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PITT'S INEQUALITY AND LOGARITHMIC UNCERTAINTY PRINCIPLE FOR THE DUNKL TRANSFORM ON $\mathbb{R}$

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Abstract. We establish Pitt's inequality and deduce Beckner's logarithmic uncertainty principle for the Dunkl transform on $\mathbb{R}$. Also, we prove Stein–Weiss inequality for the Dunkl–Bessel potentials.

1. Introduction

In the Euclidean case Pitt's inequality for the Fourier transform [1,2], is given for $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space) and $0 \leqq \beta < n$, by

$$\int_{\mathbb{R}^n} |y|^{-\beta} \big| \widehat{f}(y) \big|^2 \, dy \leqq \pi^\beta \left[\Gamma\left(\frac{n-\beta}{4}\right) \Big/ \Gamma\left(\frac{n+\beta}{4}\right) \right]^2 \int_{\mathbb{R}^n} |x|^\beta \big| f(x) \big|^2 \, dx. \tag{1}$$

This inequality plays an important role for which some uncertainty principles hold. One of these uncertainty principles is the well-known Beckner's logarithmic uncertainty principle [1], that is, for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \ln\big(|x|\big) \big| f(x) \big|^2 \, dx + \int_{\mathbb{R}^n} \ln\big(|y|\big) \big| \widehat{f}(y) \big|^2 \, dy \geqq \big(\psi(n/4) - \ln \pi\big) \int_{\mathbb{R}^n} \big| f(x) \big|^2 \, dx, \tag{2}$$

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where

$$\psi(t) = \frac{d}{dt}\big[\ln \Gamma(t)\big]\,.$$

Recently, Omri [9] has established an analogue of the Beckner logarithmic inequality for the Hankel transform.

In this paper, we consider $\alpha \geqq -1/2$ and denote by $L^p_\alpha(\mathbb{R})$, $1 \leqq p \leqq \infty$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^p_\alpha} := \begin{cases} \Big(\displaystyle\int_{\mathbb{R}} \big| f(y)\big|^p |y|^{2\alpha+1}\, dy\Big)^{1/p} < \infty, & 1 \leqq p < \infty, \\ \operatorname*{ess\,sup}_{y\in\mathbb{R}} \big| f(y)\big| < \infty, & p = \infty. \end{cases}$$

For $f \in L^1_\alpha(\mathbb{R})$ the Dunkl transform is defined (see [5]) by

$$\mathcal{F}_\alpha(f)(y) := \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{\mathbb{R}} E_\alpha(-ixy) f(x)|y|^{2\alpha+1}\, dy, \quad y \in \mathbb{R},$$

where $E_\alpha(-ixy)$ denotes the Dunkl kernel (for more details see the next section).

Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [11] and Shimeno [12] who established (by two different methods) the Heisenberg–Pauli–Weyl inequality for the Dunkl transform. Kawazoe and Mejjaoli [8] gave some related versions of the uncertainty principle for the Dunkl transform (Cowling–Price's theorem, Miyachi's theorem, Beurling's theorem and Donoho-Stark's theorem). The author [15] proved a general form of the Heisenberg–Pauli–Weyl inequality for the Dunkl transform.

Building on the ideas of Beckner [1] and Omri [9] we show a Pitt's inequality for the Dunkl transform (Theorem 1) and we deduce logarithmic uncertainty inequality for the Dunkl transform $\mathcal{F}_\alpha$ (Theorem 2). This inequality generalizes the Beckner's logarithmic uncertainty inequality given by (2).

The Pitt's inequality for the Dunkl transform also leads to a Stein–Weiss inequality in the Dunkl setting (Theorem 5).

This paper is organized as follows. In Section 2, we give Pitt's inequality for the Dunkl transform $\mathcal{F}_\alpha$ and deduce Beckner's logarithmic uncertainty inequality for $\mathcal{F}_\alpha$. The last section is devoted to prove Stein–Weiss inequality in the Dunkl setting.

