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A GENERALIZATION OF THE UNIT AND UNITARY CAYLEY GRAPHS OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with non-zero identity and G be a multiplicative subgroup of $U(R)$, where $U(R)$ is the multiplicative group of unit elements of R. Also, suppose that S is a non-empty subset of G such that $S^{-1} = \{s^{-1} \mid s \in S\} \subseteqq S$. Then we define $\Gamma(R,G,S)$ to be the graph with vertex set R and two distinct elements $x, y \in R$ are adjacent if and only if there exists $s \in S$ such that $x + sy \in G$. This graph provides a generalization of the unit and unitary Cayley graphs. In fact, $\Gamma(R,U(R),S)$ is the unit graph or the unitary Cayley graph, whenever $S = \{1\}$ or $S = \{-1\}$, respectively. In this paper, we study the properties of the graph $\Gamma(R,G,S)$ and extend some results in the unit and unitary Cayley graphs.

1. Introduction

The Cayley graph introduced by Arthur Cayley in 1878 is a useful tool for connection between group theory and the theory of algebraic graphs. Let A be an abelian group and C be a subset of A. Then the *Cayley sum graph* $\mathrm{Cay}^+(A,C)$ is the graph with vertex set A and edge set $\{\{a,b\} \mid a+b \in C\}$. Furthermore, whenever $0 \notin C$ and $-C = \{-c \mid c \in C\} \subseteqq C$, then the *Cayley graph* $\mathrm{Cay}(A,C)$ is the graph with vertex set A and edge set $\{\{a,b\} \mid a-b \in C\}$. We refer the reader to [8] for general properties of Cayley graphs. Unlike Cayley graphs, there are only a few appearances of Cayley sum graphs in the literature (see [9] and references therein). It seems that Cayley sum graphs are rather difficult to study (cf. [5] and [9]).

In recent years, for a ring R, Cayley (sum) graphs of the abelian group $(R,+)$ with respect to subsets of R have received much attention in the literature. Suppose that $Z(R)$ and $U(R)$ are the sets of zero-divisors and units

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of R, respectively. In [2], the authors defined the *total graph* of a commutative ring that is the graph $\mathrm{Cay}^+(R, Z(R))$ without loops. Also, in [12], the authors studied the chromatic number of $\mathrm{Cay}(R, Z(R)\backslash\{0\})$. Moreover, in [3], the authors defined the *unit graph* of a commutative ring that is $\mathrm{Cay}^+(R, U(R))$ without loops. In [1] and [10], the authors obtained some basic properties of $\mathrm{Cay}(R, U(R))$, which is usually called the *unitary Cayley graph*. When we compare the unit and unitary Cayley graphs, it seems that they have the same behavior, however in general, they are not isomorphic. This provides a motivation to introduce a generalization of these graphs.

Let R be a commutative ring with non-zero identity, G be a multiplicative subgroup of $U(R)$ and S be a non-empty subset of G such that $S^{-1} = \{ s^{-1} \mid s \in S\} \subseteq S$. Then $\Gamma(R, G, S)$ is the (simple) graph with vertex set R and two distinct elements $x, y \in R$ are adjacent if and only if there exists $s \in S$ such that $x + sy \in G$. If we omit the word "distinct", we obtain the graph $\overline{\Gamma}(R, G, S)$; this graph may have loops. The graph $\Gamma(R, G, S)$ provides an 'umbrella concept' which covers the unit graphs and the unitary Cayley graphs; indeed, if $G = U(R)$ and $S = \{1\}$, then $\Gamma(R, G, S)$ is the unit graph, and if $G = U(R)$ and $S = \{-1\}$, then $\Gamma(R, G, S)$ is the unitary Cayley graph. Note that the graph $\Gamma(R, G, S)$ is a subgraph of the *co-maximal graph* (cf. [13]). In this paper we study properties of the graph $\Gamma(R, G, S)$ and generalize some corresponding results in [1] and [3].

2. Some basic general properties

Throughout this article, all rings are assumed to be commutative with non-zero identity. We denote the ring of integers modulo n by $\mathbb{Z}_n$ and the field with q elements by $\mathbb{F}_q$. Also, for a ring R, the Jacobson radical of R is denoted by $J(R)$. We refer the reader to [4] for general references on ring theory.

