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TRIANGULAR CESÀRO SUMMABILITY OF TWO DIMENSIONAL FOURIER SERIES

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Abstract. It is proved that the maximal operator of the triangular Cesàro means of a two-dimensional Fourier series is bounded from the periodic Hardy space $H_p(\mathbb{T}^2)$ to $L_p(\mathbb{T}^2)$ for all $2/(2+\alpha) < p \leqq \infty$ and, consequently, is of weak type $(1,1)$. As a consequence we obtain that the triangular Cesàro means of a function $f \in L_1(\mathbb{T}^2)$ converge a.e. to f.

1. Introduction

It is known that Carleson's theorem [3] holds for two dimensions, too. More exactly,

$$s_k f(x,y) := \sum_{i,j\in\mathbb{Z},\ |i|+|j|\leqq k} \hat{f}(i,j)e^{\imath(ix+jy)} \to f(x,y)$$

for a.e. $(x,y) \in \mathbb{T}^2$ as $k \to \infty$ if $f \in L_p(\mathbb{T}^2)$ $(1 < p < \infty)$ (see Carleson [3], Fefferman [4] and Grafakos [5, p. 231]). This is false for $p = 1$. However, in the one dimensional case the well known Lebesgue's theorem says that the Fejér and more generally the Cesàro means $\sigma_n^\alpha f$ of f converge to f almost everywhere if $f \in L_1(\mathbb{T})$ (see e.g. Zygmund [13, I. p. 90–94]).

In this paper we generalize Lebesgue's theorem for the triangular Cesàro (or (C,α)) means defined by

$$\sigma_n^\alpha f(x,y) := \frac{1}{A_{n-1}^\alpha} \sum_{i,j\in\mathbb{Z},\ |i|+|j|\leqq n} A_{n-1-|i|-|j|}^\alpha \hat{f}(i,j)e^{\imath(ix+jy)}$$

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$(0<\alpha<\infty)$. If $\alpha=1$, we get the Fejér means

$$\sigma_n f(x)=\sum_{i,j\in\mathbb{Z},\ |i|+|j|\leqq n}\left(1-\frac{|i|+|j|}{n}\right)\hat f(i,j)e^{\imath(ix+jy)}.$$

Because of the complexity of the kernel function, this summability method is rarely investigated in the literature (see e.g. Herriot [6], Berens, Li and Xu [1,2,7,12] and more recently Szili and Vértesi [10]).

We will prove that $\sigma_n^\alpha f\to f$ in B-norm, where B is a homogeneous Banach space, which includes the norm convergence in $L_p(\mathbb{T}^2)$ $(1\leqq p<\infty)$ and in $C(\mathbb{T}^2)$. Next we obtain that the maximal operator σ_*^α is bounded from the Hardy space $H_p(\mathbb{T}^2)$ to $L_p(\mathbb{T}^2)$ for all $2/(2+\alpha\wedge 1)<p\leqq\infty$, where $\alpha\wedge\beta:=\min\{\alpha,\beta\}$. This implies by interpolation that σ_*^α is of weak type $(1,1)$. As a consequence we get the a.e. convergence of $\sigma_n^\alpha f$ to f, whenever $f\in L_1(\mathbb{T}^2)$. Note that this convergence result was proved with different methods by Herriot [6] for Nörlund means and by Berens, Li and Xu [1] for Fourier transforms.

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2. The kernel functions

We briefly write $L_p(\mathbb{T}^2)$ instead of the $L_p(\mathbb{T}^2,\lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p:=\left(\int_{\mathbb{T}^2}|f|^p\,d\lambda\right)^{1/p}$, $(0<p\leqq\infty)$, where $\mathbb{T}:=[-\pi,\pi]$ is the torus and λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I. The space of continuous functions with the supremum norm is denoted by $C(\mathbb{T}^2)$.

For a distribution f the nth *Fourier coefficient* is defined by $\hat f(n):=f\left(e^{-\imath(nx+my)}\right)$ $(n,m\in\mathbb{Z})$. If f is an integrable function then

$$\hat f(n,m)=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}f(x)e^{-\imath(nx+my)}\,dx,\quad\text{where}\quad \imath=\sqrt{-1}.$$

For $f\in L_1(\mathbb{T}^2)$ the kth *triangular or ℓ_1-partial sum* s_kf is introduced by

$$s_kf(x,y):=\sum_{i,j\in\mathbb{Z}}\mathbf{1}_{\{|i|+|j|\leqq k\}}\hat f(i,j)e^{\imath(ix+jy)}$$

$$=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}f(x-u,y-v)D_k(u,v)\,du\,dv,$$

where $\mathbf{1}_H$ denotes the characteristic function of the set H and

$$D_k(u,v) := \sum_{i,j\in\mathbb{Z}} \mathbf{1}_{\{|i|+|j|\leqq k\}} e^{\imath(iu+jv)} \qquad (k\in\mathbb{N})$$

is the *Dirichlet kernel.* It is easy to see that $|D_k| \leqq Ck^2$. Recently Szili and Vértesi [10] verified that $\|D_k\|_1 \sim (\log k)^2$.