2. Logarithmic uncertainty principle for the Dunkl transform

The Dunkl operator Λ_α, $\alpha \geqq -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha f(x) := \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left[\frac{f(x)-f(-x)}{2}\right],$$

is the operator devised by Dunkl in connection with a generalization of the classical theory of spherical harmonics.

For $\alpha \geqq -1/2$ and $y \in \mathbb{R}$, the initial problem:

$$\Lambda_\alpha f(x) = iyf(x), \quad f(0) = 1,$$

has a unique analytic solution $E_\alpha(ixy)$ called Dunkl kernel [4,10] given by

$$E_\alpha(ixy) = j_\alpha(xy) + \frac{ixy}{2(\alpha+1)} j_{\alpha+1}(xy), \quad x \in \mathbb{R},$$

where

$$j_\alpha(xy) := \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (xy)^{2n}}{2^{2n} n!\,\Gamma(n+\alpha+1)},$$

is the spherical Bessel function of order α (see [19]).

For $x, y \in \mathbb{R}$, the Dunkl kernel $E_\alpha(ixy)$ has the following Bochner-type representation (see [4,10])

$$E_\alpha(ixy) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_{-1}^{1} e^{ixyt} \left(1-t^2\right)^{\alpha-1/2} (1+t)\,dt.$$

In particular, we have

$$\left| E_\alpha(ixy) \right| \leqq 1, \quad x, y \in \mathbb{R}. \tag{3}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}$, and was introduced by Dunkl in [5], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu [6]. The Dunkl transform of a function f in $L^1_\alpha(\mathbb{R})$, is

$$\mathcal{F}_\alpha(f)(y) := \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{\mathbb{R}} E_\alpha(-ixy) f(x) |x|^{2\alpha+1}\,dx, \quad y \in \mathbb{R}.$$

We notice that $\mathcal{F}_{-1/2}$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ixy} f(x)\,dx, \quad y \in \mathbb{R}.$$

The Dunkl transform of a function $f \in L^1_\alpha(\mathbb{R})$ could be computed via the associated Hankel transform $\mathcal{H}_\alpha$ that is

$$\mathcal{F}_\alpha(f)(y) = \mathcal{H}_\alpha(f_e)\big(|y|\big) - iy\mathcal{H}_{\alpha+1}\left(\frac{f_o}{\cdot}\right)\big(|y|\big), \quad y \in \mathbb{R},$$

where $f_e(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$, $f_o(x) = \frac{1}{2}\big(f(x) - f(-x)\big)$ and

$$\mathcal{H}_\alpha(f_e)\big(|y|\big) := \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^\infty f_e(r)j_\alpha\big(|y|r\big)\,r^{2\alpha+1}\,dr.$$

More details for the Hankel transform are collected in [3,16].

Some of the properties of Dunkl transform $\mathcal{F}_\alpha$ are collected below (see [5,6]).

(a) The Dunkl transform $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathcal{S}(\mathbb{R})$ onto itself, and from $\mathcal{S}'(\mathbb{R})$ onto itself.

(b) For all $f \in L^1_\alpha(\mathbb{R})$, we have $\mathcal{F}_\alpha(f) \in L^\infty_\alpha(\mathbb{R})$, and

$$\big\|\mathcal{F}_\alpha(f)\big\|_{L^\infty_\alpha} \leqq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\|f\|_{L^1_\alpha}.$$

(c) *Inversion theorem:* Let $f \in L^1_\alpha(\mathbb{R})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$. Then

$$f(x) = \mathcal{F}_\alpha\big(\mathcal{F}_\alpha(f)\big)(-x) \quad \text{a.e.} \quad x \in \mathbb{R}.$$

(d) *Plancherel theorem:* The Dunkl transform $\mathcal{F}_\alpha$ extends uniquely to an isometric isomorphism of $L^2_\alpha(\mathbb{R})$ onto itself. In particular, we have

$$\|f\|_{L^2_\alpha} = \big\|\mathcal{F}_\alpha(f)\big\|_{L^2_\alpha}, \quad f \in L^2_\alpha(\mathbb{R}). \tag{4}$$