For a graph Γ, $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and edge set of Γ respectively. The set of vertices adjacent to a vertex v in the graph Γ is denoted by $N_\Gamma(v)$, or briefly by $N(v)$. The *degree* $\deg(v)$ of a vertex v in the graph Γ is the number of edges of Γ incident with v. The graph Γ is called *k-regular* if all vertices of Γ have degree k, where k is a fixed positive integer. A *complete* graph Γ is a simple graph such that all vertices of Γ are adjacent. In addition, K_n denotes a complete graph with n vertices. A graph Γ is called *bipartite* if $V(\Gamma)$ admits a partition into two classes such that vertices in the same partition class must not be adjacent. A simple bipartite graph in which every two vertices from different partition classes are adjacent is called a *complete bipartite* graph, denoted by $K_{m,n}$, where m and n are of size of the partition classes. Graphs of the form $K_{1,n}$ are called *stars*. A *refinement* of a star graph with center c is a simple graph such that all vertices are adjacent to c. A *clique* of a graph Γ is a complete subgraph of Γ. A *coclique* in a graph Γ is a set of pairwise nonadjacent vertices.

A *walk* (of length k) in a graph Γ between two vertices x, y is an alternating sequence $x = v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k = y$ of vertices and edges in G, denoted by

$$v_0 - v_1 - \cdots - v_k,$$

such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If the vertices in a walk are all distinct, it defines a *path* in Γ, denoted by P_k. If $P = v_0 - v_1 - \cdots - v_{k-1}$ is a path, then the graph $C := P + v_{k-1}v_0$ is called a cycle of length k and is denoted by C_k. The minimum length of a cycle (contained) in a graph Γ is the *girth* $\operatorname{gr}(\Gamma)$ of Γ; if Γ does not contain a cycle, we set $\operatorname{gr}(\Gamma) := \infty$. A graph Γ is called *connected* if any two of its vertices are linked by a walk in Γ. The *distance* $d(x, y)$ in Γ of two vertices x, y is the length of a shortest path between x, y in G; if no such paths exists, we set $d(x, y) := \infty$. The greatest distance between any two vertices in Γ is the *diameter* of Γ, denoted by $\operatorname{diam}(\Gamma)$. Suppose that Γ_1 and Γ_2 are two simple graphs with disjoint vertex sets. Then the *union* $\Gamma = \Gamma_1 \cup \Gamma_2$ is a graph with vertex set $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$. Also, if a graph Γ consists of k (with $k \geqq 2$) disjoint copies of a graph H, then we set $\Gamma := kH$. We refer the reader to [7] and [14] for general references on graph theory.

We begin with the following proposition.

PROPOSITION 2.1. *The following statements hold.*
(a) *If* $G = \{1\}$, *then*

$$\Gamma(R, G, S) \cong \begin{cases} \dfrac{|R|}{2} K_2 & \text{if } 2 \notin U(R), \\ \dfrac{|R| - 1}{2} K_2 \cup K_1 & \text{otherwise.} \end{cases}$$

(b) *If* $G = \{1, a\}$, *for some* $a \in R$ *with* $a \neq 1$, *and* $S = \{s\}$, *then*

(i) *whenever* $1 + s \notin U(R)$, *the graph* $\Gamma(R, G, S)$ *is* 2*-regular*; *and*,

(ii) *whenever* $1 + s \in U(R)$, *all vertices in* $\Gamma(R, G, S)$, *except two vertices, have degree* 2. *In particular*, *if* R *is finite*, *then* $\Gamma(R, G, S) \cong P_n \cup H$, *where* P_n *is a path of length* $n \geqq 2$ *and* H *is an empty graph or a* 2*-regular graph.*

PROOF. (a) It follows from the fact that, for every vertex x of $\Gamma\big(R, \{1\}, \{1\}\big)$, $N(x) = \{1 - x \mid 1 - x \neq x\}$.

(b) The first claim follows from the fact that, for every vertex x of $\Gamma\big(R, \{1, a\}, \{s\}\big)$, $N(x) = \{a - sx, 1 - sx\}$. For the second one, note that there exists a unique element z in R such that $(1+s)z=1$. Hence $N(z) = \{a-sz\}$ and $N(az)=\{1-saz\}$. Also, for a vertex x with $x \neq z$ and $x \neq az$, $N(x)=\{a-sx, 1-sx\}$. The result now immediately follows from the fact that the two elements $a-sz$ and $1-saz$ do not belong to $\{z, az\}$. $\square$

The following example shows that in the last part of Proposition 2.1, the graph H may be non-empty.

EXAMPLE 2.2. Let $R=\mathbb{Z}_{15}$. Put $G:=\{1,4\}$ and $S:=\{1\}$. Then it is not hard to see that $\Gamma(R,G,S) \cong P_4 \cup C_{10}$.

In view of the Proposition 2.1(a), we obtain the following result.

COROLLARY 2.3. *Let R be a finite ring. Then $2 \in U(R)$ if and only if $|R|$ is odd.*

Let Γ_1 and Γ_2 be two graphs without multiple edges. Recall that the *tensor product* $\Gamma = \Gamma_1 \otimes \Gamma_2$ is a graph with vertex set $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$ and two distinct vertices (u_1, u_2) and (v_1, v_2) of Γ are adjacent if and only if $\{u_1, v_1\} \in E(\Gamma_1)$ and $\{u_2, v_2\} \in E(\Gamma_2)$.