Defining

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!},$$

we know that $A_k^\alpha = O(k^\alpha)$ $(k\in\mathbb{N})$ (see Zygmund [13, I. p. 77]). For $n \geqq 1$ the *triangular Cesàro (or (C,α)) means* of a function $f \in L_1(\mathbb{T}^2)$ are defined by

$$\sigma_n^\alpha f(x,y) := \frac{1}{A_{n-1}^\alpha} \sum_{i,j\in\mathbb{Z},\ |i|+|j|\leqq n} A_{n-1-|i|-|j|}^\alpha \hat{f}(i,j) e^{\imath(ix+jy)}$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x-u,y-v) K_n^\alpha(u,v)\,du\,dv,$$

where the *Cesàro-kernel* is given by

$$K_n^\alpha(u,v) := \frac{1}{A_{n-1}^\alpha} \sum_{i,j\in\mathbb{Z},\ |i|+|j|\leqq n} A_{n-1-|i|-|j|}^\alpha e^{\imath(iu+jv)} \tag{1}$$

$$= \frac{1}{A_{n-1}^\alpha} \sum_{|i|+|j|\leqq n} \sum_{k=|i|+|j|}^{n-1} A_{n-1-k}^{\alpha-1} e^{\imath(iu+jv)} = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} D_k(u,v).$$

Then

$$\sigma_n^\alpha f(x,y) = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_k f(x,y).$$

If $\alpha = 1$, we get the Fejér means

$$\sigma_n f(x,y) = \sum_{i,j\in\mathbb{Z},\ |i|+|j|\leqq n} \left(1 - \frac{|i|+|j|}{n}\right) \hat{f}(i,j) e^{\imath(ix+jy)} = \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x,y).$$

It was proved by Herriot [6] and Berens and Xu [2,12] that

$$D_k(x,y) = 2\frac{\cos(x/2)\cos\big((k+1/2)x\big) - \cos(y/2)\cos\big((k+1/2)y\big)}{\cos x - \cos y} \tag{2}$$

$$= -\frac{\cos(x/2)\cos\big((k+1/2)x\big) - \cos(y/2)\cos\big((k+1/2)y\big)}{\sin\big((x-y)/2\big)\,\sin\big((x+y)/2\big)}.$$

In what follows we may suppose that $\pi > x > y > 0$.

LEMMA 1. *If* $0 < \alpha \leqq 1$ *and* $0 \leqq \beta \leqq 1$ *then*

$$|K_n^\alpha(x,y)| \leqq Cn^2 \tag{3}$$

$$\leqq C(x-y)^{-3/2}\big(y^{-1/2}\mathbf{1}_{\{y\leqq\pi/2\}} + (\pi-x)^{-1/2}\mathbf{1}_{\{y>\pi/2\}}\big) \tag{4}$$

$$\leqq \max_{\gamma=\alpha,1}\big\{Cn^{-\gamma}(x-y)^{-1-\beta}\big(y^{\beta-\gamma-1}\mathbf{1}_{\{y\leqq\pi/2\}} + (\pi-x)^{\beta-\gamma-1}\mathbf{1}_{\{y>\pi/2\}}\big)\big\} \tag{5}$$

$$\leqq \max_{\gamma=\alpha,1}\big\{Cn^{1-\gamma}y^{-\gamma-1}\mathbf{1}_{\{y\leqq\pi/2\}} + Cn^{1-\gamma}(\pi-x)^{-\gamma-1}\mathbf{1}_{\{y>\pi/2\}}\big\} \tag{6}$$

$$\leqq \max_{\gamma=\alpha,1}\big\{C(x-y)^{\gamma-1}y^{-\gamma-1}\mathbf{1}_{\{y\leqq\pi/2\}} + C(x-y)^{\gamma-1}(\pi-x)^{-\gamma-1}\mathbf{1}_{\{y>\pi/2\}}\big\}. \tag{7}$$

PROOF. Inequality (3) follows from (1). In (2) we use that $\sin(x\pm y)/2 \sim x\pm y$ if $y \leqq \pi/2$ and $\sin(x-y)/2 \sim x-y$, $\sin(x+y)/2 \sim 2\pi - x - y$, if $y > \pi/2$. The facts $x+y > x-y$, $x+y > y$ and $2\pi - x - y > x - y$, $2\pi - x - y > \pi - x$ imply (4). Using (2) and the inequality

$$\Big|\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} e^{\imath(k+1/2)t}\Big| = \Big|\sum_{k=0}^{n-1} A_k^{\alpha-1} e^{-\imath kt}\Big| \leqq \frac{C}{\big(\sin(t/2)\big)^\alpha} + \frac{Cn^{\alpha-1}}{\big(\sin(t/2)\big)}$$

