence, 1994), xiii+220 pp.

[9] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory,* Cambridge University Press (Cambridge, 2007), xvii+552 pp.

[10] K. Nishioka, New approach in Mahler's method, *J. Reine Angew. Math.,* **407** (1990), 202–219.

[11] W. Rudin, Some theorems on Fourier coefficients, *Proc. Amer. Math. Soc.,* **10** (1959), 855–859.

[12] H. S. Shapiro, *Extremal problems for polynomials and power series,* M.S. Thesis, Massachusetts Institute of Technology, 1951.

[13] G. Tenenbaum, Sur la non-dérivabilité de fonctions périodiques associées à certaines fonctions sommatoires, in: R. L. Graham and J. Nešetřil (eds), *The Mathematics of Paul Erdős,* Algorithms and combinatorics 13 (Springer Verlag, 1997), pp. 117–128.

[14] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres,* troisième édition, coll. Échelles (Belin, 2008), 592 pp.

[15] N. Wiener and A. Wintner, On a local L^2-variant of Ikehara's theorem, *Rev. Math. Cuyana,* **2** (1956), 53–59; Norbert Wiener: *Collected Works,* vol. II, MIT Press, Cambridge, 1979, pp. 758–764; 788–790.