REMARKS 2.4. (a) For any vertex x of $\Gamma(R,G,S)$, we have the inequalities

$$(*) \qquad |G| - 1 \leqq \deg(x) \leqq |G|\,|S|.$$

Furthermore, for any vertex x of $\overline{\Gamma}(R,G,S)$, $\deg(x) \geqq |G|$.

(b) In the light of Proposition 2.1(a), the above inequalities imply that the graph $\Gamma(R,G,S)$ has an isolated vertex if and only if $G=\{1\}$ and $2 \in U(R)$.

(c) Suppose that R_1 and R_2 are rings and, for each i with $i=1,2$, G_i is a subgroup of $U(R_i)$. Also, assume that S_i is a non-empty subset of G_i with $S_i^{-1} \subseteqq S_i$.

(i) Then $\Gamma(R_1 \times R_2, G_1 \times G_2, S_1 \times S_2) \cong \overline{\Gamma}(R_1, G_1, S_1) \otimes \overline{\Gamma}(R_2, G_2, S_2)$.

(ii) Furthermore, whenever $R_1 = R_2$, $G_1 \subseteqq G_2$ and $S_1 \subseteqq S_2$, then $\Gamma(R_1, G_1, S_1)$ is a subgraph of $\Gamma(R_2, G_2, S_2)$.

The next example shows that the bound obtained in $(*)$ is sharp.

EXAMPLE 2.5. Suppose that $R=\mathbb{Z}_8$ and $G=S=\{1,3\}$. Then $\deg(2) = \deg(6) = 4$ (see Fig. 1).

THEOREM 2.6. *Let R be a finite ring. Then the following statements hold.*

(a) *If $(R, \mathfrak{m})$ is a local ring of even order, then*

$$\Gamma(R, U(R), \{1\}) \cong \Gamma(R, U(R), \{-1\}).$$

(b) *If R is a ring of odd order, then* $\Gamma(R, U(R), \{1\}) \ncong \Gamma(R, U(R), \{-1\})$.

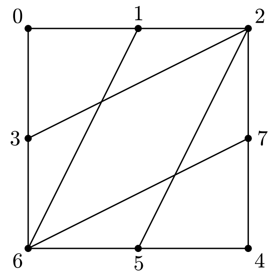

Fig. 1: $\Gamma(\mathbb{Z}_8, \{1,3\}, \{1,3\})$

PROOF. (a) Clearly, for every two elements x and y of R, $(x+y) + (x-y) = 2x$. Also, in view of Corollary 2.3, we have that $2 \in \mathfrak{m}$. Hence $x+y \in \mathfrak{m}$ if and only if $x-y \in \mathfrak{m}$. This implies that $x+y \in U(R)$ if and only if $x-y \in U(R)$, which completes the proof.

(b) By [1, Proposition 2.2], the unitary Cayley graph $\Gamma\big(R, U(R), \{-1\}\big)$ is regular. On the other hand, since R has odd order, by Corollary 2.3 and [3, Proposition 2.4(b)], the unit graph $\Gamma\big(R, U(R), \{1\}\big)$ is not regular. □

THEOREM 2.7. *The graph* $\Gamma(R,G,S)$ *is a complete graph if and only if the following statements hold.*

(a) R *is a field*;

(b) $G = U(R)$; *and*,

(c) $|S| \geqq 2$ *or* $S = \{-1\}$.

PROOF. First, assume that $\Gamma(R,G,S)$ is complete. Since every non-zero element x of R is adjacent to 0, there exists $s \in S$ such that $x + s0 = x \in G$, and so $G = R - \{0\}$. This means that R is a field with $G = U(R)$. Now, assume to the contrary that $S = \{s\}$ with $s \neq -1$. Then, for every element x of R with $x \neq 1$, x is adjacent to 1, and hence $1 + sx$ is unit. On the other hand, since s is unit, $s \notin J(R)$. Thus $1+s$ is not unit, and so $s = -1$ which is the required contradiction.

Conversely, assume that there exist two distinct non-adjacent elements x and y in $\Gamma(R,G,S)$. Without loss of generality, we may assume that $y \neq 0$. Thus, for every $s \in S$, $x + sy = 0$. Hence $s = -y^{-1}x$, and so $|S| = 1$. Therefore $S = \{-1\}$, and hence $x = -sy = y$, which is impossible. □

PROPOSITION 2.8. *The graph* $\Gamma(R,G,S)$ *is a refinement of a star graph with center* 0 *if and only if* R *is a field with* $G = U(R)$.