$(0 < \alpha \leqq 1)$ (see Zygmund [13, I. p. 94]), we conclude

$$|K_n^\alpha(x,y)| \leqq Cn^{-\alpha}(x-y)^{-1}(x+y)^{-1}y^{-\alpha} + Cn^{-1}(x-y)^{-1}(x+y)^{-1}y^{-1}$$

$$\leqq Cn^{-\alpha}(x-y)^{-1-\beta}y^{\beta-\alpha-1} + Cn^{-1}(x-y)^{-1-\beta}y^{\beta-2}$$

if $y \leqq \pi/2$, which is exactly (5). The inequality for $y > \pi/2$ can be proved in the same way.

Lagrange's theorem and (2) imply that there exists $x > \xi > y$ such that

$$D_k(x,y) = -\frac{H_k'(\xi)(x-y)}{\sin\big((x-y)/2\big)\,\sin\big((x+y)/2\big)},$$

where

$$H_k(t) = \cos(t/2)\cos\big((k+1/2)t\big).$$

Then

$$\big|K_n^\alpha(x,y)\big| \leqq C(x-y)(n+1)(x-y)^{-1}(x+y)^{-1}\big(n^{-\alpha}y^{-\alpha}+n^{-1}y^{-1}\big)$$

$$\leqq C\big(n^{1-\alpha}y^{-\alpha-1}+y^{-2}\big)$$

shows (6), if $y \leqq \pi/2$. The case $y > \pi/2$ is similar. Inequality (7) follows from (5) (with $\beta = 0$) if $n \geqq (x-y)^{-1}$ and from (6) if $n < (x-y)^{-1}$. □

In the next lemma we estimate the partial derivatives of the kernel function.

Lemma 2. *If* $0 < \alpha \leqq 1$ *and* $0 \leqq \beta \leqq 1$ *then for* $j = 1, 2$,

$$\big|\partial_j K_n^\alpha(x,y)\big| \tag{8}$$

$$\leqq \max_{\gamma=\alpha,1}\Big\{Cn^{1-\gamma}(x-y)^{-1-\beta}\big(y^{\beta-\gamma-1}\mathbf{1}_{\{y\leqq\pi/2\}}+(\pi-x)^{\beta-\gamma-1}\mathbf{1}_{\{y>\pi/2\}}\big)\Big\}.$$

Proof. By Lagrange's theorem and (2),

$$\partial_1 D_k(x,y) = \frac{1}{2}\big(\sin(x/2)\cos\big((k+1/2)x\big)$$

$$+\cos(x/2)(2k+1)\sin\big((k+1/2)x\big)\big)\sin\big((x-y)/2\big)^{-1}\sin\big((x+y)/2\big)^{-1}$$

$$+\frac{1}{2}H_k'(\xi)(x-y)\big(\sin\big((x-y)/2\big)^{-2}\sin\big((x+y)/2\big)^{-1}\cos\big((x-y)/2\big)$$

$$+\sin\big((x-y)/2\big)^{-1}\sin\big((x+y)/2\big)^{-2}\cos\big((x+y)/2\big)\big),$$

where $y < \xi < x$ is a suitable number. Using the methods above,

$$\big|\partial_1 K_n^\alpha(x,y)\big| \leqq C(x-y)^{-1}(x+y)^{-1}\big(n^{1-\alpha}y^{-\alpha}+y^{-1}\big)$$

$$+C(x+y)^{-2}\big(n^{1-\alpha}y^{-\alpha}+y^{-1}\big) \leqq C(x-y)^{-1-\beta}\big(n^{1-\alpha}y^{\beta-\alpha-1}+y^{\beta-2}\big),$$

which proves (8), if $y \leqq \pi/2$. The case $y > \pi/2$ can be shown similarly. □

3. Norm convergence of the summability

Li and Xu [7] proved for Jacobi polynomials that the L_1-norms of the kernel functions are uniformly bounded.

THEOREM 1. *For* $0<\alpha<\infty$ *we have*

$$\int_{\mathbb{T}^2} |K_n^\alpha(x)|\,dx \leqq C \qquad (n\in\mathbb{N}).$$

A Banach space B consisting of Lebesgue measurable functions on $\mathbb{T}^2$ is called a *homogeneous Banach space,* if $\|f\|_1 \leqq C\|f\|_B$, $(f\in B)$ and

(i) for all $f\in B$ and $x\in\mathbb{T}^2$, $T_xf:=f(\cdot - x)\in B$ and $\|T_xf\|_B=\|f\|_B$,

(ii) the function $x\mapsto T_xf$ from $\mathbb{T}^2$ to B is continuous for all $f\in B$.

