

HYPERSTABILITY OF A FUNCTIONAL EQUATION

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Abstract. The aim of this paper is to prove that the parametric fundamental equation of information is hyperstable on its open as well as on its closed domain, assuming that the parameter is negative. As a corollary of the main result, it is also proved that the system of equations that defines the alpha-recursive information measures is stable.

1. Introduction

The stability theory of functional equations mainly deals with the following question: Is it true that the solution of a given equation differing slightly from a given one, must necessarily be close to the solution of the equation in question? In case of a positive answer we say that the equation in question is *stable* (see e.g. Forti [4]). The investigation of the multiplicative Cauchy equation highlighted a new phenomenon, which is nowadays called *superstability* (see e.g. Moszner [11]). In this case the so-called stability inequality implies that the function in question is either bounded or it is the solution

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of the functional equation itself. Furthermore, it can happen that there is no such alternative, that is, all the solutions of the stability inequality are exactly the solutions of the functional equation. In this case we speak about *hyperstability* (see e.g. Maksa–Páles [10]).

In this paper we shall prove that the parametric fundamental equation of information, that is,

$$(1.1) \quad f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^\alpha f\left(\frac{x}{1-y}\right)$$

is hyperstable, provided that the parameter α is negative. Equation (1.1) arises in the characterization problem of the 3-semi-symmetric, α -recursive information measures (see Aczél–Daróczy [1] and Ebanks–Sahoo–Sander [3]).

Throughout this paper we will use the following notations. Let \mathbb{R} denote the set of the real numbers and for each $n = 2, 3, \dots$ define the sets D_n and \overline{D}_n by

$$D_n = \left\{ (p_1, \dots, p_n) \in \mathbb{R} \mid p_1, \dots, p_n, \sum_{i=1}^n p_i \in]0, 1[\right\}.$$

and

$$\overline{D}_n = \left\{ (p_1, \dots, p_n) \in \mathbb{R} \mid p_1, \dots, p_n \in [0, 1[, \sum_{i=1}^n p_i \leq 1 \right\},$$

respectively.

In the investigations (1.1) is supposed to hold on D_2 or on \overline{D}_2 . The general solution of equation (1.1) is contained in the following theorem (see also Maksa [8] and Ebanks–Sahoo–Sander [3]).

THEOREM 1.1. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 1$ and $f :]0, 1[\rightarrow \mathbb{R}$ be a function. Then the functional equation (1.1) holds for all $(x, y) \in D_2$ if and only if there exist $c, d \in \mathbb{R}$ such that*

$$f(x) = cx^\alpha + d(1-x)^\alpha - d$$

holds for all $x \in]0, 1[$.

In [9] Maksa proved that (1.1) is superstable on \overline{D}_2 if $1 \neq \alpha > 0$, but this method is inappropriate on the set D_2 as well as if the parameter α is negative. It was conjectured that equation (1.1) is hyperstable but finally stability could be proved (see [6]). However, this method was appropriate to prove superstability for the case $1 \neq \alpha > 0$. This unified method can be found in [5]. Recently it turned out that with a different method we can prove hyperstability for equation (1.1) on D_2 as well as on \overline{D}_2 , if $\alpha < 0$.

The next section will be devoted to this result.

All in all, equation (1.1) is superstable if $1 \neq \alpha > 0$, it is stable, provided that $\alpha = 0$ and it is hyperstable assuming that $\alpha < 0$ on the set D_2 . As to the case $\alpha = 1$, it was raised by Székelyhidi at the 28th International Symposium on Functional Equations (see [12]), and it is still open.

2. Main result

Our main result is contained in the following theorem.

THEOREM 2.1. *Let $\alpha, \varepsilon \in \mathbb{R}$, $\alpha < 0$, $\varepsilon \geq 0$ and $f :]0, 1[\rightarrow \mathbb{R}$ be a function. Assume that*

$$(2.1) \quad \left| f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) - f(y) - (1-y)^\alpha f\left(\frac{x}{1-y}\right) \right| \leq \varepsilon$$

holds for all $(x, y) \in D_2$. Then, and only then, there exist $c, d \in \mathbb{R}$ such that

$$(2.2) \quad f(x) = cx^\alpha + d(1-x)^\alpha - d$$

for all $x \in]0, 1[$.

