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A FIXED POINT RESULT IN MENGER SPACES USING A REAL FUNCTION

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Abstract. The main result of this paper is a fixed point theorem of self-mappings in Menger spaces which satisfy certain inequality. This inequality involves a class of real functions which we call Φ-functions. As a corollary we obtain a result in the corresponding metric spaces. The result is supported by an example. The class of real functions we have used is the conceptual extension of altering distance functions used in metric fixed point theory.

1. Introduction

The study of fixed point results in probabilistic metric spaces has been extensively done in the last quarter of the twentieth century and is being continued in the present time. One of the earliest works in this line of research is due to Sehgal and Bharucha-Reid [21] where they have introduced probabilistic q-contraction and proved a corresponding unique fixed point result. After that several types of contractions and associated fixed point theorems have been established in probabilistic metric spaces, especially in Menger spaces which is a special type of probabilistic metric spaces. Various aspects of this theory have been elaborately discussed in the book due to Hadzic and

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Pap [9]. Some other recent references are noted in [2], [4], [8], [10], [13], [14], [15] and [22].

A new class of fixed point problems in metric spaces was addressed by Khan, Swaleh and Sessa in [12]. They introduced a control function called altering distance function which alters the distance between any two points in a given metric space. They proved fixed point theorems for mappings satisfying certain inequalities involving this altering distance function. Afterwards a number of works have appeared in which altering distance functions and their generalizations have been used in metric spaces for obtaining fixed point results. We note some of these in references [1], [3], [5], [11], [16], [17], [18] and [19]. In [6] altering distances have also been used in the case of multi-valued and fuzzy mappings.

With a view to extending this idea of altering distances to probabilistic metric spaces in [7] a new contraction has been introduced in Menger spaces. This contraction involves a class of real functions which we call Φ-functions and generalizes the q-contraction introduced by Sehgal and Bharucha-Reid [21]. The purpose of the present work is to define new contractive inequalities with the help of Φ-functions and then to establish that any self-mapping of a complete Menger space with continuous t-norm satisfying this inequality will have a unique fixed point.

We now state some definitions which are needed for the discussion of the present topic.

Definition 1.1 (altering distance function [12]). The control function $\psi : [0,\infty) \to [0,\infty)$ is called altering distance function if it has the following properties.

(i) ψ is monotone increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.2. A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R^+ denotes the set of non-negative real numbers.

Definition 1.3 (probabilistic metric space [20]). A probabilistic metric space (PM space) is an ordered pair (S,F), where S is a non-empty set and F is a function defined on $S \times S$ to the set of distribution functions which satisfies the following conditions:

(i) $F_{xy}(0) = 0$,

(ii) $F_{xy}(t) = 1$ for all $t > 0$ iff $x = y$,

(iii) $F_{xy}(t) = F_{yx}(t)$ for all $t \in R$,

(iv) $F_{xy}(t_1) = 1$ and $F_{yz}(t_2) = 1$, imply $F_{xz}(t_1 + t_2) = 1$.

Definition 1.4 (t-norm [20]). A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ which satisfies the following:

(i) $T(1,a) = a$, $T(0,0) = 0$,

(ii) $T(a,b) = T(b,a)$,

(iii) $T(c,d) \geqq T(a,b)$ whenever $c \geqq a$ and $d \geqq b$,
(iv) $T\big(T(a,b),c\big) = T\big(a,T(b,c)\big)$.

DEFINITION 1.5 (Menger space [20]). A Menger space is a triplet (S,F,T), where S is a non-empty set, F is a function defined on $S \times S$ to the set of distribution functions and T is a t-norm such that the following are satisfied:

(i) $F_{xy}(0) = 0$ for all $x,y \in S$,
(ii) $F_{xy}(s) = 1$ for all $s > 0$ iff $x = y$,
(iii) $F_{xy}(s) = F_{yx}(s)$ for all $x,y \in S$,
(iv) $F_{xy}(u+v) \geqq T\big(F_{xz}(u), F_{zy}(v)\big)$ for all $u,v \geqq 0$ and $x,y,z \in S$.

Menger spaces are generalizations of metric spaces through an introduction of a probabilistic metric F in place of deterministic metric. The following are some definitions and concepts associated with Menger space.

DEFINITION 1.6. A sequence $\{x_n\} \subset S$ is said to converge to some point $x \in S$ if given $\varepsilon > 0$, $\lambda > 0$ we can find a positive integer $N_{\varepsilon,\lambda}$ such that for all $n > N_{\varepsilon,\lambda}$, $F_{x_n x}(\varepsilon) > 1 - \lambda$.

