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COMPLETE GENERALIZED NEIGHBORHOOD SYSTEMS

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Abstract. We define and study complete generalized neighborhood systems, and prove that a generalized neighborhood system is complete if and only if it can be generated by a generalized topology. Also we obtain some applications of complete generalized neighborhood systems.

1. Introduction

In [2], Császár introduced the notions of generalized neighborhood systems and generalized topologies. He proved that every generalized topology can be generated by a generalized neighborhood system. Naturally, we are interested in the question whether every generalized neighborhood system can be generated by a generalized topology. It will be seen from Remark 2.2 in the present paper that the answer to this question is negative. This leads us to study what kind of generalized neighborhood systems can be generated by generalized topologies. In Section 2, we define complete generalized neighborhood systems and prove that a generalized neighborhood system is complete if and only if it can be generated by a generalized topology. Also we give some characterizations of complete generalized neighborhood systems. In Section 3 and Section 4, we apply complete generalized neighborhood systems into the theory of generalized interior operators and generalized continuous maps.

We recall some basic definitions and notations. Let X be a set, and denote $\exp X$ the power set of X. We call a class $\mu \subset \exp X$ a *generalized topology* [2] (briefly GT) on X if $\emptyset \in \mu$ and any union of elements of μ be-

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longs to μ. A set with a GT is said to be a *generalized topological space* (briefly GTS). For a GTS (X, μ), the elements of μ are called μ-*open sets* and the complements of μ-open sets are called μ-*closed sets*.

An important way for obtaining GT's on a set X is to consider generalized neighborhood systems. According to [2], a map $\psi : X \to \exp(\exp X)$ is called a *generalized neighborhood system* (briefly GNS) on X if for each $x \in X$, $V \in \psi(x)$ imply $x \in V$. If ψ is a GNS on X, let μ_ψ denote the collection of all subsets $M \subset X$ such that $x \in M$ implies the existence of a set $V \in \psi(x)$ satisfying $V \subset M$. By [2, 1.2], μ_ψ is a GT on X. We call μ_ψ the GT *generated by* ψ [2]. Conversely, if μ is a GT on X, then we can define a GNS ψ_μ on X by $\psi(x) = \{A : x \in M \subset A \text{ for some } M \in \mu\}$ $(x \in X)$. We call ψ_μ the GNS *generated by* μ [2].

2. Complete generalized neighborhood systems and generalized topologies

A GNS ψ is called *ascending* [4] if for each $x \in X$ and $A \in \psi(x)$, $A \subset B \subset X$ imply $B \in \psi(x)$.

In [2, 2.3], Á. Császár proved that for each GT μ on X, $\mu = \mu_{\psi_\mu}$, and then each GT on X can be generated by some GNS on X. However, we can see from the following example that for a GNS ψ on a set X, $\psi = \psi_{\mu_\psi}$ does not always hold even when ψ is ascending.

Example 2.1 [4]. Let $X = \{a, b, c\}$, $\psi(a) = \big\{\{a, b\}, X\big\}$, $\psi(b) = \big\{\{b, c\}, X\big\}$, $\psi(c) = \big\{\{a, c\}, X\big\}$. Then ψ is an ascending GNS and clearly $\mu_\psi = \{\emptyset, X\}$, $\psi_{\mu_\psi}(a) = \psi_{\mu_\psi}(b) = \psi_{\mu_\psi}(c) = \{X\}$. Therefore $\psi \neq \psi_{\mu_\psi}$.

Remark 2.2. For the GNS ψ on X defined in the above example, there is no GT μ on X such that ψ is generated by μ. Otherwise, if $\psi = \psi_\mu$, then $\psi_{\mu_\psi} = \psi_{\mu_{\psi_\mu}} = \psi_\mu = \psi$, this is a contradiction.

Now we are going to define a class of GNS's such that each member ψ of which satisfies $\psi = \psi_{\mu_\psi}$, and then ψ can be generated by a GT.