It is easy to see that the spaces $L_p(\mathbb{T}^2)$ $(1\leqq p<\infty)$, $C(\mathbb{T}^2)$, Lorentz spaces $L_{p,q}(\mathbb{T}^2)$ $(1<p<\infty, 1\leqq q<\infty)$ and Hardy space $H_1(\mathbb{T}^2)$ are homogeneous Banach spaces.

We can extend the definition of the *triangular Cesàro means* to distributions by

$$\sigma_n^\alpha f := f * K_n^\alpha \qquad (n\in\mathbb{N}).$$

Indeed, $\sigma_n^\alpha f$ is well defined for all $f\in H_p(\mathbb{T}^2)$ $(0<p\leqq\infty)$, for all $f\in L_p(\mathbb{T}^2)$ $(1\leqq p\leqq\infty)$ and $f\in B$, where B is a homogeneous Banach space (cf. Stein [9, p. 115]).

THEOREM 2. *If* B *is a homogeneous Banach space on* $\mathbb{T}^2$ *then*

$$\|\sigma_n^\alpha f\|_B \leqq \|f\|_B \|K_n^\alpha\|_1 \qquad (n\in\mathbb{N})$$

and $\sigma_n^\alpha f\to f$ *in* B, *for all* $f\in B$ *as* $n\to\infty$.

PROOF. Since the trigonometric polynomials are dense in B, the theorem follows from Theorem 1 with standard methods. $\square$

4. Triangular summability and Hardy spaces

The *Hardy space* $H_p(\mathbb{T}^2)$ $(0<p\leqq\infty)$ consists of all distributions f for which

$$\|f\|_{H_p} := \Big\| \sup_{0<t} |f * P_t^2| \Big\|_p < \infty,$$

where

$$P_t^2(x,y) := \sum_{n,m\in\mathbb{Z}} e^{-t\sqrt{n^2+m^2}} e^{\imath(nx+my)} \qquad (x,y\in\mathbb{T},\ t>0)$$

is the two dimensional *Poisson kernel*. It is known that the Hardy spaces $H_p(\mathbb{R}^2)$ are equivalent to the $L_p(\mathbb{R}^2)$ spaces when $1 < p \leqq \infty$ (see e.g. Stein [9, p. 91] or Weisz [11, p. 68]).

The atomic decomposition is a useful characterization of Hardy spaces. A bounded function a is an $H_p(\mathbb{T}^2)$ *-atom* if there exists a cube $I \subset \mathbb{T}^2$ such that

(i) $\operatorname{supp} a \subset I$,

(ii) $\|a\|_\infty \leqq |I|^{-1/p}$,

(iii) $\int_I a(x,y)x^k y^l \, dx \, dy = 0$ for all indices k, l with $k + l \leqq M$, where $M \geqq [2(1/p - 1)]$. (Here $[x]$ denotes the integer part of $x \in \mathbb{R}$.)

The atomic decomposition theorem can be found e.g. in Stein [9, p. 107] or Weisz [11, p. 76].

Let us introduce the *maximal operator*

$$\sigma_*^\alpha f := \sup_{n \geqq 1} |\sigma_n^\alpha f|.$$

THEOREM 3. *For $0 < \alpha < \infty$ we have*

$$\|\sigma_*^\alpha f\|_p \leqq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^2))$$

for all $2/(2 + \alpha \wedge 1) < p < \infty$. In particular, if $f \in L_1(\mathbb{T}^2)$ then

$$\lambda(\sigma_*^\alpha f > \rho) \leqq \frac{C}{\rho} \|f\|_1 \qquad (\rho > 0).$$

PROOF. By a usual method (see Zygmund [13, I. p. 77]) it is enough to prove the theorem for $0 < \alpha \leqq 1$. We have to show that

$$\int_{\mathbb{T}^2} \left|\sigma_*^\alpha a(x,y)\right|^p dx\, dy \tag{9}$$

$$= \int_{\mathbb{T}^2} \sup_{n \geqq 1} \left| \int_I a(u,v) K_n^\alpha(x-u, y-v)\, du\, dv \right|^p dx\, dy \leqq C_p$$

for every p-atom a, where $2/(2 + \alpha) < p < 1$ and I is the support of the atom (see [11, p. 106]). Without loss of generality we can suppose that a is a p-atom with support $I = I_1 \times I_2$ and

$$\left[-2^{-K-2}, 2^{-K-2}\right] \subset I_j \subset \left[-2^{-K-1}, 2^{-K-1}\right] \qquad (j = 1, 2)$$

for some $K \in \mathbb{N}$. By symmetry we can assume that $\pi > x - u > y - v > 0$, and so, instead of (9), it is enough to show that