DEFINITION 1.7. A sequence $\{x_n\}$ is said to be a Cauchy sequence in S if given $\varepsilon > 0$, $\lambda > 0$ there exists a positive integer $N_{\varepsilon,\lambda}$, such that $F_{x_n x_m}(\varepsilon) > 1 - \lambda$ for all $m,n > N_{\varepsilon,\lambda}$.

DEFINITION 1.8. A Menger space (S,F,T) is said to be complete if every Cauchy sequence in it is convergent.

DEFINITION 1.9. If (S,F,T) is a Menger space with continuous t-norm then the topology induced by the family $\big\{U_{\varepsilon,\lambda}(p) : p \in S,\ \varepsilon > 0,\ \lambda > 0\big\}$ is called the $(\varepsilon - \lambda)$-topology, where $U_{\varepsilon,\lambda}(p) = \big\{q \in S : F_{pq}(\varepsilon) > 1 - \lambda\big\}$ is called the $(\varepsilon - \lambda)$-neighborhood of p.

The following category of functions was introduced in [7].

DEFINITION 1.10 (Φ-function [7]). A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\varphi(t) = 0$ if and only if $t = 0$,
(ii) $\varphi(t)$ is strictly monotone increasing and $\varphi(t) \to \infty$ as $t \to \infty$,
(iii) φ is left continuous in $(0,\infty)$,
(iv) φ is continuous at 0.

DEFINITION 1.11 [7]. Let (S,F,T) be a Menger space. A self map $f : S \to S$ is said to be φ-contractive if

$$F_{fxfy}\big(\varphi(t)\big) \geqq F_{xy}\left(\varphi\left(\frac{t}{c}\right)\right) \tag{1.1}$$

where $0 < c < 1$, $x,y \in S$ and $t > 0$ and the function φ is a Φ-function.

The following result was proved in [7].

THEOREM 1.1 [7]. *Let (S, F, T_M) be a complete Menger space with t-norm T_M given by $T_M(a,b) = \min(a,b)$. If $f : S \to S$ is φ-contractive then f has a unique fixed point.*

It may be seen that the inequality (1.1) reduces to Sehgal's q-contraction when φ is assumed to be the identity function. It has also been shown in [7] that Theorem 1.1 implies a result established in [12]. In fact the Φ-function plays the role of altering distance function (Definition 1.1) in probabilistic metric spaces [7].

The purpose of the present work is to establish a fixed point theorem in Menger spaces by use of Φ-functions. We also deduce a result in metric spaces as a corollary of our main theorem. Lastly we have supported our theorem by an example.

2. Main results

THEOREM 2.1. *Let (S, F, T) be a complete Menger space with continuous t-norm T and let $f : S \to S$ satisfy the following inequality:*

(2.1)

$$F_{fxfy}\big(\varphi(t)\big) \geqq \min\left\{F_{xy}\left(\varphi\left(\frac{t_1}{a}\right)\right), F_{xfx}\left(\varphi\left(\frac{t_2}{b}\right)\right), F_{yfy}\left(\varphi\left(\frac{t_3}{c}\right)\right)\right\}$$

where a, b, c are positive numbers such that $0 < a+b+c < 1$, $t_1, t_2, t_3 > 0$, $t_1 + t_2 + t_3 = t$ and φ is a Φ-function (Definition 1.10). Then f has a unique fixed point.

PROOF. Let $x_0 \in S$. We construct a sequence x_n in S as follows:

$$x_{n+1} = fx_n, \quad n = 0, 1, 2, 3, \dots .$$

We have $0 < a+b+c < 1$, hence we can take $\varepsilon' > 0$ such that $a+b+c+3\varepsilon' = 1$. Let $t_1 = (a+\varepsilon')t$, $t_2 = (b+\varepsilon')t$ and $t_3 = (c+\varepsilon')t$. Then $t_1 + t_2 + t_3 = t$.

We next prove that $F_{x_{n+1}x_n}(s) \to 1$ as $n \to \infty$ for all $s > 0$.

$$\begin{aligned} F_{x_{n+1}x_n}\big(\varphi(t)\big) &= F_{fx_nfx_{n-1}}\big(\varphi(t)\big) \qquad (2.2)\\ &\geqq \min\left\{F_{x_{n-1}x_n}\left(\varphi\left(\frac{(a+\varepsilon')t}{a}\right)\right), F_{x_nx_{n+1}}\left(\varphi\left(\frac{(b+\varepsilon')t}{b}\right)\right),\right.\\ &\qquad \left. F_{x_{n-1}x_n}\left(\varphi\left(\frac{(c+\varepsilon')t}{c}\right)\right)\right\}\\ &= \min\left\{F_{x_{n-1}x_n}\left(\varphi\left(\frac{t}{a/(a+\varepsilon')}\right)\right), F_{x_nx_{n+1}}\left(\varphi\left(\frac{t}{b/(b+\varepsilon')}\right)\right),\right.\\ &\qquad \left. F_{x_{n-1}x_n}\left(\varphi\left(\frac{t}{c/(c+\varepsilon')}\right)\right)\right\}. \end{aligned}$$