Definition 2.3. A GNS ψ is called *complete* if ψ is ascending and satisfies the following condition $(*)$:

$(*)$ for each $x \in X$ and $A \in \psi(x)$, there is a set O satisfying $x \in O \subset A$, and $y \in O$ implies the existence of a set $B \in \psi(y)$ with $B \subset O$.

For special cases, each neighborhood system of a topological space is a complete GNS, and we shall see from Theorem 2.4 that each GNS generated by a GT is complete.

Theorem 2.4. *For a GNS ψ on a set X, the following conditions are equivalent.*

(1) *ψ is complete.*
(2) $\psi = \psi_{\mu_\psi}$.
(3) *$\psi = \psi_\mu$ for some GT μ on X.*

PROOF. (1) $\Rightarrow$ (2). For each $x \in X$, if $A \in \psi(x)$, by condition $(*)$, we can find a set O satisfying $x \in O \subset A$, and $y \in O$ implies the existence of a set $B \in \psi(y)$ with $B \subset O$. Then, by the definition of μ_ψ, we know $O \in \mu_\psi$, and thus $A \in \psi_{\mu_\psi}(x)$. So $\psi(x) \subset \psi_{\mu_\psi}(x)$. Conversely, if $A \in \psi_{\mu_\psi}(x)$, then there is $M \in \mu_\psi$ such that $x \in M \subset A$. $M \in \mu_\psi$ implies that for each $y \in M$, $y \in V \subset M$ for some $V \in \psi(y)$. Then $M \in \psi(x)$. Since ψ is ascending, $A \in \psi(x)$, which shows that $\psi(x) \supset \psi_{\mu_\psi}(x)$. Therefore, $\psi(x) = \psi_{\mu_\psi}(x)$ for each $x \in X$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). It is easy to see that ψ is ascending. For each $x \in X$ and $A \subset \psi(x) = \psi_\mu(x)$, by the definition of ψ_μ, there is an $M \in \mu$ such that $x \in M \subset A$. Obviously, $M \in \psi_\mu(y) = \psi(y)$ for each $y \in M$. Therefore ψ satisfies $(*)$, and thus ψ is complete. $\square$

The following Corollary 2.5 and Corollary 2.6 reveal the consistency of complete GNS's and GT's. [4] proved that a GT may be generated by several GNS's. We shall see from Corollary 2.5 that among these GNS's there is a unique one which is complete.

COROLLARY 2.5. *For each GT μ on a set X, there is a unique complete GNS ψ such that $\mu = \mu_\psi$.*

PROOF. The existence of ψ comes from [4, 2.3]. It suffices to show such ψ is unique. In fact, if $\mu = \mu_\psi = \mu_{\psi'}$ for complete GNS's ψ and ψ', by Theorem 2.4 [(1) $\Rightarrow$ (2)], $\psi = \psi_{\mu_\psi} = \psi_\mu = \psi_{\mu_{\psi'}} = \psi'$. $\square$

COROLLARY 2.6. *For each complete GNS ψ on a set X, there is a unique GT μ such that $\psi = \psi_\mu$.*

PROOF. By Theorem 2.4 [(1) $\Rightarrow$ (3)], $\psi = \psi_\mu$ for some GT μ on X. Similar to the proof of Corollary 2.5, we know such μ is unique. $\square$

3. On operation ι_ψ

Let us consider another way for obtaining GT's. Let X be a set, $I : \exp X \to \exp X$ be a map. If I satisfies (1) $I(A) \subset A$ for all $A \subset X$ and (2) $A \subset B$ implies $I(A) \subset I(B)$ for all $A, B \subset X$, then we call I a *generalized interior operator* [5] on X. A generalized interior operator I on X is called *strong* [5] if $I\big(I(A)\big) = I(A)$ for all $A \subset X$. For a generalized interior operator I on X, we denote $\mu_I = \big\{A \subset X : A = I(A)\big\}$. According to [1, 1.1], μ_I is a GT on X, which we call the *GT generated by the generalized interior operator I*. Conversely, for a GT μ on a set X, we denote by $i_\mu(A)$ the union of all μ-open sets contained in A for each $A \subset X$. Then i_μ is a generalized interior operator on X, which we call the *generalized interior operator generated by the GT* μ [2].

