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ON ASCENDING GENERALIZED NEIGHBORHOOD SYSTEMS

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Abstract. Császár introduced the notions of ascending operation, ascending hull of operation and ascending generalized neighborhood system in [2]. In this paper, we introduce the notions of interior and closure operators and (ψ, ψ')-continuity on ascending generalized neighborhood systems. We characterize some properties of such notions in terms of convergence of p-stacks.

1. Introduction

Császár introduced the notions of generalized neighborhood systems and generalized topological spaces [1]. He also introduced the notions of ascending operation, ascending hull of operation and ascending generalized neighborhood system [2]. In this paper, we introduce the notions of interior and closure operators and (ψ, ψ')-continuity on ascending generalized neighborhood systems. We characterize some properties of such notions in terms of convergence of p-stacks. In particular, we show that if $f(\mathcal{E}_\psi) \subseteqq \mathcal{E}_{\psi'}$, then f is (ψ, ψ')-continuous on ascending generalized neighborhood systems ψ, ψ' iff $f(\mathcal{F})$ ψ'-converges to $f(x)$ whenever a p-stack $\mathcal{F}$ ψ-converges to x.

2. Preliminaries

We recall some notions and notations defined in [1]. Let X be a nonempty set and $\exp(X)$ the power set of X. Let $\psi : X \to \exp\big(\exp(X)\big)$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighborhood* of $x \in X$ and ψ is called a *generalized neighborhood system* (briefly GNS) on X.

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Let us consider a map $\gamma : \exp X \to \exp X$. In [2], such a map is called an *operation* on X. If γ is an operation on X, we write γA for $\gamma(A)$. An operation γ is said to be *monotonic* iff $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$, *enlarging* iff $A \subseteq X$ implies $A \subseteq \gamma A$ and *restricting* iff $\gamma A \subseteq A$ for $A \subseteq X$. Let ψ and ψ' be GNS's on X and Y, respectively. Then a function $f : X \to Y$ is said to be (ψ, ψ')-*continuous* if for $x \in X$ and $V \in \psi'\big(f(x)\big)$, there is $U \in \psi(x)$ such that $f(U) \subseteq V$. Let κ be a subset of $\exp(X)$. Then κ is said to be *ascending* [2] (or *stack* [3]) iff for $A \in \kappa$, and $A \subseteq B \subseteq X$ implies $B \in \kappa$. A GNS ψ on X is said to be *ascending* iff $\psi(x)$ is ascending whenever $x \in X$ [2]. For a subset κ of $\exp(X)$, the ascending $\kappa^+ = \{B \subseteq X\colon$ there is $A \in \kappa$ such that $A \subseteq B\}$ is called the ascending hull of κ [2].

An ascending set $\mathcal{A}$ of $\exp(X)$ is called a *p-stack* [3] if it satisfies the condition $P, Q \in \mathcal{A}$ implies $P \cap Q \neq \emptyset$.

3. Results

DEFINITION 3.1. Let X be a nonempty set and $\psi : X \to \exp\big(\exp(X)\big)$ an ascending generalized neighborhood system.

$$\iota_\psi(A) = \big\{x \in A : A \in \psi(x)\big\};$$

$$\gamma_\psi(A) = \big\{x \in X : A \cap V \neq \emptyset, \text{ for every } V \in \psi(x)\big\}.$$

THEOREM 3.2. *Let ψ be an ascending generalized neighborhood system on X. Then*

(1) $\gamma_\psi(X - A) = X - \iota_\psi(A)$.

(2) $\iota_\psi(X - A) = X - \gamma_\psi(A)$.

PROOF. (1) If $x \in \gamma_\psi(X - A)$ then $(X - A) \cap V \neq \emptyset$ for every $V \in \psi(x)$, say $\psi(x) \neq \emptyset$, hence there is no element $V \in \psi(x)$ such that $V \subseteq A$. This implies $A \notin \psi(x)$ and so $x \notin \iota_\psi(A)$. Hence $x \in X - \iota_\psi(A)$.

For the converse, let $x \notin \gamma_\psi(X - A)$. Then there is an element $V \in \psi(x)$ such that $(X - A) \cap V = \emptyset$. So $V \subseteq A$ and since $\psi(x)$ is ascending, $A \in \psi(x)$. From definition of the operation ι_ψ, we have $x \in \iota_\psi(A)$, hence $x \notin X - \iota_\psi(A)$.