Now we have $\frac{a}{a+\varepsilon'} < 1$, $\frac{b}{b+\varepsilon'} < 1$, $\frac{c}{c+\varepsilon'} < 1$, hence we can choose k such that $0 < k < 1$ and

$$\max\left\{\frac{a}{a+\varepsilon'}, \frac{b}{b+\varepsilon'}, \frac{c}{c+\varepsilon'}\right\} < k.$$

Therefore

$$\frac{t}{a/(a+\varepsilon')}, \frac{t}{b/(b+\varepsilon')}, \frac{t}{c/(c+\varepsilon')} > \frac{t}{k}. \tag{2.3}$$

Hence, we have from (2.2) and (2.3) for all $t > 0$,

$$F_{x_{n+1}x_n}\big(\varphi(t)\big) \tag{2.4}$$

$$\geqq \min\left\{F_{x_nx_{n-1}}\left(\varphi\left(\frac{t}{k}\right)\right), F_{x_{n+1}x_n}\left(\varphi\left(\frac{t}{k}\right)\right), F_{x_nx_{n-1}}\left(\varphi\left(\frac{t}{k}\right)\right)\right\}$$

[since φ is monotone increasing]

$$= \min\left\{F_{x_nx_{n-1}}\left(\varphi\left(\frac{t}{k}\right)\right), F_{x_{n+1}x_n}\left(\varphi\left(\frac{t}{k}\right)\right)\right\}.$$

Now we claim that

$$\min\left\{F_{x_nx_{n-1}}\left(\varphi\left(\frac{t}{k}\right)\right), F_{x_{n+1}x_n}\left(\varphi\left(\frac{t}{k}\right)\right)\right\} = F_{x_nx_{n-1}}\left(\varphi\left(\frac{t}{k}\right)\right) \tag{2.5}$$

for all $t > 0$. If otherwise, there exists $s > 0$ such that

$$F_{x_nx_{n-1}}\left(\varphi\left(\frac{s}{k}\right)\right) > F_{x_{n+1}x_n}\left(\varphi\left(\frac{s}{k}\right)\right). \tag{2.6}$$

By (2.4) and (2.6)

$$F_{x_{n+1}x_n}\big(\varphi(s)\big) \geqq \min\left\{F_{x_nx_{n-1}}\left(\varphi\left(\frac{s}{k}\right)\right), F_{x_{n+1}x_n}\left(\varphi\left(\frac{s}{k}\right)\right)\right\} \tag{2.7}$$

$$= F_{x_{n+1}x_n}\left(\varphi\left(\frac{s}{k}\right)\right) \geqq \min\left\{F_{x_nx_{n-1}}\left(\varphi\left(\frac{s}{k^2}\right)\right), F_{x_{n+1}x_n}\left(\varphi\left(\frac{s}{k^2}\right)\right)\right\}.$$

If

$$\min\left\{F_{x_nx_{n-1}}\left(\varphi\left(\frac{s}{k^2}\right)\right), F_{x_{n+1}x_n}\left(\varphi\left(\frac{s}{k^2}\right)\right)\right\} = F_{x_nx_{n-1}}\left(\varphi\left(\frac{s}{k^2}\right)\right)$$

then by (2.6) and (2.7) we have

$$F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k}\right)\right) > F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k}\right)\right) \geqq F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^2}\right)\right)$$
$$\geqq F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k}\right)\right),$$

which is a contradiction.

Therefore

$$F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^2}\right)\right) > F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^2}\right)\right). \tag{2.8}$$

Hence from (2.7) we have

$$F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k}\right)\right) \geqq F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^2}\right)\right). \tag{2.9}$$

Further from (2.4) we get,

$$F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^2}\right)\right) \geqq \min\left\{F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^3}\right)\right), F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^3}\right)\right)\right\}. \tag{2.10}$$

If

$$\min\left\{F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^3}\right)\right), F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^3}\right)\right)\right\} = F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^3}\right)\right)$$

then by (2.8) and (2.10) we get

$$F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^2}\right)\right) > F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^3}\right)\right)$$

which is a contradiction. Hence

$$\min\left\{F_{x_n x_{n-1}}\left(\varphi\left(\frac{s}{k^3}\right)\right), F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^3}\right)\right)\right\} = F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^3}\right)\right),$$

which implies

$$F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^2}\right)\right) \geqq F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^3}\right)\right).$$