In [9], Omri proved the following Pitt's inequality for the Hankel transform, that is, for $f \in \mathcal{S}_e(\mathbb{R})$ (the Schwartz space of even functions) and $0 \leqq \beta < 2\alpha + 2$,

$$\int_0^\infty y^{-\beta}\big|\mathcal{H}_\alpha(f)(y)\big|^2 y^{2\alpha+1}\,dy \leqq A_{\alpha,\beta}\int_0^\infty x^\beta\big|f(x)\big|^2 x^{2\alpha+1}\,dx, \tag{5}$$

where

$$A_{\alpha,\beta} = 2^{-\beta}\left[\Gamma\left(\frac{2\alpha+2-\beta}{4}\right)\Big/\Gamma\left(\frac{2\alpha+2+\beta}{4}\right)\right]^2. \tag{6}$$

In the following, we extend the Pitt's inequalities (1) and (5) to a more general case.

THEOREM 1 (Pitt's inequality). *Let* $0 \leqq \beta < 2\alpha + 2$ *and let* $f \in \mathcal{S}(\mathbb{R})$, *then*

$$\int_{\mathbb{R}} |y|^{-\beta} \big| \mathcal{F}_\alpha(f)(y) \big|^2 |y|^{2\alpha+1}\, dy \leqq M_{\alpha,\beta} \int_{\mathbb{R}} |x|^\beta \big| f(x) \big|^2 |x|^{2\alpha+1}\, dx,$$

with $M_{\alpha,\beta} = \sup(A_{\alpha,\beta}, A_{\alpha+1,\beta})$, *where* $A_{\alpha,\beta}$ *is the constant given by* (6).

PROOF. Let $0 \leqq \beta < 2\alpha + 2$ and let $f \in \mathcal{S}(\mathbb{R})$, then

$$\int_{\mathbb{R}} |y|^{-\beta} \big| \mathcal{F}_\alpha(f)(y) \big|^2 |y|^{2\alpha+1}\, dy = \int_{\mathbb{R}} |y|^{-\beta} \big| \mathcal{H}_\alpha(f_e)\big(|y|\big) \big|^2 |y|^{2\alpha+1}\, dy$$

$$+ \int_{\mathbb{R}} |y|^{-\beta} \left| \mathcal{H}_{\alpha+1}\left(\frac{f_o}{\cdot}\right)\big(|y|\big) \right|^2 |y|^{2\alpha+3}\, dy \leqq 2 \int_0^\infty y^{-\beta} \big| \mathcal{H}_\alpha(f_e)(y) \big|^2 y^{2\alpha+1}\, dy$$

$$+ 2 \int_0^\infty y^{-\beta} \left| \mathcal{H}_{\alpha+1}\left(\frac{f_o}{\cdot}\right)(y) \right|^2 y^{2\alpha+3}\, dy.$$

Then by (5) we obtain

$$\int_{\mathbb{R}} |y|^{-\beta} \big| \mathcal{F}_\alpha(f)(y) \big|^2 |y|^{2\alpha+1}\, dy \leqq 2A_{\alpha,\beta} \int_0^\infty x^\beta \big| f_e(x) \big|^2 x^{2\alpha+1}\, dx$$

$$+ 2A_{\alpha+1,\beta} \int_0^\infty x^\beta \left| \frac{f_o(x)}{x} \right|^2 x^{2\alpha+3}\, dx$$

$$\leqq 2 \sup(A_{\alpha,\beta}, A_{\alpha+1,\beta}) \int_0^\infty x^\beta \big[\big| f_e(x) \big|^2 + \big| f_o(x) \big|^2 \big] x^{2\alpha+1}\, dx.$$