PROOF. Due to Theorem 1.1, in case the function f is given by formula (2.2), then

$$f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^\alpha f\left(\frac{x}{1-y}\right)$$

holds for all $(x, y) \in D_2$. Thus inequality (2.1) is also satisfied with arbitrary $\varepsilon \geq 0$. Therefore it is enough to prove the converse direction.

Define the function $G : D_2 \rightarrow \mathbb{R}$ by

$$(2.3) \quad G(x, y) = f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) - f(x+y) \quad ((x, y) \in D_2).$$

Then inequality (2.1) immediately implies that

$$(2.4) \quad |G(x, y) - G(y, x)| \leq \varepsilon$$

for all $(x, y) \in D_2$.

Let $(x, y, z) \in D_3$, then due to the definition of the function G ,

$$G(x+y, z) = f(x+y) + (1-(x+y))^\alpha f\left(\frac{z}{1-(x+y)}\right) - f(x+y+z),$$

$$G(x, y + z) = f(x) + (1 - x)^\alpha f\left(\frac{y + z}{1 - x}\right) - f(x + y + z)$$

and

$$\begin{aligned} & (1 - x)^\alpha G\left(\frac{y}{1 - x}, \frac{z}{1 - x}\right) \\ &= (1 - x)^\alpha \left[f\left(\frac{y}{1 - x}\right) + \left(1 - \frac{y}{1 - x}\right)^\alpha f\left(\frac{\frac{z}{1 - x}}{1 - \frac{y}{1 - x}}\right) - f\left(\frac{y + z}{1 - x}\right) \right], \end{aligned}$$

that is

$$(2.5) \quad G(x, y) + G(x + y, z) = G(x, y + z) + (1 - x)^\alpha G\left(\frac{y}{1 - x}, \frac{z}{1 - x}\right)$$

holds on D_3 , where we used the identity

$$\frac{z}{1 - (x + y)} = \frac{\frac{z}{1 - x}}{1 - \frac{y}{1 - x}}$$

also.

In what follows we will show that the function G is α -homogeneous. Indeed, interchanging x and y in (2.5), we get

$$\begin{aligned} & G(y, x) + G(x + y, z) \\ &= G(y, x + z) + (1 - y)^\alpha G\left(\frac{x}{1 - y}, \frac{z}{1 - y}\right) \quad ((x, y, z) \in D_3). \end{aligned}$$

Furthermore, equation (2.5) with the substitution $(x, y, z) = (y, z, x)$ yields that

$$G(y, z) + G(y + z, x) = G(y, x + z) + (1 - y)^\alpha G\left(\frac{z}{1 - y}, \frac{x}{1 - y}\right)$$

is fulfilled for all $(x, y, z) \in D_3$.

Thus

$$\begin{aligned} (2.6) \quad & G(y, z) - (1 - x)^\alpha G\left(\frac{y}{1 - x}, \frac{z}{1 - x}\right) \\ &= \left\{ G(x, y) + G(x + y, z) - G(x, y + z) - (1 - x)^\alpha G\left(\frac{y}{1 - x}, \frac{z}{1 - x}\right) \right\} \end{aligned}$$

$$\begin{aligned}
& -G(x, y) - G(x + y, z) + G(x, y + z) \\
& + \left\{ G(y, x) + G(x + y, z) - G(y, x + z) - (1 - y)^\alpha G\left(\frac{x}{1 - y}, \frac{z}{1 - y}\right) \right\} \\
& + \left\{ G(y, z) + G(y + z, x) - G(y, x + z) - (1 - y)^\alpha G\left(\frac{z}{1 - y}, \frac{x}{1 - y}\right) \right\} \\
& - G(y + z, x) + G(y, x + z) + (1 - y)^\alpha G\left(\frac{z}{1 - y}, \frac{x}{1 - y}\right) \\
& = G(y, x) - G(x, y) + G(x, y + z) - G(y + z, x) \\
& + (1 - y)^\alpha \left(G\left(\frac{z}{1 - y}, \frac{x}{1 - y}\right) - G\left(\frac{x}{1 - y}, \frac{z}{1 - y}\right) \right)
\end{aligned}$$