Therefore from (2.9) we have

$$F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k}\right)\right) \geqq F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^2}\right)\right) \geqq F_{x_{n+1} x_n}\left(\varphi\left(\frac{s}{k^3}\right)\right).$$

Continuing in this way we obtain

$$F_{x_{n+1}x_n}\left(\varphi\left(\frac{s}{k}\right)\right) \geqq F_{x_{n+1}x_n}\left(\varphi\left(\frac{t}{k^{p'}}\right)\right) \to 1 \quad \text{as} \quad p' \to \infty$$

that is, $F_{x_{n+1}x_n}\big(\varphi\big(\frac{s}{k}\big)\big) = 1$. But by our assumption (2.6), $F_{x_nx_{n-1}}\big(\varphi\big(\frac{s}{k}\big)\big) > F_{x_{n+1}x_n}\big(\varphi\big(\frac{s}{k}\big)\big)$ that is, $F_{x_nx_{n-1}}\big(\varphi\big(\frac{s}{k}\big)\big) > 1$, which is impossible.

Therefore (2.5) holds.

Then from (2.4) we have for all $t > 0$,

$$F_{x_{n+1}x_n}\big(\varphi(t)\big) \geqq F_{x_nx_{n-1}}\left(\varphi\left(\frac{t}{k}\right)\right). \tag{2.11}$$

Applying successively,

$$F_{x_{n+1}x_n}\big(\varphi(t)\big) \geqq F_{x_nx_{n-1}}\left(\varphi\left(\frac{t}{k}\right)\right) \geqq F_{x_{n-1}x_{n-2}}\left(\varphi\left(\frac{t}{k^2}\right)\right) \geqq \cdots$$
$$\geqq F_{x_1x_0}\left(\varphi\left(\frac{t}{k^n}\right)\right) \to 1$$

as $n \to \infty$. Thus we have proved that for all $t > 0$, $F_{x_{n+1}x_n}\big(\varphi(t)\big) \to 1$ as $n \to \infty$.

By property of φ, given $s > 0$ there exists $t > 0$ such that $s > \varphi(t)$, so that

$$F_{x_{n+1}x_n}(s) \to 1 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad s > 0. \tag{2.12}$$

We now claim that $\{x_n\}$ is a Cauchy sequence. If not, then there exist $\varepsilon > 0$ and $\lambda > 0$ and subsequences $\big\{x_{m(k)}\big\}$ and $\big\{x_{n(k)}\big\}$ such that $m(k) < n(k)$ and

$$F_{x_{m(k)}x_{n(k)}}(\varepsilon) < 1 - \lambda, \tag{2.13}$$

$$F_{x_{m(k)}x_{n(k)-1}}(\varepsilon) \geqq 1 - \lambda. \tag{2.14}$$

Since

$$\big\{x : F_{xp}(\varepsilon'') \geqq 1 - \lambda\big\} \subseteqq \big\{x : F_{xp}(\varepsilon) \geqq 1 - \lambda\big\} \tag{2.15}$$

for all $p \in S$, $\lambda > 0$ and $0 < \varepsilon'' < \varepsilon$, it follows that whenever the above construction is possible for $\varepsilon > 0$, $\lambda > 0$, it is also possible to construct $\big\{x_{m(k)}\big\}$

and $\{x_{n(k)}\}$ satisfying (2.13) and (2.14) corresponding to $\varepsilon'' > 0$, $\lambda > 0$ whenever $\varepsilon'' < \varepsilon$.

Again φ is continuous at the origin and strictly monotone increasing with $\varphi(0) = 0$, so it is possible to obtain $\varepsilon_1 > 0$ such that $\varphi(\varepsilon_1) < \varepsilon$.

Then by the above argument it is possible to obtain increasing sequences of integers $m(k)$ and $n(k)$ with $m(k) < n(k)$ such that

$$F_{x_{m(k)}x_{n(k)}}\big(\varphi(\varepsilon_1)\big) < 1 - \lambda, \tag{2.16}$$

and

$$F_{x_{m(k)}x_{n(k)-1}}\big(\varphi(\varepsilon_1)\big) \geqq 1 - \lambda. \tag{2.17}$$