$$\int_{\mathbb{T}^2} \sup_{n \geqq 1} \left| \int_I a(u,v) K_n^\alpha(x-u, y-v) \mathbf{1}_{A_i}(x-u, y-v)\, du\, dv \right|^p dx\, dy \leqq C_p$$

for all $i=1,\dots,10$, where

$$A_1 := \{(x,y) : 0 < x \leqq 2^{-K+5},\ 0 < y < x < \pi,\ y \leqq \pi/2\},$$

$$A_2 := \{(x,y) : 2^{-K+5} < x < \pi,\ 0 < y \leqq 2^{-K+2},\ y \leqq \pi/2\},$$

$$A_3 := \{(x,y) : 2^{-K+5} < x < \pi,\ 2^{-K+2} < y \leqq x/2,\ y \leqq \pi/2\},$$

$$A_4 := \{(x,y) : 2^{-K+5} < x < \pi,\ x/2 < y \leqq x - 2^{-K+2},\ y \leqq \pi/2\},$$

$$A_5 := \{(x,y) : 2^{-K+5} < x < \pi,\ x - 2^{-K+2} < y < x,\ y \leqq \pi/2\}$$

$$A_6 := \{(x,y) : y > \pi/2,\ \pi - 2^{-K+5} \leqq y < \pi,\ 0 < y < x < \pi\},$$

$$A_7 := \{(x,y) : \pi/2 < y < \pi - 2^{-K+5},\ \pi - 2^{-K+2} < x < \pi\},$$

$$A_8 := \{(x,y) : \pi/2 < y < \pi - 2^{-K+5},\ (\pi+y)/2 < x \leqq \pi - 2^{-K+2}\},$$

$$A_9 := \{(x,y) : \pi/2 < y < \pi - 2^{-K+5},\ y + 2^{-K+2} < x \leqq (\pi+y)/2\},$$

$$A_{10} := \{(x,y) : \pi/2 < y < \pi - 2^{-K+5},\ y < x \leqq y + 2^{-K+2}\}.$$

First of all if $0 < x-u \leqq 2^{-K+5}$ then $-2^{-K-1} < x \leqq 2^{-K+6}$ and the same holds for y. If $\pi - 2^{-K+5} \leqq y-v < \pi$ then $\pi - 2^{-K+6} < y \leqq \pi + 2^{-K-1}$ and the same is true for x. By the definition of the atom and by Theorem 1,

$$\int_{\mathbb{T}^2} \sup_{n\geqq 1} \left| \int_I a(u,v) K_n^\alpha(x-u,y-v) \mathbf{1}_{A_1\cup A_6}(x-u,y-v)\,du\,dv \right|^p dx\,dy$$

$$\leqq C_p 2^{2K} 2^{-2K}.$$

Considering the set A_2 we have $2^{-K+5} < x-u < \pi$ and $0 < y-v \leqq 2^{-K+2}$, thus $2^{-K+4} < x < \pi + 2^{-K-1}$ and $-2^{-K-1} < y \leqq 2^{-K+3}$. Using (4) we conclude

$$\left| \int_I a(u,v) K_n^\alpha(x-u,y-v) \mathbf{1}_{A_2}(x-u,y-v)\,du\,dv \right|$$

$$\leqq C_p 2^{2K/p} \int_I (x-u-y+v)^{-3/2} (y-v)^{-1/2} \mathbf{1}_{A_2}(x-u,y-v)\,du\,dv$$

$$\leqq C_p 2^{2K/p} \mathbf{1}_{\{2^{-K+4} < x < \pi + 2^{-K-1}\}} \mathbf{1}_{\{-2^{-K-1} < y \leqq 2^{-K+3}\}}$$

$$\times \int_I \left(x - 2^{-K+3}\right)^{-3/2} (y-v)^{-1/2} \, du \, dv$$

$$\leqq C_p 2^{2K/p-3K/2} \mathbf{1}_{\{2^{-K+4}<x<\pi+2^{-K-1}\}} \mathbf{1}_{\{-2^{-K-1}<y\leqq 2^{-K+3}\}} \left(x - 2^{-K+3}\right)^{-3/2}.$$

Similarly, on A_7 $\pi/2 < y-v < \pi - 2^{-K+5}$ and $\pi - 2^{-K+2} < x-u < \pi$, thus $\pi/2 - 2^{-K-1} < y < \pi - 2^{-K+4}$ and $\pi - 2^{-K+3} < x < \pi + 2^{-K-1}$. By (4)

$$\left| \int_I a(u,v) K_n^\alpha (x-u, y-v) \mathbf{1}_{A_7}(x-u, y-v) \, du \, dv \right|$$

$$\leqq C_p 2^{2K/p} \int_I (x-u-y+v)^{-3/2} (\pi - x + u)^{-1/2} \mathbf{1}_{A_7}(x-u, y-v) \, du \, dv$$

$$\leqq C_p 2^{2K/p} \mathbf{1}_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}} \mathbf{1}_{\pi-2^{-K+3}<x<\pi+2^{-K-1}}$$

$$\times \int_I \left(\pi - 2^{-K+3} - y\right)^{-3/2} (\pi - x + u)^{-1/2} \, du \, dv$$

$$\leqq C_p 2^{2K/p-3K/2} \mathbf{1}_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}}$$

$$\times \mathbf{1}_{\pi-2^{-K+3}<x<\pi+2^{-K-1}} \left(\pi - 2^{-K+3} - y\right)^{-3/2}.$$