Since

$$\int_0^\infty x^\beta \big[\big| f_e(x) \big|^2 + \big| f_o(x) \big|^2 \big] x^{2\alpha+1}\, dx = \frac{1}{2} \int_{\mathbb{R}} |x|^\beta \big| f(x) \big|^2 |x|^{2\alpha+1}\, dx,$$

we obtain

$$\int_{\mathbb{R}} |y|^{-\beta} \big| \mathcal{F}_\alpha(f)(y) \big|^2 |y|^{2\alpha+1}\, dy \leqq \sup(A_{\alpha,\beta}, A_{\alpha+1,\beta}) \int_{\mathbb{R}} |x|^\beta \big| f(x) \big|^2 |x|^{2\alpha+1}\, dx,$$

which completes the proof. □

In [9], Omri proved the following logarithmic uncertainty principle for the Hankel transform, that is, for $f \in \mathcal{S}_e(\mathbb{R})$,

$$(7) \qquad \int_0^\infty \ln(x) \big| f(x) \big|^2 x^{2\alpha+1}\, dx + \int_0^\infty \ln(y) \big| \mathcal{H}_\alpha(f)(y) \big|^2 y^{2\alpha+1}\, dy$$

$$\geqq \left[\psi\left(\frac{\alpha+1}{2}\right)+\ln 2\right]\int_0^\infty |f(x)|^2 x^{2\alpha+1}\,dx.$$

In the next, we extend the logarithmic uncertainty principles (2) and (7) to a more general case.

THEOREM 2 (logarithmic uncertainty inequality). *For every* $f \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \ln|x|\big|f(x)\big|^2|x|^{2\alpha+1}\,dx + \int_{\mathbb{R}} \ln|y|\big|\mathcal{F}_\alpha(f)(y)\big|^2|y|^{2\alpha+1}\,dy \geqq D_\alpha \|f\|^2_{L^2_\alpha},$$

$$D_\alpha = \frac{1}{2}\left[\psi\left(\frac{\alpha+1}{2}\right)+\psi\left(\frac{\alpha+2}{2}\right)+2\ln 2\right]$$

$$-\frac{1}{2}\left|\psi\left(\frac{\alpha+1}{2}\right)-\psi\left(\frac{\alpha+2}{2}\right)\right|.$$

PROOF. Let $\varphi(x,\beta) = |x|^{\beta+2\alpha+1}\big|f(x)\big|^2$, then for every

$$\beta \in \,]-\alpha-1, \alpha+1[$$

and $0<|x|<1$, we have

$$\left|\frac{\partial}{\partial\beta}\varphi(x,\beta)\right| \leqq |x|^\alpha\big|\ln|x|\big|\,\big|f(x)\big|^2.$$

However, for every real number σ such that $0<\sigma<\alpha+1$, the function $x \to |x|^\sigma\big|\ln|x|\big|$ is bounded for $0<|x|<1$ so that

$$\int_{-1}^{1} |x|^\alpha\big|\ln|x|\big|\,\big|f(x)\big|^2\,dx < \infty.$$

In the same way, for every $\beta \in\,]-\alpha-1,\alpha+1[$ and $|x|>1$, we have

$$\left|\frac{\partial}{\partial\beta}\varphi(x,\beta)\right| \leqq |x|^{3\alpha+2}\big|\ln|x|\big|\,\big|f(x)\big|^2,$$

and since $f\in\mathcal{S}(\mathbb{R})$, then the function $x \to |x|^{3\alpha+2}\big|\ln|x|\big|\,\big|f(x)\big|^2$ is also integrable over $|x| \geqq 1$. This justifies the differentiation under the integral sign and shows that for every $\beta \in\,]-\alpha-1,\alpha+1[$

$$\frac{\partial}{\partial\beta}\left(\int_{\mathbb{R}} |x|^\beta\big|f(x)\big|^2|x|^{2\alpha+1}\,dx\right) = \int_{\mathbb{R}} |x|^\beta \ln|x|\big|f(x)\big|^2|x|^{2\alpha+1}\,dx.$$