for all $(x, y, z) \in D_3$, since the expressions in the curly brackets are zeros. Thus (2.6), (2.4) and the triangle inequality imply that

$$(2.7) \quad \left| G(y, z) - (1 - x)^\alpha G\left(\frac{y}{1 - x}, \frac{z}{1 - x}\right) \right| \leq (2 + (1 - y)^\alpha) \varepsilon$$

is fulfilled for all $(x, y, z) \in D_3$. Given any $t \in]0, 1[$, $(u, v) \in D_2$, let

$$x = 1 - t, \quad y = tu \quad \text{and} \quad z = tv.$$

Then $x, y, z \in]0, 1[$ and

$$x + y + z = 1 - t(1 - u - v) \in]0, 1[,$$

that is $(x, y, z) \in D_3$, and inequality (2.7) implies that

$$\left| G(tu, tv) - t^\alpha G(u, v) \right| \leq (2 + (1 - tu)^\alpha) \varepsilon,$$

or, after rearranging,

$$\left| \frac{G(tu, tv)}{t^\alpha} - G(u, v) \right| \leq \frac{(2 + (1 - tu)^\alpha)}{t^\alpha} \varepsilon$$

holds for arbitrary $t \in]0, 1[$ and $(u, v) \in D_2$. Taking the limit $t \rightarrow 0+$ we obtain that

$$\lim_{t \rightarrow 0+} \frac{G(tu, tv)}{t^\alpha} = G(u, v) \quad ((u, v) \in D_2),$$

since $\lim_{t \rightarrow 0+} (1 - tu)^\alpha = 1$ for all $u \in]0, 1[$ and $\lim_{t \rightarrow 0+} t^{-\alpha} = 0$, since $\alpha < 0$. This implies that the function G is α -homogeneous on D_2 . Indeed, for arbitrary $s \in]0, 1[$ and $(u, v) \in D_2$

$$(2.8) \quad \begin{aligned} G(su, sv) &= \lim_{t \rightarrow 0+} \frac{G(t(su), t(sv))}{t^\alpha} \\ &= s^\alpha \lim_{t \rightarrow 0+} \frac{G((ts)u, (ts)v)}{(ts)^\alpha} = s^\alpha G(u, v). \end{aligned}$$

At this point of the proof we will show that inequality (2.4) and equation (2.8) together imply the symmetry of the function G . Indeed, due to (2.4)

$$|G(tx, ty) - G(ty, tx)| \leq \varepsilon$$

holds for all $(x, y) \in D_2$ and $t \in]0, 1[$. Using the α -homogeneity of the function G , we obtain that

$$|t^\alpha G(x, y) - t^\alpha G(y, x)| \leq \varepsilon, \quad ((x, y) \in D_2, t \in]0, 1[)$$

or, if we rearrange this,

$$|G(x, y) - G(y, x)| \leq \frac{\varepsilon}{t^\alpha}$$

holds for all $(x, y) \in D_2$ and $t \in]0, 1[$. Taking the limit $t \rightarrow 0+$, we get that $G(x, y) = G(y, x)$ is fulfilled for all $(x, y) \in D_2$, since $\alpha < 0$. Therefore the function G is symmetric. Due to definition (2.3) this implies that

$$f(x) + (1 - x)^\alpha f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)^\alpha f\left(\frac{x}{1 - y}\right) \quad ((x, y) \in D_2),$$

i.e., the function f satisfies the parametric fundamental equation of information on D_2 . Thus by Theorem 1.1 there exist $c, d \in \mathbb{R}$ such that

$$f(x) = cx^\alpha + d(1 - x)^\alpha - d$$

holds for all $x \in]0, 1[$. \square

3. Corollaries

The first corollary of this section says that equation (1.1) is hyperstable not only on D_2 but also on \overline{D}_2 .

