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$\star$-EXTREMALLY DISCONNECTED IDEAL TOPOLOGICAL SPACES

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Abstract. The notion of $\star$-extremally disconnected ideal topological spaces is introduced and studied. Many characterizations of the space are obtained.

1. Introduction

Extremally disconnected spaces play an important role in set-theoretical topology, studying Stone–Čech compactification and the Stone space of any complete Boolean algebra, the theory of Boolean algebra, axiomatic set theory, functional analysis, C^*-algebra, studying the space $\mathrm{Seq}(\xi)$ etc. [1, 2, 10, 13]. On the other hand, the notion of ideal topological spaces was studied by Kuratowski [11] and Vaidyanathaswamy [14]. In 1990, Janković and Hamlett [9] investigated further properties of ideal topological spaces. In this paper, the class of $\star$-extremally disconnected ideal topological spaces is introduced and studied. Also, properties of $\star$-extremally disconnected ideal topological spaces are discussed.

Key words and phrases: $\star$-closure, δ-I-closure, semi*-I-open, ideal topological space, $\star$-extremally disconnected.
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2. Preliminaries

By a space, we always mean a topological space (X,τ) with no separation properties assumed. For a subset A of a topological space (X,τ), $\mathrm{Cl}\,(A)$ and $\mathrm{Int}\,(A)$ will denote the closure and interior of A in (X,τ), respectively.

An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in I$ and $B \subset A$ implies $B \in I$,

(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

Given a topological space (X,τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a local function [11] of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I,\tau) = \{x \in X : V \cap A \notin I \text{ for every } V \in \tau(x)\}$ where $\tau(x) = \{V \in \tau : x \in V\}$. A Kuratowski closure operator $\mathrm{Cl}^*(.)$ for a topology $\tau^*(I,\tau)$, called the $\star$-topology, finer than τ, is defined by $\mathrm{Cl}^*(A) = A \cup A^*(I,\tau)$ [9]. When there is no chance for confusion, we will simply write A^* for $A^*(I,\tau)$ and τ^* or $\tau^*(I)$ for $\tau^*(I,\tau)$. For any ideal space (X,τ,I), the collection $\{V\backslash J : V \in \tau \text{ and } J \in I\}$ is a basis for τ^*. If I is an ideal on X, then (X,τ,I) is called an ideal topological space or simply an ideal space.

Definition 1. A subset A of an ideal space (X,τ,I) is called

(1) pre-I-open [4] if $A \subset \mathrm{Int}\,(\mathrm{Cl}^*(A))$.

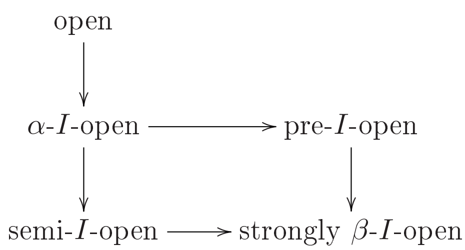
(2) semi-I-open [7] if $A \subset \mathrm{Cl}^*\,(\mathrm{Int}\,(A))$.

(3) strongly β-I-open [8] if $A \subset \mathrm{Cl}^*\,(\mathrm{Int}\,(\mathrm{Cl}^*(A)))$.

(4) α-I-open [7] if $A \subset \mathrm{Int}\,(\mathrm{Cl}^*\,(\mathrm{Int}\,(A)))$.

(5) semi-open [12] if $A \subset \mathrm{Cl}\,(\mathrm{Int}\,(A))$.

Remark 2 [7, 8]. The following diagram holds for a subset A of an ideal space (X,τ,I):



3. $\star$-extremally disconnected ideal spaces

Definition 3. An ideal space (X,τ,I) is said to be $\star$-extremally disconnected if the $\star$-closure of every open subset A of X is open.

