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GENERALIZED TOPOLOGIES, GENERALIZED NEIGHBORHOOD SYSTEMS, AND GENERALIZED INTERIOR OPERATORS

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Abstract. We give a systematic discussion on the relationship among generalized topologies, generalized neighborhood systems, and generalized interior operators. As some applications, we answer a question raised in [7] by Shen, and characterize generalized continuous maps.

1. Introduction

The theory of generalized topological spaces, which was founded by Á. Császár, is one of the most important developments of general topology in recent years. To produce generalized topologies, generalized neighborhood systems and generalized interior operators play an important role in the theory of generalized topological spaces. In [3], R. Shen discussed the complete generalized neighborhood systems, and he proved that a generalized neighborhood system is complete if and only if it can be generated by a generalized topology. Also he asked the question: for each strong generalized interior operator I on a set X, is there a unique GT μ such that $I = I_\mu$? In the present paper, we give a systematic discussion on the relationship among generalized topologies, generalized neighborhood systems, and generalized interior operators. As an application, we prove that a generalized interior operator I on a set X is strong if and only if there is a unique GT μ such that $I = I_\mu$, which answers Shen's question affirmatively. Also we apply the main results to characterize generalized continuous maps.

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Let X be a set, and denote $\exp X$ the power set of X. We call a class $\mu \subset \exp X$ a *generalized topology* (briefly GT) [1] on X if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. A set with a GT is said to be a *generalized topological space* (briefly GTS) [1]. For a GTS (X, μ), the elements of μ are called *μ-open sets* and the complements of μ-open sets are called *μ-closed sets.* A map $\psi : X \to \exp(\exp X)$ is called a *generalized neighborhood system* (briefly GNS) [1] on X if for each $x \in X$, $V \in \psi(x)$ implies $x \in V$. A GNS ψ on X is called *ascending* [1] if $V \in \psi(x)$ implies $U \in \psi(x)$ for each $x \in X$ and $V \subset U$. If ψ satisfying that for each $x \in X$ and $A \in \psi(x)$, there is a set O such that $x \in O \subset A$, and $y \in O$ implies the existence of a set $B \in \psi(y)$ with $B \subset O$, then we say ψ is *complete* [3]. A map $I : \exp X \to \exp X$ is called a *generalized interior operator* [1] if $I(A) \subset A$ for all $A \subset X$, and $A \subset B$ implies $I(A) \subset I(B)$ for all $A, B \subset X$. A GIO I on X is called *strong* [1] if $I\big(I(A)\big) = I(A)$ for all $A \subset X$.

2. On GT, GNS and GIO

Given a GT μ on X, we define $\psi_\mu(x) = \{A : x \in M \subset A \text{ for some } M \in \mu\}$ $(x \in X)$ and $I_\mu(A) = \bigcup\{M \subset A : M \in \mu\}$ $(A \subset X)$. Then ψ_μ and I_μ are respectively a GNS and a GIO on X, which we called the GNS and GIO generated by the GT μ [1]. It is not difficult to check that each GNS (GIO) generated by a GT is always complete (strong).

Given a GNS ψ on X, let μ_ψ denote the collection of all subsets $M \subset X$ such that $x \in M$ implies the existence of a set $V \in \psi(x)$ satisfying $V \subset M$. Then μ_ψ is a GT on X, which we called the GT generated by the GNS ψ [1]. Define $I_\psi : \exp X \to \exp X$ as follows: $x \in I_\psi(A)$ if and only if there is $V \in \psi(x)$ such that $V \subset A$ $(A \subset X)$. Then I_ψ is a GIO on X [2, 3.1] (in [2], I_ψ is written as ι_ψ), which we called the GIO generated by the GNS ψ.

Given a GIO I on X, let $\mu_I = \big\{A \subset X : A = I(A)\big\}$ $(A \subset X)$ and $\psi_I(x) = \big\{A \subset X : x \in I(A)\big\}$ $(x \in X)$. Then μ_I and ψ_I are respectively a GT and a GNS on X [3], which we called the GT and GNS generated by the GT μ [3].

The above relationship among them can be presented in the following diagram.