THEOREM 3.1. *Let $\alpha, \varepsilon \in \mathbb{R}$ be fixed, $\alpha < 0$, $\varepsilon \geq 0$. Then the function $f : [0, 1] \rightarrow \mathbb{R}$ satisfies the inequality (2.1) for all $(x, y) \in \overline{D}_2$ if, and only if, there exist $c, d \in \mathbb{R}$ such that*

$$(3.1) \quad f(x) = \begin{cases} 0, & \text{if } x = 0 \\ cx^\alpha + d(1-x)^\alpha - d, & \text{if } x \in]0, 1[\\ c - d, & \text{if } x = 1. \end{cases}$$

PROOF. Let $y = 0$ in (2.1). Then we have that

$$((1-x)^\alpha + 1) |f(0)| \leq \varepsilon \quad (x \in]0, 1[)$$

Since $\alpha < 0$, this yields that $f(0) = 0$. On the other hand, by Theorem 2.1,

$$f(x) = cx^\alpha + d(1-x)^\alpha - d \quad (x \in]0, 1[)$$

with some $c, d \in \mathbb{R}$. Finally, let $x \in]0, 1[$ and $y = 1 - x$ in (2.1). Then, again by Theorem 2.1, there exist $c, d \in \mathbb{R}$ such that

$$|c - d - f(1)| |x^\alpha - (1-x)^\alpha| \leq \varepsilon.$$

Since $\alpha < 0$, $f(1) = c - d$ follows.

The converse is an easy computation and it turns out that f defined by (3.1) is a solution of (1.1) on \overline{D}_2 . \square

The second corollary concerns a system of equations.

For fixed $\alpha < 0$ and $2 \leq n \in \mathbb{N}$ define the set

$$\Gamma_n^\circ = \left\{ (p_1, \dots, p_n) \mid p_i \in]0, 1[, i = 1, \dots, n, \sum_{i=1}^n p_i = 1 \right\},$$

and the function $H_n^\alpha : \Gamma_n^\circ \rightarrow \mathbb{R}$ by

$$H_n^\alpha(p_1, \dots, p_n) = (2^{1-\alpha} - 1)^{-1} \cdot \left(\sum_{i=1}^n p_i^\alpha - 1 \right).$$

The sequence (H_n^α) is called *the entropy of degree α* , and it was investigated among others by Daróczy [2], Havrda–Charvát [7] and Tsallis [13], from several points of view.

About this topic the reader could consult the monograph of Aczél and Daróczy [1]. However, we will use only the following properties.

The sequence of functions $I_n : \Gamma_n^\circ \rightarrow \mathbb{R}$ ($n = 2, 3, \dots$) is called α -recursive, if

$$\begin{aligned} & I_{n+1}(p_1, \dots, p_{n+1}) \\ &= I_n(p_1 + p_2, p_3, \dots, p_{n+1}) + (p_1 + p_2)^\alpha I_n \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right) \end{aligned}$$

holds for all $n \geq 2$, $(p_1, \dots, p_{n+1}) \in \Gamma_{n+1}^\circ$.

Furthermore, the sequence of functions $I_n : \Gamma_n^\circ \rightarrow \mathbb{R}$ ($n = 2, 3, \dots$) is called 3-semi-symmetric, if

$$I_3(p_1, p_2, p_3) = I_3(p_1, p_3, p_2)$$

holds on Γ_3° .

It is easy to see that for all $a, b \in \mathbb{R}$ the sequence of functions

$$(p_1, \dots, p_n) \mapsto aH_n^\alpha(p_1, \dots, p_n) + b(p_1^\alpha - 1) \quad ((p_1, \dots, p_n) \in \Gamma_n^\circ)$$

is α -recursive and 3-semi-symmetric. Therefore, the second corollary can be considered as a stability theorem for a system of equations.