PROOF. Let $\Gamma(R,G,S)$ be a refinement of a star graph with center 0. Then, every non-zero element x of R is adjacent to 0, and so $x \in G$. Hence $G = R - \{0\}$. The converse is obvious. □

Corollary 2.9. *$\Gamma(R,G,S)$ is a star graph with center 0 if and only if $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$ with $S = \{1\}$.*

Proof. Let $\Gamma(R,G,S)$ be a star graph with center 0. Then, by Proposition 2.8, R is a field with $G = U(R)$. If $|S| \geqq 2$ or $S = \{-1\}$, then, by Theorem 2.7, $\Gamma(R,G,S)$ is a complete graph. This implies that $R \cong \mathbb{Z}_2$. Now, if $S = \{s\}$ with $s \neq -1$, then, for every non-zero element x of R, $N(x) = R - \{x, -sx\}$, and hence $\deg(x) = |R| - 2$. Thus $R \cong \mathbb{Z}_3$ with $S = \{1\}$. The converse is obvious. □

Proposition 2.10. *Let R be a finite ring and $\Gamma(R,G,S)$ be a refinement of a star graph with non-zero center. If S is singleton, then $\Gamma(R,G,S)$ is complete.*

Proof. Put $S := \{s\}$ and consider the non-zero center c of $\Gamma(R,G,S)$. Then $c \in G$ and, for every element x of R with $x \neq 1$, xc is adjacent to c. Hence $c + csx \in G$, and so $1 + sx \in G$. This means that x is adjacent to 1, and thus 1 is a center of $\Gamma(R,G,S)$. Now, the map $f : R\backslash\{1\} \longrightarrow G$ given by $f(x) = x + s$, for all $x \in R\backslash\{1\}$, is injective. Hence, $|R| - 1 \leqq |G|$ which implies that $G = R - \{0\}$. Since $s \notin J(R)$ and, for every x of R with $x \neq 1$, $1 + sx$ is unit, we have that $1 + s$ is not unit, and so $s = -1$. Hence, by Theorem 2.7, $\Gamma(R,G,S)$ is complete. □

Theorem 2.11. *Let R be ring and $-1 \in G$. Then $\Gamma(R,G,S)$ is connected if and only if every element of R can be written as a sum of elements of G.*

Proof. ($\Rightarrow$) For every non-zero element x of R there exists a walk

$$0 = x_0 - x_1 - x_2 - \cdots - x_n = x$$

between 0 and x. Hence, for every i with $0 \leqq i \leqq n-1$, there exist $s_i \in S$ and $g_i \in G$ such that $x_{i+1} = g_i - s_i x_i$. Now an elementary inductive argument on i yields that for all i with $0 \leqq i \leqq n-1$, x_{i+1} can be written as a sum of elements of G.

($\Leftarrow$) Let $0 \neq x \in R$ and $s \in S$. Then, for each $1 \leqq i \leqq n$, there exists element g_i in G such that $sx = g_1 + g_2 + g_3 + \cdots + g_n$. So, we have the following walk between x and 0:

$$x - (-sx + g_1) - (x - s^{-1}g_1 - s^{-1}g_2) - (-sx + g_1 + g_2 + g_3)$$
$$- (x - s^{-1}g_1 - s^{-1}g_2 - s^{-1}g_3 - s^{-1}g_4) - \cdots - 0.$$

Hence $\Gamma(R,G,S)$ is connected. □

3. The case $G = U(R)$

In this section we study the properties of $\Gamma(R,G,S)$ in the case that $G = U(R)$. For simplicity of notation, we denote $\Gamma\big(R, U(R), S\big)$ by $\Gamma(R,S)$. Also, if we have no restriction on S, we denote $\Gamma(R,S)$ by $\Gamma(R)$.

REMARK 3.1. Suppose that $\{x_i + J(R)\}_{i\in I}$ is a complete set of coset representation of $J(R)$. Note that if $x \in U(R)$ and $j \in J(R)$, then $x + j \in U(R)$. Hence, whenever x_i and x_j are adjacent vertices in $\Gamma(R,S)$, then every element of $x_i + J(R)$ is adjacent to every element of $x_j + J(R)$. Moreover, if there exists $s \in S$ with $(1+s)x_i \in U(R)$ for some i, then $x_i + J(R)$ is a clique in $\Gamma(R,S)$. Otherwise $x_i + J(R)$ is a coclique in $\Gamma(R,S)$. For instance if $x_i \notin U(R)$, then $x_i + J(R)$ is a coclique in $\Gamma(R,S)$.

