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THE LAW OF THE ITERATED LOGARITHM FOR THE DISCREPANCIES OF A PERMUTATION OF $\{n_k x\}$

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Abstract. For any unbounded sequence $\{n_k\}$ of positive real numbers, there exists a permutation $\{n_{\sigma(k)}\}$ such that the discrepancies of $\{n_{\sigma(k)}x\}$ obey the law of the iterated logarithm exactly in the same way as the uniform i.i.d. sequence $\{U_k\}$.

1. Introduction

In the theory of uniform distributions, the following two types of discrepancies of a sequence $\{x_k\}$ of real numbers are frequently used:

$$D_N\{x_k\} = \sup_{0 \leqq a' < a < 1} \left| \sum_{k=1}^{N} \frac{f_{a',a}(x_k)}{N} \right|; \quad D_N^*\{x_k\} = \sup_{0 \leqq a < 1} \left| \sum_{k=1}^{N} \frac{f_{0,a}(x_k)}{N} \right|,$$

where $f_{a',a}(x) = \mathbf{1}_{[a',a)}\big(\langle x \rangle\big) - (a - a')$, $\mathbf{1}_{[a',a)}$ denotes the indicator function of $[a', a)$ and $\langle x \rangle$ denotes the fractional part $x - [x]$ of the real number x.

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We are interested in the asymptotic behavior of discrepancies as $N \to \infty$. For uniform i.i.d. $\{U_k\}$, the law of the iterated logarithm holds (cf. [4]):

$$\varlimsup_{N\to\infty} \frac{ND_N\{U_k\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{U_k\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.s.}$$

Assuming the Hadamard's gap condition $n_{k+1}/n_k > q > 1$, Philipp [5, 6] proved the following asymptotic property and solved the Erdős–Gál conjecture:

$$\frac{1}{4\sqrt{2}} < \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} \leqq C_q \quad \text{a.e.},$$

where C_q is a constant depending only on q. For the special sequence $\{2^k\}$, the exact law of the iterated logarithm

$$\varlimsup_{N\to\infty} \frac{ND_N\{2^k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{2^k x\}}{\sqrt{2N\log\log N}} = \frac{\sqrt{42}}{9} \quad \text{a.e.}$$

is proved in [3].

For uniform i.i.d. $\{U_k\}$, the law of the iterated logarithm for discrepancies holds for every permutation of $\{U_k\}$.

Recently, Berkes, Philipp and Tichy [2] made a remark that Philipp's asymptotic property above is permutation-invariant under Hadamard's gap condition, i.e., it remains valid if we permute the order of $\{n_k\}$. Relating to this remark, we show that the values of limsup itself are not permutation-invariant in general.

THEOREM. *For any unbounded sequence $\{n_k\}$ of positive real numbers, there exists a bijective transformation σ on $\mathbf{N}$ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N\left\{n_{\sigma(k)} x\right\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\left\{n_{\sigma(k)} x\right\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad a.e.$$

For the sequence $\{2^k\}$ and $a = 2, 3, \ldots$, there exists a σ with

$$\varlimsup_{N\to\infty} \frac{ND_N\left\{2^{\sigma(k)} x\right\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\left\{2^{\sigma(k)} x\right\}}{\sqrt{2N\log\log N}} = \frac{1}{2}\sqrt{\frac{(2^a+1)2^a(2^a-2)}{(2^a-1)^3}} \quad a.e.$$

In this theorem, $\{n_k\}$ may not be integers nor increasing. Informally, we get the permutation σ by considering the sequence $\{i_k\}$ and "scattering" the terms of the complementary sequence $\{j_k\}$ in this sequence, leaving large gaps between the consecutive j_k's.

2. LIL for the case of large gap

PROPOSITION. *For any sequence* $\{n_k\}$ *of positive real numbers satisfying* $n_{k+1}/n_k \to \infty$, *we have*

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad a.e.$$

PROOF. By applying the result of Berkes [1], for any $a' < a$, we have

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} f_{a',a}(n_k x) \right| = \|f_{a',a}\|_2 = \sqrt{(a-a')\bigl(1-(a-a')\bigr)}.$$

Hence we can verify

$$\varlimsup_{N\to\infty} \max_{I=1}^{2^L-1} \max_{I'=0}^{I-1} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} f_{2^{-L}I',2^{-L}I}(n_k x) \right| = \frac{1}{2} \quad \text{a.e.}$$

Put

$$\Psi_{L,I,N}(x) = \sup_{0 \leqq a < 2^{-L}} \left| \sum_{k=1}^{N} f_{2^{-L}I,2^{-L}I+a}(n_k x) \right|.$$