If $p > 2/3$ then

$$\int_{\mathbb{T}^2} \sup_{n\geqq 1} \left| \int_I a(u,v) K_n^\alpha (x-u, y-v) \mathbf{1}_{A_2 \cup A_7}(x-u, y-v) \, du \, dv \right|^p dx \, dy$$

$$\leqq C_p 2^{2K-3Kp/2} \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{-2^{-K-1}}^{2^{-K+3}} \left(x - 2^{-K+3}\right)^{-3p/2} dx \, dy$$

$$+ C_p 2^{2K-3Kp/2} \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}} \int_{\pi-2^{-K+3}}^{\pi+2^{-K-1}} \left(\pi - 2^{-K+3} - y\right)^{-3p/2} dy \, dx \leqq C_p.$$

We may suppose that the center of I is zero, in other words $I := (-\nu, \nu) \times (-\nu, \nu)$. Let

$$A_1(u,v) := \int_{-\nu}^{u} a(t,v) \, dt \qquad \text{and} \qquad A_2(u,v) := \int_{-\nu}^{v} A_1(u,t) \, dt.$$

Observe that

$$\left| A_k(u) \right| \leqq C_p 2^{K(2/p-k)}.$$

Integrating by parts we can see that

$$\int_{I_1} a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_3\cup A_8}(x-u,y-v)\,du$$

$$= A_1(\nu,v)K_n^\alpha(x-\nu,y-v)\mathbf{1}_{A_3\cup A_8}(x-\nu,y-v)$$

$$+\int_{-\nu}^{\nu} A_1(u,v)\partial_1 K_n^\alpha(x-u,y-v)\mathbf{1}_{A_3\cup A_8}(x-u,y-v)\,du,$$

because $A_1(-\nu,v)=0$. As $A_2(\nu,\nu)=\int_I a=0$, integrating the first term again by parts we obtain

$$\int_{I_1}\int_{I_2} a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_3\cup A_8}(x-u,y-v)\,du\,dv$$

$$=\int_{-\nu}^{\nu} A_2(\nu,v)\partial_2 K_n^\alpha(x-\nu,y-v)\mathbf{1}_{A_3\cup A_8}(x-\nu,y-v)\,dv$$

$$+\int_{I_1}\int_{I_2} A_1(u,v)\partial_1 K_n^\alpha(x-u,y-v)\mathbf{1}_{A_3\cup A_8}(x-u,y-v)\,du\,dv.$$

Note that $x-u-y+v>(x-u)/2$ on A_3 and $x-u-y+v>(\pi-y+v)/2$ on A_8. If $n\leqq 2^K$, we get from (8) that

$$\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_3}(x-u,y-v)\,du\,dv\right|$$

$$\leqq C_p n^{1-\gamma}2^{2K/p-2K}\int_{I_2}(x-\nu)^{-1-\beta}(y-v)^{\beta-\gamma-1}\mathbf{1}_{A_3}(x-\nu,y-v)\,dv$$

$$+C_p n^{1-\gamma}2^{2K/p-K}\int_I (x-u)^{-1-\beta}(y-v)^{\beta-\gamma-1}\mathbf{1}_{A_3}(x-u,y-v)\,du\,dv$$

$$\leqq C_p 2^{2K/p-2K-K\gamma}\mathbf{1}_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}\mathbf{1}_{\{2^{-K+1}<y\leqq x/2+2^{-K}\}}$$

$$\times\left(x-2^{-K-1}\right)^{-1-\beta}\left(y-2^{-K-1}\right)^{\beta-\gamma-1}$$

and

$$\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_8}(x-u,y-v)\,du\,dv\right|$$

$$\leqq C_p n^{1-\gamma}2^{2K/p-2K}\int_{I_2}(\pi-y+v)^{-1-\beta}(\pi-x+\nu)^{\beta-\gamma-1}$$

$$\times \mathbf{1}_{A_8}(x-\nu,y-v)\,dv + C_p n^{1-\gamma} 2^{2K/p-K}$$

$$\times \int_I (\pi-y+v)^{-1-\beta}(\pi-x+u)^{\beta-\gamma-1}\mathbf{1}_{A_8}(x-u,y-v)\,du\,dv$$

$$\leqq C_p 2^{2K/p-2K-K\gamma}\mathbf{1}_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}}\mathbf{1}_{(\pi+y)/2-2^{-K}<x<\pi-2^{-K+1}}$$

$$\times \left(\pi-y-2^{-K-1}\right)^{-1-\beta}\left(\pi-x-2^{-K-1}\right)^{\beta-\gamma-1}.$$