PROPOSITION 3.2. *Let $\mathfrak{m}$ be a maximal ideal of R such that $|R/\mathfrak{m}| = 2$. Then the graph $\Gamma(R,S)$ is bipartite. Furthermore, if R is a local ring, then $\Gamma(R,S)$ is a complete bipartite graph.*

PROOF. Set $V_1 := \mathfrak{m}$ and $V_2 := 1 + \mathfrak{m}$. It is clear that V_1 is a coclique in $\Gamma(R,S)$. Since, for all $s \in S$, $V_2 = -s + \mathfrak{m}$, it is not hard to see that V_2 is another coclique in $\Gamma(R,S)$. Whenever R is local, then, by Remark 3.1, $\Gamma(R,S)$ is complete bipartite, because 0 is adjacent to 1. $\square$

LEMMA 3.3 (cf. [3, Remark 2.5]). *Let $(R,\mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| > 2$. Then, for every elements x and y in R, $N_{\overline{\Gamma(R,S)}}(x) \cap N_{\overline{\Gamma(R,S)}}(y) \neq \emptyset$.*

PROOF. If $x, y \in \mathfrak{m} = J(R)$, then x and y are adjacent to 1. Now, suppose that $x, y \notin \mathfrak{m}$. Then $x, y \in U(R)$, and so x and y are adjacent to 0. Hence, we may assume that $x \in \mathfrak{m}$ and $y \notin \mathfrak{m}$. This implies that x is adjacent to y. Now, if there exists $s \in S$ such that $1 + s \in U(R)$, it is easy to see that $y \in N_{\overline{\Gamma(R,S)}}(x) \cap N_{\overline{\Gamma(R,S)}}(y)$. Thus we may also suppose that $1 + s \in \mathfrak{m}$ for every $s \in S$. Hence in the quotient ring $R/\mathfrak{m}$, $y + \mathfrak{m} = -sy + \mathfrak{m}$ for every $s \in S$. On the other hand, since $|R/\mathfrak{m}| > 2$, there exists $z \in R\backslash\mathfrak{m}$ such that $z + \mathfrak{m} \neq y + \mathfrak{m}$. Now, it is routine to check that $z \in N_{\overline{\Gamma(R,S)}}(x) \cap N_{\overline{\Gamma(R,S)}}(y)$. This completes the proof. $\square$

LEMMA 3.4. *Let R be an Artinian ring. Then the following statements hold:*

(a) *If $R/J(R)$ does not contain a summand isomorphic to $\mathbb{Z}_2$, then*

$$\operatorname{diam}\big(\Gamma(R,S)\big) \leqq 2.$$

(b) *If $R/J(R)$ contains exactly one summand isomorphic to $\mathbb{Z}_2$, then*

$$\operatorname{diam}\big(\Gamma(R,S)\big) \leqq 3.$$

Furthermore, if R is not local, then $\operatorname{diam}\big(\Gamma(R,S)\big)=3$.

(c) *If $R/J(R)$ has $\mathbb{Z}_2\times\mathbb{Z}_2$ as a summand, then $\Gamma(R,S)$ is disconnected. Furthermore,* $\operatorname{diam}\big(\Gamma(R,S)\big)=\infty$.

(d) $\operatorname{diam}\big(\Gamma(R,S)\big)\in\{1,2,3,\infty\}$.

PROOF. (a) By [10, Theorem 1.1], every element of R is a sum of two units. Now, suppose that x and y are distinct elements of R and $s\in S$. Then there exist two units u_1 and u_2 such that $sx-sy=u_1+u_2$. Thus we have the walk $x-(-sx+u_1)-y$ between x and y. This shows that $\operatorname{diam}\big(\Gamma(R,S)\big)\leqq 2$.

(b) By [4, Theorem 8.7], we can write $R\cong R_1\times R_2\times\ldots\times R_n$, where each R_i is a local ring with maximal ideal $\mathfrak{m}_i$, for all i with $1\leqq i\leqq n$. Without loss of generality we may assume that $R_1/\mathfrak{m}_1\cong\mathbb{Z}_2$. Let $x:=(x_1,x_2,\ldots,x_n)$ and $y:=(y_1,y_2,\ldots,y_n)$ be two distinct arbitrary vertices in $\Gamma(R,S)$ and $s:=(s_1,s_2,\ldots,s_n)\in S$. If either $x_1,y_1\in\mathfrak{m}_1$ or $x_1,y_1\notin\mathfrak{m}_1$, then, by Proposition 3.2, there exists $z_1\in R$ such that x_1 and y_1 are adjacent to z_1. Now, by Lemma 3.3, there exists $z_i\in N_{\overline{\Gamma}(R,S)}(x_i)\cap N_{\overline{\Gamma}(R,S)}(y_i)$, because $|R_i/\mathfrak{m}_i|>2$ for all $i\geqq 2$. Set $z:=(z_1,z_2,\ldots,z_n)$. Thus $x-z-y$ is a path between x and y. Hence $d(x,y)\leqq 2$. Otherwise, $s_1x_1+y_1\notin\mathfrak{m}_1$, and so, in view of the proof of [10, Lemma 4.1], $sx+y$ is a sum of three units. Thus there exist units u_1, u_2 and u_3 such that $sx+y=u_1+u_2+u_3$. So, we have the following walk between x and y:

$$x-(-sx+u_1)-\big(x-s^{-1}u_1-s^{-1}u_2\big)-y$$

This means that $d(x,y)\leqq 3$. Thus $\operatorname{diam}\big(\Gamma(R,S)\big)\leqq 3$. Furthermore, if R is not local, then, by Proposition 3.2, it is not hard to see that $(0,0,\ldots,0),(1,0,\ldots,0)\in R$ are neither adjacent nor have a common neighbor. Hence in this situation $\operatorname{diam}\big(\Gamma(R,S)\big)=3$.

(c) By Theorem 2.11, $\Gamma(R,S)$ is connected if and only if R can be generated by its units. Now, by [11, Corollary 7], this is equivalent to the quotient ring $R/J(R)$ having at most one summand isomorphic to $\mathbb{Z}_2$. So, if $R/J(R)$ has $\mathbb{Z}_2\times\mathbb{Z}_2$ as a summand, then $\Gamma(R,S)$ is disconnected. Furthermore, $\operatorname{diam}\big(\Gamma(R,S)\big)=\infty$.

(d) It is an immediate consequence from (a), (b) and (c). □

THEOREM 3.5. *Let R be an Artinian ring. Then the following statements are true.*

(a) $\operatorname{diam}\big(\Gamma(R,S)\big)=1$ *if and only if R is a field with $|S|\geqq 2$ or $S=\{-1\}$.*

(b) $\operatorname{diam}\big(\Gamma(R,S)\big)=2$ *if and only if one of the following conditions hold.*

(i) *R is a field with $S=\{s\}$, where $s\neq -1$.*

(ii) *R is not a field and $R/J(R)$ can not have $\mathbb{Z}_2$ as a summand.*

(iii) *$(R,\mathfrak{m})$ is a local ring with $|R/\mathfrak{m}|=2$ and $R\ncong\mathbb{Z}_2$.*

(c) diam $(\Gamma(R,S)) = 3$ *if and only if* $R/J(R)$ *has exactly one summand isomorphic to* $\mathbb{Z}_2$ *and* R *is not local.*

(d) diam $(\Gamma(R,S)) = \infty$ *if and only if* $R/J(R)$ *has* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *as a summand.*

PROOF. First assume that R is a field. Then, by Proposition 2.8, $\Gamma(R,S)$ is a refinement of a star graph, and hence diam $(\Gamma(R,S)) \leqq 2$. Now, we consider the following cases:

Case 1: $|S| \geqq 2$ or $S = \{-1\}$. Then, by Theorem 2.7, $\Gamma(R,S)$ is complete. This means that diam $(\Gamma(R,S)) = 1$.

Case 2: $S = \{s\}$ with $s \neq -1$. Then, in view of Theorem 2.7, diam $(\Gamma(R,S)) = 2$.

Now, assume that R is not a field. Again, by Theorem 2.7, diam $(\Gamma(R,S)) \geqq 2$. We consider the following situations:

(α) The quotient ring $R/J(R)$ does not contain a summand isomorphic to $\mathbb{Z}_2$. Hence, by Lemma 3.4(a), diam $(\Gamma(R,S)) = 2$.

(β) Suppose that $R/J(R)$ contains exactly one summand isomorphic to $\mathbb{Z}_2$. Then, by Lemma 3.4(b), diam $(\Gamma(R,S)) \leqq 3$. Moreover, Lemma 3.4(b) implies that, whenever R is not local, diam $(\Gamma(R,S)) = 3$. Also, if R is local, in view of Proposition 3.2, diam $(\Gamma(R,S)) = 2$.

(γ) Finally, if $R/J(R)$ has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a summand, then, by Lemma 3.4(c), diam $(\Gamma(R,S)) = \infty$. This completes the proof. $\square$

REMARK 3.6. Let F be a field with $|F| \geqq 4$. If $-1 \in S$, then $\Gamma(F,S)$ is complete. Otherwise, for all elements $s \in S$ and $y \in F - \{1, -s\}$, y is adjacent to 1 and $-s$. Thus, there always exists a path of length two with non-zero vertices in $\Gamma(F,S)$. Now, by slight modifications in the proof of [3, Proposition 5.10], one can conclude that, whenever R is an Artinian ring, gr $(\Gamma(R,S)) \in \{3,4,6,\infty\}$.