THEOREM 4. *For an ideal space (X,τ,I), the following properties are equivalent:*

(1) *X is ⋆-extremally disconnected.*

(2) $\operatorname{Int}^*(A)$ *is closed for every closed subset A of X.*

(3) $\operatorname{Cl}^*\big(\operatorname{Int}(A)\big) \subset \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$ *for every subset A of X.*

(4) *Every semi-I-open set is pre-I-open.*

(5) *The ⋆-closure of every strongly β-I-open subset of X is open.*

(6) *Every strongly β-I-open set is pre-I-open.*

(7) *For every subset A of X, A is α-I-open if and only if it is semi-I-open.*

PROOF. (1) ⇒ (2). Let $A \subset X$ be a closed set. Then $X\backslash A$ is open. By (1), $\operatorname{Cl}^*(X\backslash A) = X\backslash \operatorname{Int}^*(A)$ is open. Thus, $\operatorname{Int}^*(A)$ is closed.

(2) ⇒ (3). Let A be any set of X. Then $X\backslash \operatorname{Int}(A)$ is closed in X and by (2) $\operatorname{Int}^*\big(X\backslash \operatorname{Int}(A)\big)$ is closed in X. Therefore, $\operatorname{Cl}^*\big(\operatorname{Int}(A)\big)$ is open in X and hence, $\operatorname{Cl}^*\big(\operatorname{Int}(A)\big) \subset \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$.

(3) ⇒ (4). Let A be semi-I-open. By (3), we have $A \subset \operatorname{Cl}^*\big(\operatorname{Int}(A)\big) \subset \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$. Thus, A is pre-I-open.

(4) ⇒ (5). Let A be a strongly β-I-open set. Then $\operatorname{Cl}^*(A)$ is semi-I-open. By (4), $\operatorname{Cl}^*(A)$ is pre-I-open. Thus, $\operatorname{Cl}^*(A) \subset \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$ and hence $\operatorname{Cl}^*(A)$ is open.

(5) ⇒ (6). Let A be strongly β-I-open. By (5), $\operatorname{Cl}^*(A) = \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$. Thus, $A \subset \operatorname{Cl}^*(A) = \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$ and hence A is pre-I-open.

(6) ⇒ (7). Let A be a semi-I-open set. Since a semi-I-open set is strongly β-I-open, then by (6) it is pre-I-open. Since A is semi-I-open and pre-I-open, A is α-I-open.

(7) ⇒ (1). Let A be an open set of X. Then $\operatorname{Cl}^*(A)$ is semi-I-open and by (7) $\operatorname{Cl}^*(A)$ is α-I-open. Therefore, $\operatorname{Cl}^*(A) \subset \operatorname{Int}\Big(\operatorname{Cl}^*\Big(\operatorname{Int}\big(\operatorname{Cl}^*(A)\big)\Big)\Big) = \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$ and hence, $\operatorname{Cl}^*(A) = \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$. Hence $\operatorname{Cl}^*(A)$ is open and X is ⋆-extremally disconnected. □

THEOREM 5. *The following are equivalent for an ideal space (X,τ,I):*

(1) *X is ⋆-extremally disconnected,*

(2) $\operatorname{Cl}^*(A) \cap \operatorname{Cl}(B) = \emptyset$ *for every open set A and every ⋆-open set B with $A \cap B = \emptyset$,*

(3) $\operatorname{Cl}^*(A) \cap \operatorname{Cl}(B) \subset \operatorname{Cl}(A \cap B)$ *for every open set A and every ⋆-open set B,*

(4) $\operatorname{Cl}^*\Big(\operatorname{Int}\big(\operatorname{Cl}^*(A)\big)\Big) \cap \operatorname{Cl}(B) = \emptyset$ *for every subset $A \subset X$ and every ⋆-open set B with $A \cap B = \emptyset$.*

PROOF. (2) ⇒ (1). Suppose that $\operatorname{Cl}^*(A) \cap \operatorname{Cl}(B) = \emptyset$ for every open set A and every ⋆-open set B with $A \cap B = \emptyset$. Let U be an open subset of X. Since U and $X\backslash \operatorname{Cl}^*(U)$ are disjoint open and ⋆-open sets, respectively, then $\mathrm{Cl}^*(U) \cap \mathrm{Cl}\left(X \setminus \mathrm{Cl}^*(U)\right) = \emptyset$. This implies that $\mathrm{Cl}^*(U) \subset \mathrm{Int}\left(\mathrm{Cl}^*(U)\right)$. Thus, $\mathrm{Cl}^*(U)$ is open and hence X is $\star$-extremally disconnected.