Similarly, if $n>2^K$, then we get from (5) that

$$\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_3}(x-u,y-v)\,du\,dv\right|$$

$$\leqq C_p n^{-\gamma}2^{2K/p}\int_I (x-u)^{-1-\beta}(y-v)^{\beta-\gamma-1}\mathbf{1}_{A_3}(x-u,y-v)\,du\,dv$$

$$\leqq C_p 2^{2K/p-2K-K\gamma}\mathbf{1}_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}\mathbf{1}_{\{2^{-K+1}<y\leqq x/2+2^{-K}\}}$$

$$\times \left(x-2^{-K-1}\right)^{-1-\beta}\left(y-2^{-K-1}\right)^{\beta-\gamma-1}$$

and

$$\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_8}(x-u,y-v)\,du\,dv\right|$$

$$\leqq C_p n^{-\gamma}2^{2K/p}\int_I (\pi-y+v)^{-1-\beta}(\pi-x+u)^{\beta-\gamma-1}\mathbf{1}_{A_8}(x-u,y-v)\,du\,dv$$

$$\leqq C_p 2^{2K/p-2K-K\gamma}\mathbf{1}_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}}\mathbf{1}_{(\pi+y)/2-2^{-K}<x<\pi-2^{-K+1}}$$

$$\times \left(\pi-y-2^{-K-1}\right)^{-1-\beta}\left(\pi-x-2^{-K-1}\right)^{\beta-\gamma-1}.$$

Choosing $\beta=\gamma/2$ we conclude

$$\int_{\mathbb{T}^2}\sup_{n\geqq 1}\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_3\cup A_8}(x-u,y-v)\,du\,dv\right|^p dx\,dy$$

$$\leqq C_p 2^{2K-2Kp-K\gamma p}$$

$$\times \Bigg(\int_{2^{-K+4}}^{\pi+2^{-K-1}}\int_{2^{-K+1}}^{x/2+2^{-K}}\left(x-2^{-K-1}\right)^{-p(1+\gamma/2)}\left(y-2^{-K-1}\right)^{-p(1+\gamma/2)}\,dx\,dy$$

$$+\int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}}\int_{(\pi+y)/2-2^{-K}}^{\pi-2^{-K+1}}\left(\pi-y-2^{-K-1}\right)^{-p(1+\gamma/2)}$$

$$\times\left(\pi-x-2^{-K-1}\right)^{-p(1+\gamma/2)}\,dx\,dy\Big)$$

$$\leqq C_p 2^{2K-2Kp-K\gamma p}2^{-K(1-p(1+\gamma/2))}2^{-K(1-p(1+\gamma/2))}\leqq C_p,$$

whenever $p>2/(2+\gamma)$. Recall that $\gamma=\alpha$ or $\gamma=1$.

Since $y-v>(x-u)/2$ on A_4 and $\pi-x+u>(\pi-y+v)/2$ on the set A_9, (8) implies

$$\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_4}(x-u,y-v)\,du\,dv\right| \tag{10}$$

$$\leqq C_p n^{1-\gamma}2^{2K/p-2K}\int_{I_2}(x-\nu-y+v)^{-1-\beta}(x-u)^{\beta-\gamma-1}$$

$$\times\mathbf{1}_{A_4}(x-\nu,y-v)\,dv+C_p n^{1-\gamma}2^{2K/p-K}$$

$$\times\int_I(x-u-y+v)^{-1-\beta}(x-u)^{\beta-\gamma-1}\mathbf{1}_{A_4}(x-u,y-v)\,du\,dv$$

$$\leqq C_p 2^{2K/p-2K-K\gamma}\mathbf{1}_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}\mathbf{1}_{\{x/2-2^{-K}<y\leqq x-2^{-K+1}\}}$$

$$\times\left(x-y-2^{-K}\right)^{-1-\beta}\left(x-2^{-K-1}\right)^{\beta-\gamma-1}$$

and

$$\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_9}(x-u,y-v)\,du\,dv\right| \tag{11}$$

$$\leqq C_p 2^{2K/p-2K-K\gamma}\mathbf{1}_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}}\mathbf{1}_{y+2^{-K+1}<x<(\pi+y)/2+2^{-K}}$$

$$\times\left(x-y-2^{-K}\right)^{-1-\beta}\left(\pi-y-2^{-K-1}\right)^{\beta-\gamma-1},$$