(2) It is similar to the proof of (1). $\square$

THEOREM 3.3. *Let ψ be an ascending generalized neighborhood system on X. Then ι_ψ is monotonic and restricting.*

PROOF. If $A \subseteq B$ and $x \in \iota_\psi(A)$, then $A \in \psi(x)$, since $\psi(x)$ is ascending, $B \in \psi(x)$ and so $x \in \iota_\psi(B)$. Hence ι_ψ is monotonic.

Let $x \in \iota_\psi(A)$ for $A \subseteq X$. Then $A \in \psi(x)$. Since ψ is a generalized neighborhood system, $x \in A$, so that ι_ψ is restricting. $\square$

THEOREM 3.4. *Let ψ be an ascending generalized neighborhood system on X. Then γ_ψ is monotonic and enlarging.*

PROOF. If $A \subseteqq B$ and $x \in \gamma_\psi(A)$, then from definition of γ_ψ, obviously γ_ψ is monotonic.

Let $x \notin \gamma_\psi(A)$ for $A \subseteqq X$. Then there is a $V \in \psi(x)$ such that $V \cap A = \emptyset$. Since $x \in V$, $x \notin A$, so that γ_ψ is enlarging. $\square$

For ascending sets κ and λ of $\exp(X)$, if there are $F \in \kappa$, $G \in \lambda$ such that $G \cap F = \emptyset$, we say that κ and λ are *disjoint.*

LEMMA 3.5. *For ascending sets κ and λ of* $\exp(X)$*, if κ and λ are not disjoint, then $\kappa \vee \lambda = \{F \cap G : F \in \kappa, G \in \lambda\}^+$ is an ascending set containing κ, λ.*

DEFINITION 3.6. Let ψ be an ascending generalized neighborhood system on X. A p-stack $\mathcal{F}$ on X ψ-converges to x if $\psi(x) \subseteqq \mathcal{F}$.

THEOREM 3.7. *Let X be a nonempty set and $\psi : X \to \exp\big(\exp(X)\big)$ an ascending generalized neighborhood system.*

(1) $\iota_\psi(A) = \{x \in A : A \in \mathcal{F}$*, for every p-stack $\mathcal{F}$ ψ-converging to* $x\}$*;*

(2) $\gamma_\psi(A) = \{x \in X :$ *there is a p-stack $\mathcal{F}$ such that $\mathcal{F}$ ψ-converges to x and* $A \in \mathcal{F}\}$.

PROOF. (1) If $x \in \iota_\psi(A)$ and a p-stack $\mathcal{F}$ ψ-converges to x, then $A \in \psi(x)$ and $\psi(x) \subseteqq \mathcal{F}$, so $x \in A$ and $A \in \mathcal{F}$.

For the converse, let $\psi(x)$ be nonempty for $x \in X$. Then $\psi(x)$ is a p-stack and it ψ-converges to x. By hypothesis, $A \in \psi(x)$. Hence $x \in \iota_\psi(A)$.

(2) Let $x \in \gamma_\psi(A)$. Then $V \cap A \neq \emptyset$ for every $V \in \psi(x)$. Put

$$\mathcal{F} = \begin{cases} \{A\}^+, & \text{if } \ \psi(x) = \emptyset, \\ \psi(x) \vee \{A\}^+, & \text{if } \ \psi(x) \neq \emptyset, \end{cases}$$

then $\mathcal{F}$ is a p-stack such that it ψ-converges to x and $A \in \mathcal{F}$.

Suppose there is a p-stack $\mathcal{F}$ such that it ψ-converges to x and $A \in \mathcal{F}$. Since $\psi(x) \subseteqq \mathcal{F}$ and $\mathcal{F}$ is a p-stack, $V \cap A \neq \emptyset$ for every $V \in \psi(x)$, so this implies $x \in \gamma_\psi(A)$. $\square$

DEFINITION 3.8. A function $f : (X, \psi) \to (Y, \psi')$ on ascending generalized neighborhood systems ψ, ψ' is said to be (ψ, ψ')-*continuous* if for $x \in X$ and for each $V \in \psi'\big(f(x)\big)$, $f^{-1}(V) \in \psi(x)$.