As $a < 1$ it is possible to find a $v > 0$ such that $a + v < 1$. It is also possible to choose $\eta_1 > 0$, $\eta_2 > 0$ such that

$$\begin{cases} \dfrac{v}{a+v}\varepsilon_1 > \eta_1 + \eta_2, \quad \dfrac{\varepsilon_1}{a} - \dfrac{\varepsilon_1}{a+v} > \dfrac{\eta_1 + \eta_2}{a}, \\ \dfrac{\varepsilon_1}{a} - \dfrac{\eta_1 + \eta_2}{a} > \dfrac{\varepsilon_1}{a+v}, \quad \dfrac{\varepsilon_1 - \eta_1 - \eta_2}{a} > \dfrac{\varepsilon_1}{a+v}. \end{cases} \tag{2.18}$$

Also we can choose $\eta > 0$ such that

$$0 < \eta < \varphi\left(\frac{\varepsilon_1}{a+v}\right) - \varphi(\varepsilon_1)$$

(since φ is strictly increasing), that is,

$$\varphi\left(\frac{\varepsilon_1}{a+v}\right) - \eta > \varphi(\varepsilon_1). \tag{2.19}$$

By (2.12) for $\lambda_1 < \lambda < 1$ it is possible to find a positive integer N_1 such that for all $k > N_1$

$$F_{x_{m(k)}x_{m(k)-1}}\left(\varphi\left(\frac{\eta_1}{b}\right)\right) \geqq 1 - \lambda_1, \tag{2.20}$$

$$F_{x_{n(k)}x_{n(k)-1}}\left(\varphi\left(\frac{\eta_2}{c}\right)\right) \geqq 1 - \lambda_1. \tag{2.21}$$

Again by (2.17)–(2.19),

$$F_{x_{m(k)}x_{n(k)-1}}\left(\varphi\left(\frac{\varepsilon_1 - \eta_1 - \eta_2}{a}\right) - \eta\right) \tag{2.22}$$

$$\geqq F_{x_{m(k)}x_{n(k)-1}}\left(\varphi\left(\frac{\varepsilon_1}{a+v}\right)\right) - \eta) \geqq F_{x_{m(k)}x_{n(k)-1}}\big(\varphi(\varepsilon_1)\big) \geqq 1 - \lambda.$$

Now,

$$F_{x_{m(k)-1}x_{n(k)-1}}\left(\varphi\left(\frac{\varepsilon_1-\eta_1-\eta_2}{a}\right)\right) \tag{2.23}$$

$$\geqq T\left\{F_{x_{m(k)-1}x_{m(k)}}(\eta), F_{x_{m(k)}x_{n(k)-1}}\left(\varphi\left(\frac{\varepsilon_1-\eta_1-\eta_2}{a}\right)-\eta\right)\right\}.$$

Let $0<\lambda_2<1$ be arbitrary. Then by (2.12) there exists a positive integer N_2 such that for all $k>N_2$,

$$F_{x_{m(k)-1}x_{m(k)}}(\eta)\geqq 1-\lambda_2. \tag{2.24}$$

Using (2.22) and (2.24), we have from (2.23) for all $k>\max\{N_1,N_2\}$,

$$F_{x_{m(k)-1}x_{n(k)-1}}\left(\varphi\left(\frac{\varepsilon_1-\eta_1-\eta_2}{a}\right)\right)\geqq T\{1-\lambda_2,1-\lambda\}.$$

As λ_2 is arbitrary and T being continuous, we have

$$F_{x_{m(k)-1}x_{n(k)-1}}\left(\varphi\left(\frac{\varepsilon_1-\eta_1-\eta_2}{a}\right)\right)\geqq T(1,1-\lambda)=1-\lambda. \tag{2.25}$$

Using (2.1), (2.16), (2.20), (2.21) and (2.25) we have

$$1-\lambda>F_{x_{m(k)}x_{n(k)}}\big(\varphi(\varepsilon_1)\big)$$

$$\geqq \min\left\{F_{x_{m(k)-1}x_{n(k)-1}}\left(\varphi\left(\frac{\varepsilon_1-\eta_1-\eta_2}{a}\right)\right), F_{x_{m(k)-1}x_{m(k)}}\left(\varphi\left(\frac{\eta_1}{b}\right)\right),\right.$$
$$\left.F_{x_{n(k)-1}x_{n(k)}}\left(\varphi\left(\frac{\eta_2}{c}\right)\right)\right\}\geqq \min\{1-\lambda,1-\lambda_1,1-\lambda_1\}=1-\lambda,$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. As (S,F,T) is complete, we have $x_n\to z$ as $n\to\infty$ for some $z\in S$.