On the other hand, in [4] Á. Császár defined an operation $\iota_\psi : \exp X \to \exp X$ for each GNS ψ on a set X as: $x \in \iota_\psi(A)$ if and only if there is $V \in \psi(x)$ such that $V \subset A$ $(A \subset X)$. By [4, 3.1], if ψ is a GNS on X, then ι_ψ is a generalized interior operator, which we call the *generalized interior operator generated by the GNS* ψ. [4, 3.8] proved that if ψ is a GNS on X and $A \subset X$ then $i_{\mu_\psi}(A) \subset \iota_\psi(A)$. The subsequent Theorem 3.3 explains when $i_{\mu_\psi} = \iota_\psi$ hold. First we give two lemmas.

Lemma 3.1 [5, 3.3]. *For each GT μ on a set X, i_μ is strong.*

Lemma 3.2. *For each GNS ψ on a set X, $\mu_\psi = \mu_{\iota_\psi}$.*

Proof. For each $A \in \mu_\psi$ and $x \in A$, there is a set $V \in \psi(x)$ such that $V \subset A$. By the definition of ι_ψ, $x \in \iota_\psi(A)$. So $A = \iota_\psi(A)$, thus $A \in \mu_{\iota_\psi}$. Therefore $\mu_\psi \subset \mu_{\iota_\psi}$. Conversely, if $A \in \mu_{\iota_\psi}$, then $A \subset \iota_\psi(A)$. For each $x \in A$, we have $x \in \iota_\psi(A)$, that is, there is a set $V \in \psi(x)$ such that $V \subset A$. So $A \in \mu_\psi$. Therefore $\mu_\psi = \mu_{\iota_\psi}$. □

Theorem 3.3. *For an ascending GNS ψ on a set X, the following conditions are equivalent.*

(1) *ψ is complete.*

(2) *ι_ψ is strong.*

(3) *$\iota_\psi = i_\mu$ for some GT μ on X.*

(4) $\iota_\psi = i_{\mu_\psi}$.

(5) $\iota_\psi = i_{\mu_{\iota_\psi}}$.

Proof. (4) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (2) comes from Lemma 3.1. (4) $\Leftrightarrow$ (5) comes from Lemma 3.2. So it suffices to show (2) $\Rightarrow$ (1) $\Rightarrow$ (4).

(2) $\Rightarrow$ (1). For each $x \in X$ and $A \in \psi(x)$, put $O = \iota_\psi(A)$. By the definition of ι_ψ, $x \in O$. For each $y \in O$, $y \in \iota_\psi(A) = \iota_\psi\big(\iota_\psi(A)\big)$. Then there is a set $V \in \psi(y)$ such that $V \subset \iota_\psi(A) = O$. Therefore ψ satisfies the condition $(*)$.

(1) $\Rightarrow$ (4). By [4, 3.8], it only needs to show $\iota_\psi(A) \subset i_{\mu_\psi}(A)$ for each $A \subset X$. Let $x \in \iota_\psi(A)$. By the definition of ι_ψ, there is a set $V \in \psi(x)$ such that $V \subset A$. By the condition $(*)$, there is a set O satisfying that $x \in O \subset V \subset A$, and $y \in O$ implies the existence of a set $B \in \psi(y)$ with $B \subset O$. Then $O \in \mu_\psi$. So $x \in i_{\mu_\psi}(A)$, which shows $\iota_\psi(A) \subset i_{\mu_\psi}(A)$. □

Remark 3.4. Since there exists a GNS ψ on a set X which is ascending but not complete, $i_{\mu_\psi} = \iota_\psi$ does not always hold in general.

To complete the relationship among GNS, GT and generalized interior operator, we give a way for obtaining GNS's generated by generalized interior operators, which can be seen as the inverse operation of ι. In the following, all GNS's are assumed to be ascending. For a generalized interior operator I on a set X, we define $\psi_I : X \to \exp(\exp X)$ as $A \in \psi_I(x)$ if and only if $x \in I(A)$.

