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PRODUCT OF GENERALIZED TOPOLOGIES

Á. CSÁSZÁR*

Department of Analysis, Eötvös Loránd University, H-1117 Budapest, Pázmány P. sétány 1/C

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Abstract. The definition of the product of topologies is generalized in such a way that topologies are replaced by generalized topologies in the sense of [3].

0. Introduction

Let $X = \prod_{k \in K} X_k$ be a nonempty set. A well-known construction in general topology is the following: given a topology τ_k on X_k, a topology τ on X, called the *product* of the topologies τ_k, is constructed (see e.g. [1]).

A generalization of the concept of topology is that of generalized topology; according to [3], a *generalized topology* (briefly GT) on X is a subset μ of the power set $\exp X$ such that $\emptyset \in \mu$ and every union of some elements of μ belongs to μ. Of course every topology is a GT.

The purpose of the present paper is to show how the definition of the product of topologies can be modified in order to define the product of GT's. We also consider some properties of the product GT and present some applications.

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1. Preliminaries

Let μ be a GT on a set $X \neq \emptyset$. Observe that $X \in \mu$ must not hold; if all the same $X \in \mu$ then we say that the GT μ is *strong* (see [5]). In general, let M_μ denote the union of all elements of μ; of course, $M_\mu \in \mu$, and $M_\mu = X$ iff μ is a strong GT.

We call $\gamma : \exp X \to \exp X$ an *operation* (see [7]) iff it is *monotonic* (i.e. $A \subset B \subset X$ implies $\gamma A \subset \gamma B$; for an operation γ we write γA instead of $\gamma(A)$). If both γ and γ' are operations, we write simply $\gamma\gamma'$ instead of $\gamma \circ \gamma'$.

Let μ be a GT on X. We say that $M \subset X$ is *μ-open* iff $M \in \mu$; $N \subset X$ is *μ-closed* iff $X - N \in \mu$. If $A \subset X$ then $i_\mu A$ denotes the union of all μ-open sets contained in A and $c_\mu A$ is the intersection of all μ-closed sets containing A (see [6]). Both i_μ and c_μ are idempotent operations (where the operation γ is said to be *idempotent* iff $\gamma\gamma A = \gamma A$ for $A \subset X$) (see [6]). According to [9], for $A \subset X$ and $x \in X$, we have $x \in c_\mu A$ iff $x \in M \in \mu$ implies $M \cap A \neq \emptyset$.

Let $\mathfrak{B} \subset \exp X$ satisfy $\emptyset \in \mathfrak{B}$. Then all unions of some elements of $\mathfrak{B}$ constitute a GT $\mu(\mathfrak{B})$, and $\mathfrak{B}$ is said to be a *base* for $\mu(\mathfrak{B})$ (see [8]).

Let μ be a GT on X and μ' a GT on X', moreover $f : X \to X'$. We say that the map f is *(μ, μ')-continuous* iff $M' \in \mu'$ implies $f^{-1}(M') \in \mu$ (see [3]), and *(μ, μ')-open* iff $M \in \mu$ implies $f(M) \in \mu'$ (cf. [4]). If f is bijective and (μ, μ')-continuous, moreover f^{-1} is (μ', μ)-continuous, then it is natural to say that f is a *(μ, μ')-homeomorphism.*

2. Product of GT's

Now let $K \neq \emptyset$ be an index set, $X_k \neq \emptyset$ for $k \in K$, and $X = \prod_{k\in K} X_k$ the Cartesian product of the sets X_k. We denote by p_k the *projection* $p_k : X \to X_k$.

Suppose that, for $k \in K$, μ_k is a given GT on X_k. Let us consider all sets of the form $\prod_{k\in K} M_k$ where $M_k \in \mu_k$ and, with the exception of a finite number of indices k, $M_k = Z_k = M_{\mu_k}$. We denote by $\mathfrak{B}$ the collection of all these sets. Clearly $\emptyset \in \mathfrak{B}$ so that we can define a GT $\mu = \mu(\mathfrak{B})$ having $\mathfrak{B}$ for base. We call μ the *product* of the GT's μ_k and denote it by $\mathbf{P}_{k\in K}\mu_k$.

If each μ_k is a topology then clearly μ is the product topology of the factors μ_k (see [1]).

Let us write $i = i_\mu$, $c = c_\mu$, $i_k = i_{\mu_k}$, $c_k = c_{\mu_k}$.

Consider in the following $A_k \subset X_k$, $A = \prod_{k\in K} A_k$, $x \in X$ and $x_k = p_k(x)$.