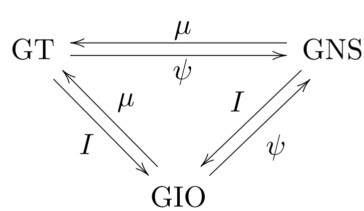

Diagram I

Is Diagram I commutative? We now give a systematic discussion on this question.

THEOREM 2.1. *For each GT μ on a set X, we have*
(1) $\mu = \mu_{I_\mu}$.
(2) $\mu = \mu_{\psi_\mu}$.
(3) $I_\mu = I_{\psi_\mu}$.
(4) $\psi_\mu = \psi_{I_\mu}$.
(5) $\mu = \mu_{I_{\psi_\mu}}$.
(6) $\mu = \mu_{\psi_{I_\mu}}$.

PROOF. (1) If $A \in \mu$, then $I_\mu(A) = \bigcup\{M \in \mu : M \subset A\} = A$. Thus $A \in \mu_{I_\mu}$. Conversely, if $A \in \mu_{I_\mu}$, then $I_\mu(A) = A$. By the definition of $I_\mu(A)$, $A \in \mu$. So $\mu = \mu_{I_\mu}$.

(2) See [1, 2.3].

(3) For each $A \subset X$, $I_\mu(A) = \bigcup\{M \subset A : M \in \mu\}$. Then $x \in I_\mu(A)$ if and only if there is a $M \in \mu$ such $x \in M \in A$. By the definition of ψ_μ, this is equivalent to the condition that there is an $M \in \psi_\mu(x)$ such that $M \subset A$. So $x \in I_\mu(A)$ if and only if $x \in I_{\psi\mu}(A)$, and then $I_\mu = I_{\psi_\mu}$.

(4) For each $x \in A$, $A \in \psi_\mu(x) \Leftrightarrow$ there is a $M \in \mu$, such that $x \in M \subset A$ $\Leftrightarrow$ $x \in I_\mu(A)$ $\Leftrightarrow$ $A \in \psi_{I_\mu}(x)$. So $\psi_\mu(x) = \psi_{I_\mu}(x)$.

(1) and (3) imply (5). (2) and (4) imply (6). $\square$

LEMMA 2.2 [3, 3.5]. *For a GIO I on a set x, we have*
(1) $I = I_{\psi_I}$,
(2) $\mu_I = \mu_{\psi_I}$.

For each GT μ on X, we have proved $\mu = \mu_{I_\mu}$, and then each GT on X can be generated by some GIO on X. However, we can see from the following example that for a GIO I on a set X, $I = I_{\mu_I}$ does not always hold.

EXAMPLE 2.3. Let $X = \{a, b, c\}$, $I(\{a, b, c\}) = \{a, b, c\}$, $I(a) = I(b) = I(c) = I(\emptyset) = \emptyset$, $I(\{a, b\}) = a$, $I(\{a, c\}) = c$, $I(\{b, c\}) = b$. Clearly $\mu_I = \{\emptyset, \{a, b, c\}\}$, whence $I_{\mu_I}(\{b, c\}) = I_{\mu_I}(\{a, c\}) = I_{\mu_I}(\{a, b\}) = \emptyset$. Therefore $I \neq I_{\mu_I}$.

REMARK 2.4. For the GIO I on X defined in the above example, there is no GT μ on X such that I is generated by μ. Otherwise, if $I = I_\mu$ for some GT μ, then $I_{\mu_I} = I_{\mu_{I_\mu}} = I_\mu = I$ by Theorem 2.1, a contradiction. So, it is natural to study when a GIO on X can be generated by some GT. In [3], Shen asked the question: for each strong generalized interior operator I on a set X, is there a unique GT μ such that $I = I_\mu$? Corollary 2.6 below will give an affirmative answer to this question.

THEOREM 2.5. *For a GIO I on a set X, the following conditions are equivalent*:
(1) *I is strong.*
(2) *ψ_I is complete.*
(3) *$\psi_I = \psi_\mu$ for some GT μ on X.*
(4) *$\psi_I = \psi_{\mu_I}$.*
(5) *$\psi_I = \psi_{\mu_{\psi_I}}$.*
(6) *$I = I_{\mu_I}$.*
(7) *$I = I_{\mu_{\psi_I}}$.*
(8) *$I = I_{\psi_{\mu_I}}$.*

PROOF. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (4) is proved in [3, 3.5]. (4) $\Rightarrow$ (3) $\Rightarrow$ (2) is obvious. (4) $\Leftrightarrow$ (5) comes from Lemma 2.2.