THEOREM 3.9. *Let $f : (X, \psi) \to (Y, \psi')$ be a function on ascending generalized neighborhood systems ψ, ψ'. Then the following are equivalent:*

(1) *f is (ψ, ψ')-continuous.*

(2) *For $x \in X$ and for each $V \in \psi'\big(f(x)\big)$, there is $U \in \psi(x)$ such that $f(U) \subseteqq V$.*

(3) $f^{-1}\big(\iota_{\psi'}(B)\big) \subseteqq \iota_\psi\big(f^{-1}(B)\big)$ *for* $B \subseteqq Y$.
(4) $\gamma_\psi\big(f^{-1}(B)\big) \subseteqq f^{-1}\big(\gamma_{\psi'}(B)\big)$ *for* $B \subseteqq Y$.
(5) $f\big(\gamma_\psi(A)\big) \subseteqq \gamma_{\psi'}\big(f(A)\big)$ *for* $A \subseteqq X$.

PROOF. (1) $\Rightarrow$ (2). For $x \in X$, let $V \in \psi'\big(f(x)\big)$. From (ψ, ψ')-continuity of f, $f^{-1}(V) \in \psi(x)$. Put $U = f^{-1}(V)$; then $U \in \psi(x)$ and $f(U) \subseteqq V$. Thus (2) is obtained.

(2) $\Rightarrow$ (3). For $B \subseteqq Y$, let $x \in f^{-1}\big(\iota_{\psi'}(B)\big)$. Then $f(x) \in \iota_{\psi'}(B)$ and so $B \in \psi'\big(f(x)\big)$. By (2), there is $U \in \psi(x)$ such that $f(U) \subseteqq B$, since $\psi(x)$ is ascending, it is $f^{-1}(B) \in \psi(x)$. Hence $x \in \iota_\psi\big(f^{-1}(B)\big)$.

(3) $\Rightarrow$ (1). For $x \in X$, let $V \in \psi'\big(f(x)\big)$. Then we know that $f(x) \in \iota_{\psi'}(V)$ and so $x \in f^{-1}\big(\iota_{\psi'}(V)\big)$. It follows $x \in \iota_\psi\big(f^{-1}(V)\big)$ from (3). This implies $f^{-1}(V) \in \psi(x)$. Hence f is (ψ, ψ')-continuous.

(3) $\Leftrightarrow$ (4). It is obtained from Theorem 3.2.

(4) $\Leftrightarrow$ (5). Obvious. □

Let $f : X \to Y$ be a function. For an ascending set $\mathcal{F}$ of $\exp(X)$, let us define $f(\mathcal{F}) = \big\{ f(K) : K \in \mathcal{F} \big\}^+$.

LEMMA 3.10. *Let* $f : X \to Y$ *be a function and* $\mathcal{F}$ *an ascending set of* $\exp(X)$. *Then* $U \in f(\mathcal{F})$ *iff there exists* $K \in \mathcal{F}$ *such that* $f(K) \subseteqq U$.

PROOF. Obvious. □

THEOREM 3.11. *Let* $f : (X, \psi) \to (Y, \psi')$ *be a function on ascending generalized neighborhood systems* ψ, ψ'. *Then if* f *is* (ψ, ψ')*-continuous, then* $f(\mathcal{F})$ ψ'*-converges to* $f(x)$ *whenever a* p*-stack* $\mathcal{F}$ ψ*-converges to* x.

PROOF. Suppose f is (ψ, ψ')-continuous and a p-stack $\mathcal{F}$ ψ-converges to x. Let $\psi'\big(f(x)\big) \neq \emptyset$. Then for each $V \in \psi'\big(f(x)\big)$, by Theorem 3.9 (2), there is some $U \in \psi(x)$ such that $f(U) \subseteqq V$. Since $\mathcal{F}$ ψ-converges to x, we can say that there is $U \in \mathcal{F}$ such that $f(U) \subseteqq V$. So by Lemma 3.10, V is an element in the ascending set $f(\mathcal{F})$. Hence $f(\mathcal{F})$ ψ'-converges to $f(x)$. □

In Theorem 3.11, the converse is not always true as shown in the next example.