We now show that z is a fixed point of f, that is, $fz=z$. Let $\varepsilon_2>0$ be arbitrary. As φ is strictly monotone increasing, we can take a positive number ε_3 and k such that $c<k<1$, $\varepsilon_3<\varepsilon_2$ and $\frac{\varepsilon_2}{k}<\frac{\varepsilon_3}{c}$. Now $\eta'>0$ is chosen in such a way that

$$\eta'<\min\left\{\varphi(\varepsilon_2)-\varphi(\varepsilon_3),\varphi\left(\frac{\varepsilon_3}{c}\right)-\varphi\left(\frac{\varepsilon_2}{k}\right)\right\}. \tag{2.26}$$

(The choice is possible since φ is strictly monotone increasing.)

By virtue of left continuity of φ, we can choose positive numbers α_1, α_2, α_3 in such a way that

(2.27)

$$\alpha_1+\alpha_2+\alpha_3=\varepsilon_3 \quad \text{and} \quad \varphi\left(\frac{\alpha_3}{c}\right)=\varphi\left(\frac{\varepsilon_3}{c}-\frac{\alpha_1+\alpha_2}{c}\right)>\varphi\left(\frac{\varepsilon_3}{c}\right)-\eta'.$$

Again by (2.26) we have $\varphi\bigl(\frac{\varepsilon_3}{c}\bigr)-\eta'>\varphi\bigl(\frac{\varepsilon_2}{k}\bigr)\bigr)$, and by (2.1), (2.27) and (2.26),

$$F_{zfz}\bigl(\varphi(\varepsilon_2)\bigr) \geqq T\Bigl\{F_{zx_n}(\eta'),F_{x_nfz}\bigl(\varphi(\varepsilon_2)\bigr)-\eta'\Bigr\}$$

$$\geqq T\Bigl\{F_{zx_n}(\eta'),F_{x_nfz}\bigl(\varphi(\varepsilon_3)\bigr)\Bigr\}=T\Bigl\{F_{zx_n}(\eta'),F_{fx_{n-1}fz}\bigl(\varphi(\varepsilon_3)\bigr)\Bigr\}$$

(2.28)

$$\geqq T\{F_{zx_n}(\eta'),\min\left\{F_{x_{n-1}z}\left(\varphi\left(\frac{\alpha_1}{a}\right)\right),F_{x_{n-1}x_n}\left(\varphi\left(\frac{\alpha_2}{b}\right)\right),F_{zfz}\left(\varphi\left(\frac{\alpha_3}{c}\right)\right)\right\}$$

(2.29)
$$\geqq T\{F_{zx_n}(\eta'),\min\left\{F_{x_{n-1}z}\left(\varphi\left(\frac{\alpha_1}{a}\right)\right),F_{x_{n-1}x_n}\left(\varphi\left(\frac{\alpha_2}{b}\right)\right),\right.$$
$$\left.F_{zfz}\left(\varphi\left(\frac{\varepsilon_3}{c}-\eta'\right)\right)\right\}$$

$$\geqq T\{F_{zx_n}(\eta'),\min\left\{F_{x_{n-1}z}\left(\varphi\left(\frac{\alpha_1}{a}\right)\right),F_{x_{n-1}x_n}\left(\varphi\left(\frac{\alpha_2}{b}\right)\right),F_{zfz}\left(\varphi\left(\frac{\varepsilon_2}{k}\right)\right)\right\}.$$

As T is continuous, taking limit as $n\to\infty$, using (2.12) and the fact that $x_n\to z$, we have

$$F_{zfz}\bigl(\varphi(\varepsilon_2)\bigr) \geqq T\left(1,\min\left\{1,1,F_{zfz}\left(\varphi\left(\frac{\varepsilon_2}{k}\right)\right)\right\}\right)$$

$$\geqq T\left(1,F_{zfz}\left(\varphi\left(\frac{\varepsilon_2}{k}\right)\right)\right)=F_{zfz}\left(\varphi\left(\frac{\varepsilon_2}{k}\right)\right).$$

Since $\varepsilon_2>0$ is arbitrary, we successively apply the above inequality and we obtain

$$F_{zfz}\bigl(\varphi(\varepsilon_2)\bigr) \geqq F_{zfz}\left(\varphi\left(\frac{\varepsilon_2}{k^2}\right)\right)\geqq F_{zfz}\left(\varphi\left(\frac{\varepsilon_2}{k^3}\right)\right)\geqq\cdots$$

$$\geqq F_{zfz}\left(\varphi\left(\frac{\varepsilon_2}{k^{q'}}\right)\right)\to 1$$

as $q'\to\infty$. By properties (i) and (ii) of φ, given $s_1>0$ we can have $r_1>0$ such that $\varphi(r_1)<s_1$. Therefore, $F_{zfz}(s_1)>F_{zfz}\bigl(\varphi(r_1)\bigr)=1$ for all $s_1>0$. Hence $z=fz$. Therefore z is a fixed point of f.