(1) $\Rightarrow$ (6). For each set $A \subset X$,

$$I_{\mu_I}(A) = \bigcup\{M \in \mu_I : M \subset A\} = \bigcup\{\, M \subset A : I(M) = M\} \subset I(A).$$

Conversely, since I is strong, then $I\bigl(I(A)\bigr) = I(A) \subset A$, so $I(A) \in \{\, M \subset A : I(M) = M\}$. By the definition of $I_{\mu_I}(A)$, $I_{\mu_I}(A) \supset I(A)$. Therefore $I = I_{\mu_I}$.

(6) $\Rightarrow$ (1). For each $A \subset X$, $I\bigl(I(A)\bigr) \subset I(A)$ is obvious. Conversely, if $x \in I(A) = I_{\mu_I}(A) = \bigcup\{M \in \mu_I : M \subset A\}$, then there is a set $M \in \mu_I$ such that $x \in M \subset A$. Since $I(M) = M$, so $x \in I\bigl(I(M)\bigr) \subset I\bigl(I(A)\bigr)$. Therefore $I\bigl(I(A)\bigr) \supset I(A)$.

(6) $\Leftrightarrow$ (7) comes from Lemma 2.2 (2).

(4) $\Rightarrow$ (8) comes from Lemma 2.2 (1).

(8) $\Rightarrow$ (1). Since μ_I is a GT, then ψ_{μ_I} is complete. By [3, 3.3], $I = I_{\psi_{\mu_I}}$ is strong. $\square$

COROLLARY 2.6. *For each strong GIO I on a set X, there is a unique GT μ such that $I = I_\mu$.*

PROOF. The existence of μ comes from Theorem 2.5(6). It suffices to show such μ is unique. In fact, if $I = I_\mu = I_{\mu'}$ for strong GIO I and I', by Theorem 2.1, $\mu = \mu_{I_\mu} = \mu_{I_{\mu'}} = \mu'$. $\square$

Some of the following results are known, we list here for completeness.

THEOREM 2.7. *For an ascending GNS ψ on a set X, we have*
(1) *$\psi = \psi_{I_\psi}$.*
(2) *$\mu_\psi = \mu_{I_\psi}$.*

PROOF. (1) is proved in [3, 3.5].

(2) If $M \in \mu_\psi$, then for each $x \in M$, there is a set $A \in \psi(x)$ such that $A \subset M$. $I_\psi(M) = \{\, x \in M :$ there is a set $V \in \psi(x)$ such that $V \subset M\}$.

Then $I_\psi(M) = M$. So $M \in \mu_{I_\psi}$. Therefore $\mu_\psi \subset \mu_{I_\psi}$. Conversely, if $M \in \mu_{I_\psi}$, we know $M = I_\psi(M)$. That is to say for each $x \in M$, there is a set $V \in \psi(x)$ such that $V \subset M$. So $M \in \mu_\psi$. Therefore $\mu_\psi \supset \mu_{I_\psi}$. □

THEOREM 2.8. *For an ascending GNS ψ on a set X, the following conditions are equivalent.*

(1) *ψ is complete.*
(2) *I_ψ is strong.*
(3) *$I_\psi = I_\mu$ for some GT μ on X.*
(4) *$I_\psi = I_{\mu_\psi}$.*
(5) *$I_\psi = I_{\mu_{I_\psi}}$.*
(6) *$\psi = \psi_{\mu_\psi}$.*
(7) *$\psi = \psi_{I_{\mu_\psi}}$.*
(8) *$\psi = \psi_{\mu_{I_\psi}}$.*

PROOF. The equivalence of (1)–(4) is given in [3, 3.3]. By Theorem 2.7(2), we have (4) ⇔ (5) and (6) ⇔ (8). (1) ⇔ (6) comes from [3, 2.4]. (4) ⇒ (7) comes from Theorem 2.7(1).