A graph is said to be planar if it can be drawn in the plane such that its edges intersect only at their ends. An elementary subdivision of a graph is a graph obtained from it by removing some edge $e = \{u,v\}$ and adding new vertex w and edges $\{u,w\}$ and $\{w,v\}$. A subdivision of a graph is a graph obtained from it by a succession of elementary subdivisions. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [14, Theorem 6.2.2]).

THEOREM 3.7. *Let* R *be an Artinian ring. Then* $\Gamma(R,S)$ *is planar if and only if one of the following conditions hold.*

(a) $R \cong (\mathbb{Z}_2)^\ell \times T$, *where* $\ell \geqq 0$ *and* T *is isomorphic to one of the following rings:* $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$.

(b) $R \cong \mathbb{F}_4$.

(c) $R \cong (\mathbb{Z}_2)^\ell \times \mathbb{F}_4$, *where* $\ell > 0$ *with* $S = \{1\}$.

(d) $R \cong \mathbb{Z}_5$ *with* $S = \{1\}$.
(e) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ *with* $S = \{(1,1)\}$, $S = \{(1,-1)\}$ *or* $S = \{(-1,1)\}$.

Proof. Assume that the graph $\Gamma(R,S)$ is planar. Hence $\Gamma(R,S)$ contains a vertex with degree at most five. By Remark 2.4(a), the degree of all vertices in $\Gamma(R,S)$ are at least $|U(R)| - 1$. Thus $|U(R)| \leqq 6$. Now, by an argument similar to that used in the proof of [3, Theorem 5.14], $|J(R)| \leqq 2$. Since R is Artinian, we can write $R \cong R_1 \times R_2 \times \ldots \times R_n$, where each R_i is local with maximal ideal $\mathfrak{m}_i$, for all i with $1 \leqq i \leqq n$. On the other hand, for all $1 \leqq i \leqq n$, $|U(R_i)| = |R_i \backslash \mathfrak{m}_i| \leqq 6$ and $|\mathfrak{m}_i| \leqq 2$. Hence we have that $|R_i| \leqq 8$. Furthermore, if R_i is a field, then $|R_i| \leqq 7$.

Now, suppose that R is local with maximal ideal $\mathfrak{m}$. In this case, since the number of elements of a finite local ring is a power of a prime number, $|R| = 2,3,4,5,7$ or 8. If $|R| \leqq 4$, then $\Gamma(R,S)$ is planar. If R is a field with $|R| \geqq 5$ and $|S| \geqq 2$ or $S = \{-1\}$, then, by Theorem 2.7, $\Gamma(R,S)$ is complete, and so in these situations $\Gamma(R,S)$ are not planar. Furthermore, $\Gamma(\mathbb{Z}_7, \{1\})$ is not planar, because it contains a subdivision of K_5. Also clearly $\Gamma(\mathbb{Z}_5, \{1\})$ is planar. If R is not a field and $|R| = 8$, then, in view of [6, p. 687], $|\mathfrak{m}| = 4$, which is impossible.

Now, assume that R is not local. Since $|U(R)| \leqq 6$ and $|J(R)| \leqq 2$, we obtain the following candidates for R:

(i) $(\mathbb{Z}_2)^\ell \times T$, where $\ell > 0$ and T is isomorphic to one of the following rings:

$$\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{F}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_5 \ \text{ or } \ \mathbb{Z}_7.$$

(ii) $(\mathbb{Z}_2)^\ell \times \mathbb{Z}_3 \times T$, where $\ell \geqq 0$ and T is isomorphic to one of the following rings:

$$\mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{F}_4 \ \text{ or } \ \mathbb{Z}_2[x]/(x^2).$$

(iii) $(\mathbb{Z}_2)^\ell \times \mathbb{F}_4 \times T$, where $\ell \geqq 0$ and $T \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

Suppose that R is isomorphic to one of the rings in (i). Since $\Gamma(\mathbb{Z}_2) = \overline{\Gamma}(\mathbb{Z}_2) \cong K_2$, $\Gamma(\mathbb{Z}_2 \times T) = \overline{\Gamma}(\mathbb{Z}_2 \times T)$ and $\Gamma(\mathbb{Z}_2 \times T)$ is bipartite, by Remark 2.4(c)(i) and [1, Lemma 8.1], $\Gamma(R) \cong 2^{\ell-1}\Gamma(\mathbb{Z}_2 \times T)$. So it is sufficient to check the planarity of $\Gamma(\mathbb{Z}_2 \times T)$. By Proposition 2.1(a), $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong 2K_2$, and so it is planar. It is easy to check that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3, U(\mathbb{Z}_2 \times \mathbb{Z}_3))$ is planar, and hence, by Remark 2.4(c)(ii), $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3, S)$ is planar. Moreover, it is routine to check that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4, S) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), S) \cong 2C_4$ which is planar. Also, if $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$, then S is one of the following sets:

$$\{(1,1)\}, \{(1,\alpha),(1,\alpha^2)\} \quad \text{or} \quad U(R).$$

If $|S| \geqq 2$, then, in view of the proof of Theorem 2.7, all vertices in $\overline{\Gamma}(\mathbb{F}_4 \backslash \{0\})$ are adjacent. Hence, by Remark 2.4(c)(i), $\Gamma\big(\mathbb{Z}_2 \times (\mathbb{F}_4 \backslash \{0\}), S\big)$ is isomorphic to the complete bipartite $K_{3,3}$. Thus in this case $\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4, S)$ is not planar. Finally, by [3, Theorem 5.14], $\Gamma\big(\mathbb{Z}_2 \times \mathbb{F}_4, \{(1,1)\}\big)$ is planar.

Now, whenever $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5, S)$ is planar, in view of [14, Theorem 6.1.23], $\big|E\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5, S)\big)\big| \leqq 2\big|V\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5, S)\big)\big| - 4 = 16$, because $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5, S)$ is bipartite. However, since all vertices in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5, S)$ have degree at least 4, then $\big|E\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5, S)\big)\big| \geqq 20$, which is impossible. This means that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5, S)$ is not planar. Also, in the light of Remark 2.4(a) and the fact that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_7, S) = \overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_7, S)$ it is easy to see that the graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_7, S)$ has no vertices of degree at most 5. This implies that it is not planar.

Now, suppose that R is isomorphic to one of the rings in (ii). If $\ell > 0$, then the degree of all vertices in $\Gamma(R, S)$ is at least $\big|U(R)\big|$. Since in this case $\Gamma(R, S)$ is bipartite, thus planarity of $\Gamma(R, S)$ forces that $\big|E\big(\Gamma(R, S)\big)\big| \leqq 2\big|V\big(\Gamma(R, S)\big)\big| - 4$. However, this does not happen. Thus in this case all these rings are ruled out.

Now, assume that $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $|S| > 1$ or S is one of the following sets:

$$\{(1,1)\},\ \{(-1,-1)\},\ \{(1,-1)\}\ \text{ or }\ \{(-1,1)\}.$$

If $S = \{(1,1)\}$, then, by [3, Theorem 5.14], $\Gamma(R, S)$ is planar. Also, if $S = \{(-1,-1)\}$, then $\Gamma(R, S)$ contains a subdivision of $K_{3,3}$, and so it is not planar. Moreover, in the case that $S = \{(1,-1)\}$, $\Gamma(R, S)$ is

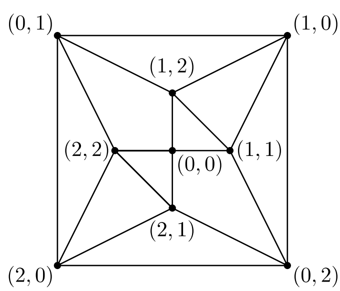

Fig. 2: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,-1)\})$

planar (see Fig. 2). Since, by Remark 2.4(c)(i), $\Gamma\big(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(-1,1)\}\big) \cong \Gamma\big(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,-1)\}\big)$, $\Gamma\big(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(-1,1)\}\big)$ is also planar. Now suppose that $|S| > 1$. If $(-1,-1) \in S$, then, by Remark 2.4(c)(ii), $\Gamma(R, S)$ is not planar. Otherwise $\big|E\big(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, S)\big)\big| \geqq 3\big|V\big(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, S)\big)\big| - 6$, which implies that $\Gamma(R, S)$ is not planar (cf. [14, Theorem 6.1.23]).

Whenever $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$, since $\Gamma(\mathbb{Z}_4) = \overline{\Gamma}(\mathbb{Z}_4) \cong C_4$ and C_4 is bipartite, the graph $\Gamma(R, S)$ is also bipartite. Now, by Remark 2.4(a), all vertices in $\Gamma(R, S)$ have degree at least 4, and so it is not planar. Also, in the case that $R \cong \mathbb{Z}_3 \times \mathbb{F}_4$, it is not hard to see that $\Gamma(R, S)$ has no vertex of degree at most 5, and hence it is not planar.

Finally, suppose that R is isomorphic to one of the rings in (iii). Since $\Gamma(T) = \overline{\Gamma}(T) \cong C_4$, by Remark 2.4(a), the degree of every vertex of $\Gamma(R, S)$ is at least 6. Hence in this situation $\Gamma(R, S)$ is not planar. This completes the proof. □

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