By the same way, one can easily see that for every $\beta \in \,]-\alpha-1, \alpha+1[$

$$\frac{\partial}{\partial\beta}\Big(\int_{\mathbb{R}} |y|^{-\beta}\big|\mathcal{F}_\alpha(f)(y)\big|^2 |y|^{2\alpha+1}\,dy\Big)$$

$$= -\int_{\mathbb{R}} |y|^{-\beta}\ln|y|\big|\mathcal{F}_\alpha(f)(y)\big|^2 |y|^{2\alpha+1}\,dy.$$

Now let ϕ be the function defined on $]-\alpha-1, \alpha+1[$ by

$$\phi(\beta) = \int_{\mathbb{R}} |y|^{-\beta}\big|\mathcal{F}_\alpha(f)(y)\big|^2 |y|^{2\alpha+1}\,dy - M_{\alpha,\beta}\int_{\mathbb{R}} |x|^{\beta}\big|f(x)\big|^2 |x|^{2\alpha+1}\,dx.$$

Then ϕ is well defined on $]-\alpha-1, \alpha+1[$, however Pitt's inequality implies that $\phi(\beta) \leqq 0$ for every $0 \leqq \beta < \alpha+1$, and according to Plancherel's theorem, $\phi(0) = 0$. Since ϕ is differentiable at 0^+, then it follows that $\phi'(0^+) \leqq 0$, and by a basic calculus, we get

$$\int_{\mathbb{R}} \ln|x|\big|f(x)\big|^2 |x|^{2\alpha+1}\,dx + \int_{\mathbb{R}} \ln|y|\big|\mathcal{F}_\alpha(f)(y)\big|^2 |y|^{2\alpha+1}\,dy$$

$$\geqq -\frac{\partial M_{\alpha,\beta}}{\partial\beta}\Big|_{\beta=0^+} \|f\|^2_{L^2_\alpha}.$$

But

$$\frac{\partial M_{\alpha,\beta}}{\partial\beta}\Big|_{\beta=0^+} = \frac{1}{2}\lim_{\beta\to0^+}\frac{A_{\alpha,\beta}+A_{\alpha+1,\beta}-2}{\beta} + \frac{1}{2}\lim_{\beta\to0^+}\Big|\frac{A_{\alpha,\beta}-A_{\alpha+1,\beta}}{\beta}\Big|$$

$$= \frac{1}{2}\Big(\frac{\partial A_{\alpha,\beta}}{\partial\beta} + \frac{\partial A_{\alpha+1,\beta}}{\partial\beta}\Big)_{\beta=0^+} + \frac{1}{2}\Big|\frac{\partial A_{\alpha,\beta}}{\partial\beta} - \frac{\partial A_{\alpha+1,\beta}}{\partial\beta}\Big|_{\beta=0^+}$$

$$= -\frac{1}{2}\Big[\psi\Big(\frac{\alpha+1}{2}\Big) + \psi\Big(\frac{\alpha+2}{2}\Big) + 2\ln 2\Big] + \frac{1}{2}\Big|\psi\Big(\frac{\alpha+1}{2}\Big) - \psi\Big(\frac{\alpha+2}{2}\Big)\Big|,$$

which completes the proof. □

3. Stein–Weiss inequality for the Dunkl–Bessel potentials

The Dunkl transform allows us to define a generalized translation operators on $L^2_\alpha(\mathbb{R})$ by setting

$$\mathcal{F}_\alpha(\tau_x f)(y) = E_\alpha(ixy)\mathcal{F}_\alpha(f)(y), \quad y \in \mathbb{R}^d.$$

This is the definition of Thangavelu and Xu given in [17].