(7) ⇒ (1). Since μ_ψ is a GT, then I_{μ_ψ} is strong. By Theorem 2.5(2), $\psi = \psi_{I_{\mu_\psi}}$ is complete. □

Now we can summarize the relationship among generalized topologies, generalized neighborhood systems, and generalized interior operators as the following theorem.

THEOREM 2.9. *The following Diagram II commutes, where CGNS means complete generalized neighborhood system and SGIO means strong generalized interior operator.*

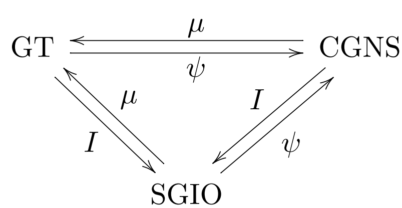

Diagram II

3. On generalized continuous maps

Let us consider a GNS ψ, a GIO I and a GT μ on a set X, a GNS ψ', a GIO I' and a GT μ' on a set X'. A map $f : X \to X'$ is called *(μ, μ')-continuous* if $U \in \mu'$ implies that $f^{-1}(U) \in \mu$. $f : X \to X'$ is called *(ψ, ψ')-continuous* if $x \in X$ and $V \in \psi'(f(x))$ imply $f^{-1}(V) \in \psi(x)$.

$f\colon X \to X'$ is called (I, I')-*continuous* if each $M \subset X'$ implies that $f^{-1}(I'(B)) \subset I(f^{-1}(B))$.

THEOREM 3.1 [3]. *For a GNS ψ on a set X, and a complete GNS ψ' on a set X', $f : X \to X'$, the following are equivalent.*
(1) *f is (ψ, ψ')-continuous.*
(2) *f is $(\mu_\psi, \mu_{\psi'})$-continuous.*
(3) *f is $(I_\psi, I_{\psi'})$-continuous.*

THEOREM 3.2. *For a GIO I on a set X, and a strong GIO I' on a set X', $f : X \to X'$, the following are equivalent.*
(1) *f is (I, I')-continuous.*
(2) *f is $(\mu_I, \mu_{I'})$-continuous.*
(3) *f is $(\psi_I, \psi_{I'})$-continuous.*

PROOF. (1) $\Rightarrow$ (2). If $M \in \mu_{I'}$, then $I(M) = M$. Since f is (I, I')-continuous, $f^{-1}(M) = f^{-1}(I'(M)) \subset I(f^{-1}(M) \subset f^{-1}(M)$. Thus $f^{-1}(M) = I(f^{-1}(M)$. Therefore $f^{-1}(M) \in \mu_I$. So f is $(\mu_I, \mu_{I'})$-continuous.

(2) $\Rightarrow$ (3). Suppose that $x \in X$ and $V \in \psi_{I'}(f(x))$. Then $f(x) \in I'(V)$. Since I' is strong, $I'(V) \in \mu_{I'}$. By (2), $f^{-1}(I'(V)) \in \mu_I$. That is to say $f^{-1}(I'(V)) = I\big(f^{-1}(I'(V))\big)$. Then $x \in f^{-1}(I'(V)) = I\big(f^{-1}(I'(V))\big) \subset I(f^{-1}(V))$, so $f^{-1}(V) \in \psi_I(x)$. f is $(\psi_I, \psi_{I'})$-continuous.

(1) $\Leftrightarrow$ (3) comes from Lemma 2.2(1), Theorem 2.5(2) and Theorem 3.1 (1) $\Leftrightarrow$ (3). $\square$

THEOREM 3.3. *For a GT μ on a set X, and a GT μ' on a set X', $f : X \to X'$, the following are equivalent.*
(1) *f is (μ, μ')-continuous.*
(2) *f is $(\psi_\mu, \psi_{\mu'})$-continuous.*
(3) *f is $(I_\mu, I_{\mu'})$-continuous.*

PROOF. (1) $\Leftrightarrow$ (2) comes from Theorem 2.1(2) and Theorem 3.1 (1) $\Leftrightarrow$ (2).

(1) $\Leftrightarrow$ (3) comes from Theorem 2.1(1) and Theorem 3.3 (1) $\Leftrightarrow$ (2). $\square$

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