THEOREM 3.5. *For each generalized interior operator I and GNS ψ on a set X, we have*

(1) $I = \iota_{\psi_I}$;
(2) $\psi = \psi_{\iota_I}$;
(3) *I is strong if and only if ψ_I is complete;*
(4) $\mu_I = \mu_{\psi_I}$;
(5) $\psi_I = \psi_{\mu_I}$ *if I is strong.*

PROOF. (1) and (2) come directly from the definitions of ι_ψ and ψ_I. (3) comes from (1), (2) and Theorem 3.3.

(4) By (1) and Lemma 3.2, $\mu_I = \mu_{\iota_{\psi_I}} = \mu_{\psi_I}$.

(5) By (4), $\psi_{\mu_I} = \psi_{\mu_{\psi_I}}$. Since I is strong, ψ_I is complete, so $\psi_{\mu_{\psi_I}} = \psi_I$ by Theorem 2.4, therefore $\psi_I = \psi_{\mu_I}$. □

COROLLARY 3.6. *For each GT μ on a set X, there is a unique strong generalized interior operator I such that $\mu = \mu_I$.*

PROOF. Put $I = \iota_{\psi_\mu}$. Then by Lemma 3.2, $\mu_I = \mu_{\iota_{\psi_\mu}} = \mu_{\psi_\mu} = \mu$. By Theorem 2.4 and Theorem 3.3, I is strong. Let $\mu = \mu_I = \mu_I'$ for strong generalized interior operators I and I' on X. Then by Theorem 3.5 (1) and (5), $I = \iota_{\psi_I} = \iota_{\psi_{\mu_I}} = \iota_{\psi_{\mu_{I'}}} = \iota_{\psi_{I'}} = I'$. □

In contrast to Corollary 2.6, we raise the following question.

QUESTION 3.7. *For each complete strong generalized interior operator I on a set X, is there a unique GT ψ such that $I = i_\mu$?*

4. On continuous maps

Let us consider a GNS ψ and a GT μ on a set X, a GNS ψ' and a GT μ' on a set X'. A map $f : X \to X'$ is called to be (μ, μ')-*continuous* [2] if $U \in \mu'$ implies that $f^{-1}(U) \in \mu$. $f : X \to X'$ is called to be (ψ, ψ')-*continuous* [2] if given $x \in X$ and $V' \in \psi'\big(f(x)\big)$, there is $V \in \psi(x)$ with $f(V) \subset V'$. [2, 2.1] proved that each (ψ, ψ')-continuous map is (μ_ψ, μ'_ψ)-continuous, and inversely not.

THEOREM 4.1. *For a GNS ψ on a set X, and a complete GNS ψ' on a set X', $f : X \to X'$, the following are equivalent.*

(1) *f is (ψ, ψ')-continuous.*
(2) *f is $(\mu_\psi, \mu_{\psi'})$-continuous.*
(3) $f^{-1}(\iota_{\psi'}B) \subset \iota_\psi\big(f^{-1}(B)\big)$ *for each $B \subset X$.*

Proof. (1) $\Leftrightarrow$ (3) by [4, 4.1]. [2, 2.1] proved that (1) $\Rightarrow$ (2). Let us prove (2) $\Rightarrow$ (1).

Suppose that $x \in X$ and $V' \in \psi'\big(f(x)\big)$. Since ψ' is complete, then there is a set $O' \subset X'$ satisfying $f(x) \in O' \subset V'$, and $y' \in O'$ implies the existence of a set $B' \in \psi'(y')$ with $B \subset O$. So $O' \in \mu_{\psi'}$. Put $V = f^{-1}(O')$, then $x \in V \in \mu_\psi$, and thus $V \in \psi(x)$. It is easy to see that $f(V) = f\big(f^{-1}(O')\big) \subset O' \subset V'$. Therefore f is (ψ, ψ')-continuous. $\square$

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