Next we prove the uniqueness of fixed point. Let z and z_1 be two fixed point of f that is $fz = z$ and $fz_1 = z_1$. Then for any $t > 0$,

$$F_{zz_1}\big(\varphi(t)\big) = F_{fzfz_1}\big(\varphi(t)\big)$$
$$\geqq \min\left\{F_{zz_1}\left(\varphi\left(\frac{t_1}{a}\right)\right), F_{zfz}\left(\varphi\left(\frac{t_2}{b}\right)\right), F_{z_1fz_1}\left(\varphi\left(\frac{t_3}{c}\right)\right)\right\}$$

where $t_1 + t_2 + t_3 = t$ and $0 < a + b + c < 1$.

We take v in such a way that $a + b + c + 3v = 1$. Let $t_1 = (a + v)t$, $t_2 = (b + v)t$ and $t_3 = (c + v)t$. Then $t_1 + t_2 + t_3 = t$. Thus

$$F_{zz_1}\big(\varphi(t)\big) = F_{fzfz_1}\big(\varphi(t)\big) \geqq \min\left\{F_{zz_1}\left(\varphi\left(\frac{(a+v)t}{a}\right)\right),\right.$$
$$\left. F_{zfz}\left(\varphi\left(\frac{(b+v)t}{b}\right)\right), F_{z_1fz_1}\left(\varphi\left(\frac{(c+v)t}{c}\right)\right)\right\}$$
$$= \min\left\{F_{zz_1}\left(\varphi\left(\frac{t}{\frac{a}{a+v}}\right)\right), F_{zz}\left(\varphi\left(\frac{t}{\frac{b}{b+v}}\right)\right), F_{z_1z_1}\left(\varphi\left(\frac{t}{\frac{c}{c+v}}\right)\right)\right\}$$
$$= \min\left\{F_{zz_1}\left(\varphi\left(\frac{t}{\frac{a}{a+v}}\right)\right), 1, 1\right\} = F_{zz_1}\left(\varphi\left(\frac{t}{\frac{a}{a+v}}\right)\right)$$
$$= F_{zz_1}\left(\varphi\left(\frac{t}{\mu}\right)\right) \quad \text{where} \quad 0 < \mu = \frac{a}{a+v} < 1.$$

Applying successively the above inequality

$$F_{zz_1}\big(\varphi(t)\big) \geqq F_{zz_1}\left(\varphi\left(\frac{t}{\mu}\right)\right) \geqq F_{zz_1}\left(\varphi\left(\frac{t}{\mu^2}\right)\right) \geqq \cdots$$
$$\geqq F_{zz_1}\left(\varphi\left(\frac{t}{\mu^n}\right)\right) \to 1$$

as $n \to \infty$. By properties (i) and (ii) of φ, given $t > 0$, we can have $r > 0$ such that $\varphi(r) < t$. Therefore, $F_{zz_1}(t) \geqq F_{zz_1}\big(\varphi(r)\big) = 1$ for all $t > 0$. Hence $z = z_1$. This proves the uniqueness of fixed point.

It is well known that any metric space may be considered as a Menger space if we assume $F_{xy}(t) = H\big(t - d(x,y)\big)$, where H is the Heaviside function given by

$$H(s) = \begin{cases} 1, & \text{if } s > 0, \\ 0, & \text{if } s \leqq 0 \end{cases}$$

and $T(a,b) = T_M(a,b) = \min\{a,b\}$.

If we take (S,d) as a complete metric space then the corresponding Menger space (S,F,T_M), where T_M is the minimum t-norm, is also complete.

If ψ is an altering distance function as in Definition 1.1 with the additional property $\psi(t) \to \infty$, as $t \to \infty$ then it is easily verified that the function defined by

$$\varphi(t) = \begin{cases} \inf\big\{\alpha : \psi(\alpha) \geqq t\big\}, & \text{for } t > 0, \\ 0, & \text{for } t = 0 \end{cases}$$

is a Φ-function (Definition 1.10).

We next show that the inequality (2.1) in this case implies

$$\psi\big(d(fx,fy)\big) \leqq a\psi\big(d(x,y)\big) + b\psi\big(d(x,fx)\big) + c\psi\big(d(y,fy)\big)$$

in a metric space.