Note that from (3) and (4), the definition makes sense, and we have

$$\|\tau_x f\|_{L^2_\alpha} \leqq \|f\|_{L^2_\alpha}, \quad f \in L^2_\alpha(\mathbb{R}).$$

Rösler [10] introduced the Dunkl translation operators for f in $L^1_\alpha(\mathbb{R})$, by

$$\tau_x f(y) = \begin{cases} \displaystyle\int_0^\pi \Big[f_e\big((x,y)_\theta\big) + f_o\big((x,y)_\theta\big) \frac{x+y}{(x,y)_\theta} \Big] \, d\nu_{x,y}(\theta), & (x,y) \neq (0,0) \\ f(y), & x = 0, \end{cases}$$

where $(x,y)_\theta = \sqrt{x^2 + y^2 - 2|xy| \cos\theta}$ and

$$d\nu_{x,y}(\theta) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\big(\alpha + \frac{1}{2}\big)} \big[1 - \operatorname{sgn}(xy)\cos\theta\big] \sin^{2\alpha}\theta \, d\theta.$$

More details for the Dunkl translation operators are collected in [13,14].

Let β be a real number such that $0 < \beta < 2\alpha + 2$. The Dunkl-type Riesz potentials $I_{\alpha,\beta} f$ are defined by (see [18]):

$$I_{\alpha,\beta} f(x) := (d_{\alpha,\beta})^{-1} \int_{\mathbb{R}} \tau_x\big(|.|^{\beta-2\alpha-2}\big)(-y) f(y) |y|^{2\alpha+1} \, dy, \quad f \in \mathcal{S}(\mathbb{R}), \quad x \in \mathbb{R},$$

where

$$d_{\alpha,\beta} := 2^{-\alpha-1+\beta} \frac{\Gamma(\beta/2)}{\Gamma\big(\alpha + 1 - \frac{\beta}{2}\big)}.$$

Thangavelu and Xu established the following relation between the Dunkl-type Riesz potentials $I_{\alpha,\beta}$ and the Dunkl transform $\mathcal{F}_\alpha$.

THEOREM 3 ([18], Proposition 4.1). *Let* $0 < \beta < 2\alpha + 2$. *The identity*

$$\mathcal{F}_\alpha(I_{\alpha,\beta} f)(y) = |y|^{-\beta} \mathcal{F}_\alpha(f)(y) \tag{8}$$

holds in the sense that

$$\int_{\mathbb{R}} I_{\alpha,\beta} f(x) g(x) |x|^{2\alpha+1} \, dx = \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(y) |y|^{-\beta} \mathcal{F}_\alpha(g)(y) |y|^{2\alpha+1} \, dy,$$

whenever $f, g \in \mathcal{S}(\mathbb{R})$.

COROLLARY 1. *Let* $\beta, \gamma > 0$ *such that* $\beta + \gamma < 2\alpha + 2$. *Then for every* $f \in \mathcal{S}(\mathbb{R})$, *the Dunkl-type Riesz potential* $I_{\alpha,\beta}$ *satisfies the semigroup property*

$$I_{\alpha,\beta}(I_{\alpha,\gamma} f) = I_{\alpha,\beta+\gamma}(f). \tag{9}$$

Theorem 4 ([7], Theorem 1.1). *Let* $0 < \beta < 2\alpha + 2$ *and let* $1 < p < \frac{2\alpha+2}{\beta}$. *Then the Dunkl-type Riesz potential* $I_{\alpha,\beta}$ *is bounded from* $L^p_\alpha(\mathbb{R})$ *into* $L^q_\alpha(\mathbb{R})$ *if and only if*

$$\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha + 2}.$$

Corollary 2. *Let* $f, g \in \mathcal{S}(\mathbb{R})$ *and let* $0 < \beta < 2\alpha + 2$. *The Dunkl-type Riesz potential* $I_{\alpha,\beta}$ *satisfies the duality property*

$$\int_{\mathbb{R}} I_{\alpha,\beta} f(x) \overline{g(x)} |x|^{2\alpha+1}\, dx = \int_{\mathbb{R}} f(x) \overline{I_{\alpha,\beta} g(x)} |x|^{2\alpha+1}\, dx. \tag{10}$$