PROPOSITION 2.1. *$iA \subset \prod_{k\in K} i_k A_k$.*

PROOF. If $x \in iA$ then there is $M \in \mu$ such that $x \in M \subset A$. Then there are sets $M_k \in \mu_k$ such that $x \in \prod_{k\in K} M_k \subset M \subset A$. For $p_k(x) = x_k$, we have $x_k \in M_k$ so that $M_k \neq \emptyset$ and therefore $\prod_{k\in K} M_k \subset \prod_{k\in K} A_k$ implies $M_k \subset A_k$ for each k. Thus $x_k \in M_k \subset A_k$ shows that $x_k \in i_k A_k$. □

The converse is valid if K is finite:

PROPOSITION 2.2. *If K is finite then $iA = \prod_{k\in K} i_k A_k$.*

PROOF. $x \in \prod_{k\in K} i_k A_k$ implies $x_k \in i_k A_k$ for $x_k = p_k(x)$. Hence there are sets $M_k \in \mu_k$ satisfying $x_k \in M_k \subset A_k$ involving $x \in \prod_{k\in K} M_k \subset \prod_{k\in K} A_k = A$ and, K being finite, $\prod_{k\in K} M_k \in \mu$. Hence $x \in iA$. By this we have proved $\supset$ and $\subset$ follows from 2.1. □

PROPOSITION 2.3. $cA = \prod_{k\in K} c_k A_k$.

PROOF. Assume $x \in cA$. Fix $l \in K$ and choose $x_l \in M_l \in \mu_l$. For $k \in K$, $k \neq l$, let $M_k = Z_k$. Then $x \in \prod_{k\in K} M_k \in \mathfrak{B}$ implies $\prod_{k\in K} M_k \cap A \neq \emptyset$, hence $M_l \cap A_l \neq \emptyset$. This being valid for each $M_l \in \mu_l$ such that $x_l \in M_l \in \mu_l$, we have $x_l \in c_l A_l$ for each $l \in K$. Finally $x \in \prod_{k\in K} c_k A$.

Assume now $x \in \prod_{k\in K} c_k A_k$ and $x \in M \in \mu$. Then $x \in \prod_{k\in K} M_k \subset M$ for a suitable set $\prod_{k\in K} M_k \in \mathfrak{B}$ so that $x_k \in M_k \in \mu_k$ for each $k \in K$. As $M_k \in \mu_k$, we have $M_k \cap A_k \neq \emptyset$ for $k \in K$ and $\prod_{k\in K} M_k \cap \prod_{k\in K} A_k \neq \emptyset$, hence $M \cap A \neq \emptyset$. Thus $x \in cA$. □

PROPOSITION 2.4. *The projection p_k is (μ, μ_k)-open.*

PROOF. It suffices to show that $p_k(M) \in \mu_k$ if $M \in \mathfrak{B}$. If $M = \prod_{k\in K} M_k$ and $M_k \in \mu_k$ for each k then $p_k(M) = M_k$ shows the statement. □

In general, p_k need not be (μ, μ_k)-continuous:

EXAMPLE 2.5. Let $X = X_1 \times X_2$, $\mu = \mathbf{P}_{\{1,2\}}\mu_k$, $X_1 = \{a, b\}$, $\mu_1 = \{\emptyset, X_1\}$, $X_2 = \{c, d\}$, $\mu_2 = \big\{\emptyset, \{c\}\big\}$. Then $X_1 \in \mu_1$ and $p_1^{-1}(X_1) = X \notin \mu$ since $\mathfrak{B} = \big\{\emptyset, X_1 \times \{c\}\big\}$.

For a statement in the opposite direction, we need a lemma:

LEMMA 2.6. $M_\mu = \prod_{k\in K} Z_k$.

PROOF. If $M \in \mu$ then $p_k(M) \in \mu_k$ by 2.4. Hence $M \subset \prod_{k\in K} p_k(M) \subset \prod_{k\in K} Z_k \in \mu$ as $\prod_{k\in K} Z_k \in \mathfrak{B} \subset \mu$. Therefore $\prod_{k\in K} Z_k$ is the largest set in μ. □

Now we can say:

PROPOSITION 2.7. *If every μ_k is strong then μ is strong and p_k is (μ, μ_k)-continuous for $k \in K$.*

PROOF. By 2.6 $M_\mu = \prod_{k\in K} Z_k = \prod_{k\in K} X_k = X$ so that $X \in \mu$.