Inequality (2.1) will be violated if there exists $t > 0$, $t_1 + t_2 + t_3 = t$ and $t_1, t_2, t_3 > 0$ such that $F_{fxfy}\big(\varphi(t)\big) = 0$ and all of $F_{xy}\big(\varphi\big(\frac{t_1}{a}\big)\big)$, $F_{xfx}\big(\varphi\big(\frac{t_2}{b}\big)\big)$, $F_{yfy}\big(\varphi\big(\frac{t_3}{c}\big)\big)$ are equal to 1. Now $F_{fxfy}\big(\varphi(t)\big) = 0$ implies

$$H\big(\big(\varphi(t) - d(fx,fy)\big) = 0, \quad \varphi(t) - d(fx,fy)\big) \leqq 0,$$

$$\varphi(t) \leqq d(fx,fy), \quad \inf\big\{\alpha : \psi(\alpha) \geqq t\big\} \leqq d(fx,fy),$$

that is, by virtue of continuity of ψ

$$\psi\big(d(fx,fy)\big) \geqq t. \tag{2.30}$$

Again $t = t_1 + t_2 + t_3$. Therefore $\psi\big(d(fx,fy)\big) \geqq t_1 + t_2 + t_3$.

Also $F_{xy}\big(\varphi\big(\frac{t_1}{a}\big)\big) = 1$ implies

$$H\left(\varphi\left(\frac{t_1}{a}\right) - d(x,y)\right) = 1, \quad \varphi\left(\frac{t_1}{a}\right) - d(x,y) > 0, \quad \varphi\left(\frac{t_1}{a}\right) > d(x,y),$$

whence $\inf\big\{\beta : \psi(\beta) \geqq \frac{t_1}{a}\big\} > d(x,y)$. By continuity of ψ, $d(x,y) \notin \big\{\beta : \psi(\beta) \geqq \frac{t_1}{a}\big\}$ that is, $\psi\big(d(x,y)\big) < \frac{t_1}{a}$ which implies

$$t_1 > a\psi\big(d(x,y)\big). \tag{2.31}$$

Similarly, from $F_{xfx}\big(\varphi\big(\frac{t_2}{b}\big)\big) = 1$ and $F_{yfy}\big(\varphi\big(\frac{t_3}{c}\big)\big) = 1$ we have respectively

$$t_2 > b\psi\big(d(x,fx)\big), \tag{2.32}$$

$$t_3 > c\psi\bigl(d(y, fy)\bigr). \tag{2.33}$$

Thus we have

$$\psi\bigl(d(fx, fy)\bigr) > a\psi\bigl(d(x, y)\bigr) + b\psi\bigl(d(x, fx)\bigr) + c\psi\bigl(d(y, fy)\bigr). \tag{2.34}$$

Hence we can say that inequality (2.1) implies

$$\psi\bigl(d(fx, fy)\bigr) \leqq a\psi\bigl(d(x, y)\bigr) + b\psi\bigl(d(x, fx)\bigr) + c\psi\bigl(d(y, fy)\bigr)$$

in the corresponding metric space.

We have then the following result.

COROLLARY 2.1. *Let (S, d) be a complete metric space, f be a self map on S and ψ be an altering distance function with the additional property $\psi(t) \to \infty$ as $t \to \infty$. If f satisfies the inequality*

$$\psi\bigl(d(fx, fy)\bigr) \leqq a\psi\bigl(d(x, y)\bigr) + b\psi\bigl(d(x, fx)\bigr) + c\psi\bigl(d(y, fy)\bigr) \tag{2.35}$$

for all $x, y \in S$ with $x \neq y$ and $0 < a + b + c < 1$, then f has a unique fixed point.

If we take $b = c$ in Corollary 2.1 we obtain a result in [12].

NOTE. In Corollary 2.1 the additional requirement on the altering distance function $\psi(t) \to \infty$ as $t \to \infty$ can be omitted if we modify the definition of the Φ-function by making these into extended real valued functions and thus allowing these functions to assume $+\infty$. Then an altering distance function generates a Φ-function in the same way, that is

$$\varphi(t) = \begin{cases} \inf\bigl\{\alpha : \psi(\alpha) \geqq t\bigr\}, & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}$$

All our results are valid under such modification and also the proofs remain identical.

EXAMPLE 2.1. Let (S, F, T) be a complete Menger space with continuous t-norm T, where $S = [2, 4]$, $F_{xy}(t) = \frac{t}{t+|x-y|}$ and let $f : S \to S$ be defined as follows:

$$fx = \frac{4 + x}{3}, \qquad 2 \leqq x \leqq 4.$$

If $\varphi(t) = t^2$, $a = \frac{3}{4}$, $b = \frac{1}{9}$, $c = \frac{1}{9}$ then it satisfies all the conditions of Theorem 2.1. It is seen that $x = 2$ is the fixed point of f.

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