In the same way as in the proof in Section 3 of [3], which originated from [5], we can prove

$$\varlimsup_{N\to\infty} \frac{\Psi_{L,I,N}(x)}{\sqrt{2N\log\log N}} \leqq C2^{-L/8} \quad \text{a.e.} \quad \bigl(L \in \mathbf{N},\ I = 0,\dots,2^L-1\bigr).$$

On the other hand, we can easily verify the approximation inequality

$$\left| \sup_{0 \leqq a' < a < 1} \sum_{k=1}^{N} f_{a',a}(n_k x) - \max_{I=1}^{2^L-1} \max_{I'=0}^{I-1} \sum_{k=1}^{N} f_{2^{-L}I',2^{-L}I}(n_k x) \right| \leqq 2 \max_{I=0}^{2^L-1} \Psi_{L,I,N}(x).$$

By combining these and letting $L \to \infty$, we have the conclusion. As to D_N^*, we can prove our result in the same way.

3. Proof of the Theorem

Since $\{n_k\}$ is unbounded, we can take a subsequence $\{n_{i_k}\}$ such that $n_{i_{k+1}}/n_{i_k} \geqq k$. We make a subsequence $\{n_{j_k}\}$ by removing $\{n_{i_k}\}$ from $\{n_k\}$. Hence we have divided $\{n_k\}$ into two subsequences $\{n_{i_k}\}$ and $\{n_{j_k}\}$.

Let us define σ as follows. Put $\sigma(j_k) = k(k+1)/2$. For $l = 1, 2, \ldots$ and k satisfying $(l-1)l/2 < k \leqq l(l+1)/2$, put $\sigma(i_k) = k + l$, which varies over $l(l+1)/2 + 1, \ldots, (l+1)(l+2)/2 - 1$. We can verify that σ is a bijective transformation on $\mathbf{N}$, and that $b_N := {}^{\#}\{k \mid \sigma(j_k) \leqq N\} = O\big(\sqrt{N}\,\big)$.

By definition of σ, we have

$$\sum_{k=1}^{N} f_{a',a}\big(n_{\sigma(k)}x\big) = \sum_{k=1}^{N-b_N} f_{a',a}(n_{i_k}x) + \sum_{k=1}^{b_N} f_{a',a}(n_{j_k}x).$$

Since the last term is $O(b_N) = O\big(\sqrt{N}\,\big)$, by definition of D_N, we have

$$ND_N\big\{n_{\sigma(k)}x\big\} = (N - b_N)D_{N-b_N}\{n_{i_k}x\} + O\big(\sqrt{N}\,\big).$$

This also holds for D_N^*. Since $\{n_{i_k}\}$ satisfies the condition of Proposition, we have

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_{i_k}x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_{i_k}x\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e.}$$

Thanks to $b_N = O\big(\sqrt{N}\,\big)$, we have

$$N - b_N \sim N, \quad \sqrt{N} = o\Big(\sqrt{2N\log\log N}\,\Big),$$

$$\sqrt{2N\log\log N} \sim \sqrt{2(N-b_N)\log\log{(N-b_N)}},$$

whence the conclusion.

For the binary sequence $\{2^k\}$, put $i_k = ak$ and define $\{j_k\}$ and σ as above. As to the sequence $\{2^{ak}\}$ $(a = 2, 3, \ldots)$, we have the LIL

$$\varlimsup_{N\to\infty} \frac{ND_N\{2^{ak}x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{2^{ak}x\}}{\sqrt{2N\log\log N}} = \frac{1}{2}\sqrt{\frac{(2^a+1)2^a(2^a-2)}{(2^a-1)^3}} \text{ a.e.}$$

(cf. [3]). Hence we have the conclusion in the same way.

References

[1] I. Berkes, On the asymptotic behavior of $\sum f(n_k x)$. I. Main Theorems, II. Applications, *Z. Wahr. verv. Geb.,* **34** (1976), 319–345, 347–365.

[2] I. Berkes, W. Philipp and R. Tichy, Metric discrepancy results for sequence $\{n_k x\}$ and diophantine equations, in: *Diophantine Approximations, Festschrift for Wolfgang Schmidt* (eds. R. Tichy, H. Schlickewei, K. Schmidt), Development in Mathematics **17**, Springer (2008), pp. 95–105.

[3] K. Fukuyama, The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$, *Acta Math. Hungar.,* **118** (2008), 155–170.

[4] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences,* Wiley-Interscience (New York, 1974); Dover (New York, 2006).

[5] W. Philipp, Limit theorems for lacunary series and uniform distribution mod 1, *Acta Arithm.,* **26** (1975), 241–251.

[6] W. Philipp, A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables, *Ann. Probab.,* **5** (1977), 319–350.