Now $Z_k = X_k$ for each k. $M_k \in \mu_k$ implies $p_k^{-1}(M_k) = \prod_{l\in K} N_l$ where $N_k = M_k$ and $N_l = X_l$ for $l \neq k$. Hence $\prod_{l\in K} N_l \in \mathfrak{B} \subset \mu$. □

3. Associativity of the product of GT's

Consider X_{jk} for index sets $j \in J$ and $k \in K_j$ with pairwise disjoint sets K_j. We construct first $X_j = \prod_{k \in K_j} X_{jk}$ and then $X = \prod_{j \in J} X_j$. We also consider $K = \bigcup_{j \in J} K_j$ and $Y = \prod_{jk \in K} X_{jk}$. We obtain the projections $p_j : X \to X_j$, $p_{jk} : X_j \to X_{jk}$ and $q_{jk} : Y \to X_{jk}$. Clearly there is a bijective map $h : X \to Y$ satisfying $p_{jk} \circ p_j = q_{jk} \circ h$ for each $j \in J$ and $k \in K_j$: when $x \in X$, $y = h(x)$, then $p_{jk}\big(p_j(x)\big) = q_{jk}(y) = y_{jk}$ determines y, and conversely y determines $y_{jk} = q_{jk}(y)$ and hence $p_{jk}\big(p_j(x)\big)$, so $p_j(x)$ and x.

Assume that a GT μ_{jk} is given on X_{jk} for $j \in J$, $k \in K_j$. Then we can construct $\nu_j = \mathbf{P}_{k \in K_j} \mu_{jk}$ as a GT on X_j, $\nu = \mathbf{P}_{j \in J} \nu_j$ as a GT on X and $\mu = \mathbf{P}_{jk \in K} \mu_{jk}$ as a GT on Y. We can prove with the above hypotheses:

THEOREM 3.1. *The map h is a (ν, μ)-homeomorphism.*

PROOF. For the (μ, ν)-continuity of h^{-1} it suffices to show that $h(N) \in \mu$ whenever N belongs to the base for ν. So let $N = \prod_{j \in J} N_j$ and $N_j \in \nu_j$, moreover $N_j = Z_j = M_{\nu_j}$ with the exception of a finite number of indices j. According to 2.6, $Z_j = \prod_{k \in K_j} Z_{jk}$ where $Z_{jk} = M_{\mu_{jk}}$. For the remaining finite number of indices j, N_j is the union of sets of the form $\prod_{k \in K_j} M_{jk}$ with $M_{jk} \in \mu_{jk}$ and $M_{jk} = Z_{jk}$ with the exception of a finite number of indices k for each j. We can assume that N_j is itself of this form. Then $N = \prod_{j \in J} \prod_{k \in K_j} M_{jk}$ with $M_{jk} \in \mu_{jk}$ and $M_{jk} = Z_{jk}$ except a finite number of pairs jk. Thus $h(N) = \prod_{jk \in K} M_{jk}$ belongs to the base for μ.

Conversely we must show that $M \in \mu$ implies $h^{-1}(M) \in \nu$. We can assume that M belongs to the base for μ, i.e. $M = \prod_{jk \in K} M_{jk}$ and $M_{jk} = Z_{jk}$ with the exception of a finite number of pairs jk. Necessarily, with the exception of a finite number of indices j, this equality is valid for all $k \in K_j$, and then $\prod_{k \in K_j} M_{jk} = Z_j$ by 2.6, so that $\prod_{k \in K_j} M_{jk} \in \nu_j$ for these j. For the remaining indices j, $M_{jk} = Z_{jk}$ with the exception of a finite number of indices k and therefore $\prod_{k \in K_j} M_{jk} \in \nu_j$. Finally $\prod_{k \in K_j} M_{jk} \in \nu_j$ for every $j \in J$ and $\prod_{k \in K_j} M_{jk} = Z_j$ with the exception of a finite number of indices j. Hence $h^{-1}(M) = \prod_{j \in J} \prod_{k \in K_j} M_{jk} \in \nu$. □

4. Products of semi-open, preopen, α-open, β-open, ζ-open sets

Consider an operation γ on the set X. We say that $A \subset X$ is γ-*open* iff $A \subset \gamma A$ (see [2]). A large literature is devoted to γ-open sets if μ is a GT on X and $\gamma = c_\mu i_\mu$ (*semi-open* sets), $\gamma = i_\mu c_\mu$ (*preopen* sets), $\gamma = i_\mu c_\mu i_\mu$

(α-*open* sets), $\gamma = c_\mu i_\mu c_\mu$ (β-*open* sets) or $\gamma A = c_\mu i_\mu A \cup i_\mu c_\mu A$ (ζ-*open* sets) (see [6]). The collection of the corresponding sets is denoted by $\sigma(\mu)$, $\pi(\mu)$, $\alpha(\mu)$, $\beta(\mu)$, $\zeta(\mu)$, respectively.

In the following, we assume that $X = \prod_{k\in K} X_k$, μ_k is a given GT on X_k, $\mu = \mathbf{P}_{k\in K}\mu_k$, and examine the question whether $A = \prod_{k\in K} A_k$ belongs to $\gamma(\mu)$ (where $\gamma = \sigma, \pi, \alpha, \beta, \zeta$) provided each A_k is $\gamma(\mu)$-open. We use again the notations i, c, i_k, c_k.