THEOREM 3.2. *Let $n \geq 2$ be a fixed positive integer, (I_n) be the sequence of functions $I_n : \Gamma_n^\circ \rightarrow \mathbb{R}$ and suppose that there exist a sequence (ε_n) of non-negative real numbers and a real number $\alpha < 0$ such that*

$$(3.2) \quad \left| I_n(p_1, \dots, p_n) - I_{n-1}(p_1 + p_2, p_3, \dots, p_n) - (p_1 + p_2)^\alpha I_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right) \right| \leq \varepsilon_{n-1}$$

holds for all $n \geq 3$ and $(p_1, \dots, p_n) \in \Gamma_n^\circ$, and

$$(3.3) \quad |I_3(p_1, p_2, p_3) - I_3(p_1, p_3, p_2)| \leq \varepsilon,$$

holds on Γ_3° . Then there exist $a, b \in \mathbb{R}$ such that

$$(3.4) \quad \left| I_n(p_1, \dots, p_n) - (aH_n^\alpha(p_1, \dots, p_n) + b(p_1^\alpha - 1)) \right| \leq \sum_{k=2}^{n-1} \varepsilon_k$$

for all $n \geq 2$ and $(p_1, \dots, p_n) \in \Gamma_n^\circ$, where the convention $\sum_{k=2}^1 \varepsilon_k = 0$ is adopted.

PROOF. As in Maksa [9], it can be proved that, due to (3.2) and (3.3), for the function f defined by $f(x) = I_2(1 - x, x)$, $x \in]0, 1[$ we get that

$$\left| f(x) + (1 - x)^\alpha f\left(\frac{y}{1 - x}\right) - f(y) - (1 - y)^\alpha f\left(\frac{x}{1 - y}\right) \right| \leq 2\varepsilon_2 + \varepsilon_1$$

for all $(x, y) \in D_2$, i.e., (2.1) holds with $\varepsilon = 2\varepsilon_2 + \varepsilon_1$. Therefore, applying Theorem 2.1, we obtain (2.2) with some $c, d \in \mathbb{R}$, i.e.,

$$I_2(1 - x, x) = cx^\alpha + d(1 - x)^\alpha - d \quad (x \in]0, 1[)$$

i.e., (3.4) holds for $n = 2$ with $a = (2^{1-\alpha} - 1)c$, $b = d - c$.

We continue the proof by induction on n . Suppose that (3.4) holds and, for the sake of brevity, introduce the notation

$$J_n(p_1, \dots, p_n) = aH_n^\alpha(p_1, \dots, p_n) + b(p_1^\alpha - 1)$$

for all $n \geq 2$, $(p_1, \dots, p_n) \in \Gamma_n^\circ$. It can be easily seen that (3.2) and (3.3) hold on Γ_n° for J_n instead of I_n ($n \geq 3$) with $\varepsilon_n = 0$ ($n \geq 2$). Thus for all $(p_1, \dots, p_{n+1}) \in \Gamma_{n+1}^\circ$, we get that

$$\begin{aligned} & I_{n+1}(p_1, \dots, p_{n+1}) - J_{n+1}(p_1, \dots, p_{n+1}) \\ &= I_{n+1}(p_1, \dots, p_{n+1}) - J_n(p_1 + p_2, p_3, \dots, p_{n+1}) \\ &\quad - (p_1 + p_2)^\alpha J_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) \\ &= I_{n+1}(p_1, \dots, p_{n+1}) - I_n(p_1 + p_2, p_3, \dots, p_{n+1}) \\ &\quad - (p_1 + p_2)^\alpha I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) \\ &\quad + I_n(p_1 + p_2, p_3, \dots, p_{n+1}) - J_n(p_1 + p_2, p_3, \dots, p_{n+1}) \\ &\quad + (p_1 + p_2)^\alpha I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) - (p_1 + p_2)^\alpha J_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right). \end{aligned}$$

Therefore (3.4) with $n = 2$ and the induction hypothesis imply that

$$\left| I_{n+1}(p_1, \dots, p_{n+1}) - J_n(p_1, \dots, p_{n+1}) \right| \leq \varepsilon_n + \sum_{k=2}^{n-1} \varepsilon_k = \sum_{k=2}^n \varepsilon_k,$$

that is, (3.4) holds for $n + 1$ in place of n . \square

COROLLARY 3.3. *Applying Theorem 3.2 with the choice $\varepsilon_n = 0$ for all $n \in \mathbb{N}$, we get the α -recursive, 3-semi-symmetric information measures. Hence the previous theorem says that the system of α -recursive and 3-semi-symmetric information measures is stable.*

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