In particular, for every $f \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x) \overline{I_{\alpha,\beta} f(x)} |x|^{2\alpha+1}\, dx = \int_{\mathbb{R}} \big| I_{\alpha,\beta/2} f(x) \big|^2 |x|^{2\alpha+1}\, dx. \tag{11}$$

Proof. Since $f \in \mathcal{S}(\mathbb{R})$ then f belongs to $L^2_\alpha(\mathbb{R}) \cap L_\alpha^{\frac{2\alpha+2}{\alpha+1+\beta}}(\mathbb{R})$, so that according to Theorem 4, $I_{\alpha,\beta} f$ belongs to $L^2_\alpha(\mathbb{R})$. Therefore by Plancherel formula (4), we deduce that

$$\int_{\mathbb{R}} I_{\alpha,\beta} f(x) \overline{g(x)} |x|^{2\alpha+1}\, dx = \int_{\mathbb{R}} |y|^{-\beta} \mathcal{F}_\alpha(f)(y) \overline{\mathcal{F}_\alpha(g)(y)} |y|^{2\alpha+1}\, dy$$

$$= \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(y) \overline{\mathcal{F}_\alpha(I_{\alpha,\beta} g)(y)} |y|^{2\alpha+1}\, dy = \int_{\mathbb{R}} f(x) \overline{I_{\alpha,\beta} g(x)} |x|^{2\alpha+1}\, dx.$$

The relation (11) is an immediate consequence of the semigroup and duality properties given by (9) and (10). □

In the Euclidean case [1] and in the Hankel setting [9], Pitt's inequality is derived from Stein–Weiss inequality. In the following we show the opposite.

Theorem 5 (Stein–Weiss inequality). *Let* $0 < \beta < 2\alpha + 2$. *Then for every* $f \in \mathcal{S}(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \tau_x \big(|.|^{\beta - 2\alpha - 2} \big) (-y) \overline{f(y)} \big(|x|\, |y| \big)^{-\beta/2 + 2\alpha + 1}\, dx\, dy \leqq B_{\alpha,\beta} \|f\|^2_{L^2_\alpha},$$

where

$$B_{\alpha,\beta} = \frac{\Gamma(\beta/2) M_{\alpha,\beta}}{2^{\alpha - \beta + 1} \Gamma\big(\alpha + 1 - \frac{\beta}{2}\big)}.$$

PROOF. According to (8) and Plancherel formula (4), we have

$$\int_{\mathbb{R}} |y|^{-\beta}\big|\mathcal{F}_\alpha(f)(y)\big|^2|y|^{2\alpha+1}\,dy = \int_{\mathbb{R}} \big|I_{\alpha,\beta/2}f(x)\big|^2|x|^{2\alpha+1}\,dx.$$

By using relation (11) we deduce that

$$\int_{\mathbb{R}} |y|^{-\beta}\big|\mathcal{F}_\alpha(f)(y)\big|^2|y|^{2\alpha+1}\,dy = \int_{\mathbb{R}} f(x)\overline{I_{\alpha,\beta}f(x)}|x|^{2\alpha+1}\,dx$$

$$= (d_{\alpha,\beta})^{-1}\int_{\mathbb{R}} f(x)\Big[\int_{\mathbb{R}} \tau_x\big(|.|^{\beta-2\alpha-2}\big)(-y)\overline{f(y)}|y|^{2\alpha+1}\,dy\Big]|x|^{2\alpha+1}\,dx.$$

However, since $f \in \mathcal{S}(\mathbb{R})$ then f and $I_{\alpha,\beta}f$ belong to $L^2_\alpha(\mathbb{R})$. Therefore by Hölder's inequality, we deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \big|f(x)\tau_x\big(|.|^{\beta-2\alpha-2}\big)(-y)\overline{f(y)}\big|\big(|x|\,|y|\big)^{2\alpha+1}\,dy\,dx$$

$$\leqq d_{\alpha,\beta}\|f\|_{L^2_\alpha}\big\|I_{\alpha,\beta}|f|\big\|_{L^2_\alpha} < \infty.$$