THEOREM 4.1. *If K is finite and every $A_k \in \sigma(\mu_k)$ then $A \in \sigma(\mu)$.*

PROOF. By 2.2 $iA = \prod_{k\in K} i_k A_k$ and by 2.3 $cA = \prod_{k\in K} c_k A_k$. Consequently $A_k \subset c_k i_k A_k$ implies

$$ciA = c\Bigl(\prod_{k\in K} i_k A_k\Bigr) = \prod_{k\in K} c_k i_k A_k \supset \prod_{k\in K} A_k = A. \quad \square$$

Similar calculations show:

THEOREM 4.2. *If K is finite and $A_k \in \pi(\mu_k)$ for $k \in K$ then $A \in \pi(\mu)$.*

THEOREM 4.3. *If K is finite and $A_k \in \alpha(\mu_k)$ for $k \in K$ then $A \in \alpha(\mu)$.*

THEOREM 4.4. *If K is finite and $A_k \in \beta(\mu_k)$ for $k \in K$ then $A \in \beta(\mu)$.*

A similar statement is not valid for ζ instead of σ:

EXAMPLE 4.5. Let $X_1 = X_2 = \mathbb{R}$ and both μ_1 and μ_2 denote the Euclidean topology. Hence μ is the Euclidean topology on the plane X. Define $A_1 = (0,1) \cap \mathbb{Q}$, $A_2 = [0,1]$ with $\mathbb{Q}$ denoting the set of all rational numbers.

Now $c_1 A_1 = [0,1]$, $i_1 c_1 A_1 = (0,1) \supset A_1$ and $A_1 \in \zeta(\mu_1)$, further $i_2 A_2 = (0,1)$, $c_2 i_2 A_2 = [0,1] \supset A_2$ and $A_2 \in \zeta(\mu_2)$.

However, $A = A_1 \times A_2$ is the union of all closed intervals $[0,1]$ located on the vertical line with abscissa $r \in \mathbb{Q}$ for $0 < r < 1$. Hence $iA = \emptyset$, $ciA = \emptyset$, $cA = [0,1] \times [0,1]$, $icA = (0,1) \times (0,1)$ and, for $a = (1/2, 0)$, we have $a \in A$ but $a \notin ciA \cup icA$ so that $A \notin \zeta(\mu)$.

We also can easily prove the converses of the statements 4.1 to 4.4:

THEOREM 4.6. *If K is finite and $A \in \sigma(\mu)$ ($A \in \pi(\mu)$, $A \in \alpha(\mu)$, $A \in \beta(\mu)$) then either $A = \emptyset$ or $A_k \in \sigma(\mu_k)$ ($A_k \in \pi(\mu_k)$, $A_k \in \alpha(\mu_k)$, $A_k \in \beta(\mu_k)$) for $k \in K$.*

PROOF. We consider the statement for σ.

If $A \neq \emptyset$, then $A \subset ciA = c\bigl(\prod_{k\in K} i_k A_k\bigr) = \prod_{k\in K} c_k i_k A_k$. This implies $A_k \subset c_k i_k A_k$ as $A_k \neq \emptyset$ for $k \in K$. $\square$

If K is infinite, the statements 4.1 to 4.4 need not be valid:

EXAMPLE 4.7. Let $K = \mathbb{N}$, $X_k = \mathbb{R}$ and μ_k denote the Euclidean topology for every k. Consider $A_k = (-1/k, 1/k)$ for $k \in K$ so that $A_k \in \mu_k$.

So $A_k \in \gamma(\mu_k)$ for $\gamma = \sigma, \pi, \alpha, \beta, \zeta$ (see [6]). However, if $A = \prod_{k\in K} A_k$ then $A \notin \beta(\mu)$ for $\mu = \mathbf{P}_{k\in K}\mu_k$. In fact, by 2.3

$$cA = \prod_{k\in K} c_k A_k = \prod_{k\in K} [-1/k, 1/k]$$

and $icA = \emptyset$ as $M \subset cA$ is impossible if $M \in \mu$, $M \neq \emptyset$ because M of this kind cannot contain any subset of the form $\prod_{k\in K} M_k$ where M_k is nonempty, μ_k-open and $M_k = X_k$ for at least one k. Thus $cicA = \emptyset$ as μ is a strong GT by 2.7 involving that $\emptyset$ is μ-closed. Consequently $A \notin \sigma(\mu), \pi(\mu), \alpha(\mu), \beta(\mu), \zeta(\mu)$ (see [6]).

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