EXAMPLE 3.12. Let $X = \{a, b, c\}$. Consider two ascending generalized neighborhood systems ψ and ψ' defined as the following:

$$\psi(a) = \emptyset, \quad \psi(b) = \{X\}, \quad \psi(c) = \{X\};$$

$$\psi'(a) = \{X\}, \quad \psi'(b) = \{X\}, \quad \psi'(c) = \{X\}.$$

Let us consider the identity function $f : (X, \psi) \to (X, \psi')$. Then for every $x \in X$, $f(\mathcal{F})$ ψ'-converges to $f(x)$ whenever a p-stack $\mathcal{F}$ ψ-converges to x. But f is not (ψ, ψ')-continuous at a.

Let X be a nonempty set and $\psi : X \to \exp\big(\exp(X)\big)$ an ascending generalized neighborhood system. Set $\mathcal{E}_\psi = \big\{ x \in X : \psi(x) = \emptyset \big\}$.

THEOREM 3.13. *Let $f : (X, \psi) \to (Y, \psi')$ be a function on ascending generalized neighborhood systems ψ, ψ' and $f(\mathcal{E}_\psi) \subseteqq \mathcal{E}_{\psi'}$. Then f is (ψ, ψ')-continuous if and only if $f(\mathcal{F})$ ψ'-converges to $f(x)$ whenever a p-stack $\mathcal{F}$ ψ-converges to x.*

PROOF. From Theorem 3.11, it is sufficient to show that if $f(\mathcal{F})$ ψ'-converges to $f(x)$ whenever a p-stack $\mathcal{F}$ ψ-converges to x, then f is (ψ, ψ')-continuous. For the proof, suppose $\psi'\big(f(x)\big) \neq \emptyset$ and $V \in \psi'\big(f(x)\big)$. Then from $f(\mathcal{E}_\psi) \subseteqq \mathcal{E}_{\psi'}$, we can say $\psi(x) \neq \emptyset$. Since $\psi(x)$ ψ-converges to x, by hypothesis, we have $V \in \psi'\big(f(x)\big) \subseteqq f\big(\psi(x)\big)$. Thus for $V \in f\big(\psi(x)\big)$, by Lemma 3.10, there is $U \in \psi(x)$ such that $f(U) \subseteqq V$. Hence from Theorem 3.9 (2), f is (ψ, ψ')-continuous. □

Let g be a collection of subsets of X. Then g is called a *generalized topology* [1] on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in g$. The elements of g are called *g-open* sets and the complements are called *g-closed* sets. Let ψ be a GNS on X and $G \in g_\psi$ iff $G \subset X$ satisfies: if $x \in G$ then there is $V \in \psi(x)$ such that $V \subset G$. Let g and g' be generalized topologies on X and Y, respectively. Then a function $f : X \to Y$ is said to be *(g, g')-continuous* [1] if $G' \in g'$ implies that $f^{-1}(G') \in g$.

THEOREM 3.14. *Let $f : (X, \psi) \to (Y, \psi')$ be a function on ascending generalized neighborhood systems ψ, ψ'. Then if f is (ψ, ψ')-continuous, then it is $(g_\psi, g_{\psi'})$-continuous.*

PROOF. If $V \in g_{\psi'}$, then for each $x \in f^{-1}(V)$, there is $U \in \psi'\big(f(x)\big)$ such that $f(x) \in U \subseteqq V$. Since f is (ψ, ψ')-continuous, $f^{-1}(U) \in \psi(x)$ and from $f^{-1}(U) \subseteqq f^{-1}(V)$, we know that $f^{-1}(V) \in g_\psi$. Hence f is $(g_\psi, g_{\psi'})$-continuous. □

EXAMPLE 3.15. Let $X = \{a, b, c, d\}$. Consider two ascending generalized neighborhood systems ψ and ψ' defined as the following:

$$\psi(a) = \{X\}, \quad \psi(b) = \{X\}, \quad \psi(c) = \{X\}, \quad \psi(d) = \emptyset;$$

$$\psi'(a) = \big\{ \{a, b\}, X \big\}, \quad \psi'(b) = \{X\}, \quad \psi'(c) = \{X\}, \quad \psi'(d) = \emptyset.$$

Then $g_\psi = g_{\psi'} = \big\{ \emptyset, \{a, b, c\} \big\}$.

Let us consider the identity function $f : (X, \psi) \to (X, \psi')$. Then f is $(g_\psi, g_{\psi'})$-continuous but not (ψ, ψ')-continuous.

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