Hence, by Fubini's theorem and Theorem 1, we deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\tau_x\big(|.|^{\beta-2\gamma-2}\big)(-y)\overline{f(y)}\big(|x|\,|y|\big)^{2\alpha+1}\,dx\,dy$$

$$\leqq \frac{\Gamma(\beta/2)M_{\alpha,\beta}}{2^{\alpha-\beta+1}\Gamma(\alpha+1-\frac{\beta}{2})}\int_{\mathbb{R}} |x|^\beta\big|f(x)\big|^2|x|^{2\alpha+1}\,dx.$$

Replacing f by $|x|^{-\beta/2}f$ in the preceding inequality we obtain the desired result. □

References

[1] W. Beckner, Pitt's inequality and the uncertainty principle, *Proc. Amer. Math. Soc.,* **123** (1995), 1897–1905.

[2] W. Beckner, Pitt's inequality with sharp convolution estimates, *Proc. Amer. Math. Soc.,* **136** (2008), 1871–1885.

[3] J. J. Betancor and L. Rodriguez-Mesa, On Hankel transformation, convolution operators and multipliers on Hardy type spaces, *J. Math. Soc. Japan,* **53** (2001), 687–709.

[4] C. F. Dunkl, Integral kernels with reflection group invariance, *Canad. J. Math.,* **43** (1991), 1213–1227.

[5] C. F. Dunkl, Hankel transforms associated to finite reflection groups, *Contemp. Math.,* **138** (1992), 123–138.

[6] M. F. E. de Jeu, The Dunkl transform, *Invent. Math.,* **113** (1993), 147–162.

[7] S. Hassani, S. Mustapha and M. Sifi, Riesz potentials and fractional maximal function for the Dunkl transform, *J. Lie Theory,* **19** (2009), 725–734.

[8] T. Kawazoe and H. Mejjaoli, Uncertainty principles for the Dunkl transform, *Hiroshima Math. J.,* **40** (2010), 241–268.

[9] S. Omri, Logarithmic uncertainty principle for the Hankel tansform, *Int. Trans. Spec. Funct.,* **22** (2011), 655–670.

[10] M. Rösler, Bessel-type signed hypergroups on $\mathbb{R}$, in: *Probability Measures on Groups and Related Structures,* XI (Oberwolfach, 1994), Editors H. Heyer and A. Mukherjea (Oberwolfach, 1994), World Sci. Publ., River Edge, NJ (1995), pp. 292–304.

[11] M. Rösler, An uncertainty principle for the Dunkl transform, *Bull. Austral. Math. Soc.,* **59** (1999), 353–360.

[12] N. Shimeno, A note on the uncertainty principle for the Dunkl transform, *J. Math. Sci. Univ. Tokyo,* **8** (2001), 33–42.

[13] F. Soltani, L^p-Fourier multipliers for the Dunkl operator on the real line, *J. Funct. Anal.,* **209** (2004), 16–35.

[14] F. Soltani, Sonine transform associated to the Dunkl kernel on the real line, *SIGMA,* **4** (2008), 1–14.

[15] F. Soltani, A general form of Heisenberg–Pauli–Weyl uncertainty inequality for the Dunkl transform, *Int. Trans. Spec. Funct.,* **24** (2013), 401–409.

[16] K. Stempak, La théorie de Littlewood–Paley pour la transformation de Fourier–Bessel, *C.R. Acad. Sci. Paris,* 303 Serie I (1) (1986), 15–18.

[17] S. Thangavelu and Y. Xu, Convolution operator and maximal function for the Dunkl transform, *J. Anal. Math.,* **97** (2005), 25–56.

[18] S. Thangavelu and Y. Xu, Riesz transform and Riesz potentials for Dunkl transform, *J. Comput. Appl. Math.,* **199** (2007), 181–195.

[19] G. N. Watson, *A Treatise on Theory of Bessel Functions,* Cambridge University Press (Cambridge, 1966).