whenever $n\leqq 2^K$. If $n>2^K$ then (10) and (11) follow from (5). Choosing again $\beta=\gamma/2$ we obtain

$$\int_{\mathbb{T}^2}\sup_{n\geqq 1}\left|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_4\cup A_9}(x-u,y-v)\,du\,dv\right|^p dx\,dy$$

$$\leqq C_p 2^{2K-2Kp-K\gamma p}\bigg(\int_{2^{-K+4}}^{\pi+2^{-K-1}}\int_{x/2-2^{-K}}^{x-2^{-K+1}}\big(x-y-2^{-K}\big)^{-p(1+\gamma/2)}$$

$$\times\big(x-2^{-K-1}\big)^{-p(1+\gamma/2)}\,dx\,dy+\int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}}\int_{y+2^{-K+1}}^{(\pi+y)/2+2^{-K}}$$

$$\times\big(x-y-2^{-K}\big)^{-p(1+\gamma/2)}\big(\pi-y-2^{-K-1}\big)^{-p(1+\gamma/2)}\,dx\,dy\bigg)$$

$$\leqq C_p 2^{2K-2Kp-K\gamma p}2^{-K(1-p(1+\gamma/2))}2^{-K(1-p(1+\gamma/2))}\leqq C_p,$$

if $p>2/(2+\gamma)$.

Finally, inequality (7) implies

$$\bigg|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_5}(x-u,y-v)\,du\,dv\bigg|$$

$$\leqq C_p 2^{2K/p}\int_I (x-u-y+v)^{\gamma-1}(y-v)^{-\gamma-1}\mathbf{1}_{A_5}(x-u,y-v)\,du\,dv$$

$$\leqq C_p 2^{2K/p-K\gamma}\mathbf{1}_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}\mathbf{1}_{\{x-2^{-K+3}<y\leqq x+2^{-K}\}}\int_{I_2}(y-v)^{-\gamma-1}\,dv$$

$$\leqq C_p 2^{2K/p-K\gamma-K}\mathbf{1}_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}\mathbf{1}_{\{x-2^{-K+3}<y\leqq x+2^{-K}\}}$$

$$\times\big(y-2^{-K-1}\big)^{-\gamma-1}$$

and

$$\bigg|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_{10}}(x-u,y-v)\,du\,dv\bigg|$$

$$\leqq C_p 2^{2K/p-K\gamma-K}\mathbf{1}_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}}\mathbf{1}_{y-2^{-K}<x<y+2^{-K+3}}$$

$$\times\big(\pi-x-2^{-K-1}\big)^{-\gamma-1},$$

hence

$$\int_{\mathbb{T}^2}\sup_{n\geqq 1}\bigg|\int_I a(u,v)K_n^\alpha(x-u,y-v)\mathbf{1}_{A_5\cup A_{10}}(x-u,y-v)\,du\,dv\bigg|^p\,dx\,dy$$

$$\leqq C_p 2^{2K-K\gamma p-Kp}\int_{2^{-K+4}}^{\pi+2^{-K-1}}\int_{x-2^{-K+3}}^{x+2^{-K}}\big(y-2^{-K-1}\big)^{-p(\gamma+1)}\,dy\,dx$$

$$+ C_p 2^{2K-K\gamma p-Kp} \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}} \int_{y-2^{-K}}^{y+2^{-K+3}} \left(\pi - x - 2^{-K-1}\right)^{-p(\gamma+1)} dx\, dy$$

$$\leqq C_p 2^{2K-K\gamma p-Kp} \int_{2^{-K+3}}^{\pi+2^{-K+5}} \int_{y-2^{-K}}^{y+2^{-K+3}} \left(y - 2^{-K-1}\right)^{-p(\gamma+1)} dx\, dy$$

$$+C_p 2^{2K-K\gamma p-Kp} \int_{\pi/2-2^{-K+1}}^{\pi-2^{-K+3}} \int_{x+2^{-K}}^{x-2^{-K+3}} \left(\pi - x - 2^{-K-1}\right)^{-p(\gamma+1)} dy\, dx \leqq C_p,$$

which finishes the proof of the theorem. □

We suspect that in case $0 < \alpha \leqq 1$, $2/(2+\alpha)$ is the best possible constant, in other words, if $p \leqq 2/(2+\alpha)$ then σ_*^α is not bounded from $H_p(\mathbb{T}^2)$ to $L_p(\mathbb{T}^2)$.

COROLLARY 1. *If* $f \in L_1(\mathbb{T}^2)$ *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad a.e.$$

PROOF. Since the trigonometric polynomials are dense in $L_1(\mathbb{T}^2)$, the corollary follows from Theorem 3 and the usual density argument due to Marcinkiewicz and Zygmund [8]. □

Note that this convergence result was proved by Herriot [6] with different methods.

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