(1) $\Rightarrow$ (3). Let A and B be open and $\star$-open sets, respectively. Since $\mathrm{Cl}^*(A)$ is open and B is $\star$-open, then

$$\mathrm{Cl}^*(A) \cap \mathrm{Cl}\,(B) \subset \mathrm{Cl}\left(\mathrm{Cl}^*(A) \cap B\right) \subset \mathrm{Cl}\left(\mathrm{Cl}^*(A \cap B)\right) \subset \mathrm{Cl}\,(A \cap B).$$

Thus, $\mathrm{Cl}^*(A) \cap \mathrm{Cl}\,(B) \subset \mathrm{Cl}\,(A \cap B)$.

(3) $\Rightarrow$ (2). Let A and B be open and $\star$-open sets, respectively with $A \cap B = \emptyset$. By (3), we have $\mathrm{Cl}^*(A) \cap \mathrm{Cl}\,(B) \subset \mathrm{Cl}\,(A \cap B) = \mathrm{Cl}\,(\emptyset) = \emptyset$. Thus, $\mathrm{Cl}^*(A) \cap \mathrm{Cl}\,(B) = \emptyset$.

(2) $\Rightarrow$ (4). Let $A \subset X$ and B be a $\star$-open set with $A \cap B = \emptyset$. Since $\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)$ is open and $\mathrm{Int}\left(\mathrm{Cl}^*(A)\right) \cap B = \emptyset$, by (2), $\mathrm{Cl}^*\left(\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)\right) \cap \mathrm{Cl}\,(B) = \emptyset$.

(4) $\Rightarrow$ (2). Let A and B be an open and a $\star$-open set, respectively with $A \cap B = \emptyset$. By (4), we have $\mathrm{Cl}^*\left(\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)\right) \cap \mathrm{Cl}\,(B) = \emptyset$. Since $\mathrm{Cl}^*(A) \subset \mathrm{Cl}^*\left(\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)\right)$, then $\mathrm{Cl}^*(A) \cap \mathrm{Cl}\,(B) = \emptyset$. □

Definition 6. An ideal space (X, τ, I) is called $\star$-normal if for any two disjoint open and $\star$-open sets A and B, respectively, there exist disjoint $\star$-closed and closed sets M and N, respectively, such that $A \subset M$ and $B \subset N$.

Theorem 7. *The following are equivalent for an ideal space* (X, τ, I)*:*

(1) X *is* $\star$*-normal,*

(2) X *is* $\star$*-extremally disconnected.*

Proof. (1) $\Rightarrow$ (2). Let (X, τ, I) be $\star$-normal and A be an open subset of X. Then, A and $B = X \setminus \mathrm{Cl}^*(A)$ are disjoint open and $\star$-open sets, respectively. This implies that there exist disjoint $\star$-closed and closed sets M and N, respectively, such that $A \subset M$ and $B \subset N$. Since $\mathrm{Cl}^*(A) \subset \mathrm{Cl}^*(M) = M \subset X \setminus N \subset X \setminus B = \mathrm{Cl}^*(A)$, then $\mathrm{Cl}^*(A) = M$. Since $B \subset N \subset X \setminus M = B$, then $B = N$. Thus, $\mathrm{Cl}^*(A) = X \setminus N$ is open. Hence, X is $\star$-extremally disconnected.

(2) $\Rightarrow$ (1). Let X be $\star$-extremally disconnected. Let A and B be two disjoint open and $\star$-open sets, respectively. Then $\mathrm{Cl}^*(A)$ and $X \setminus \mathrm{Cl}^*(A)$ are disjoint $\star$-closed and closed sets containing A and B, respectively. Thus, (X, τ, I) is $\star$-normal. □

4. R-I-open sets

Definition 8. A subset A of an ideal space (X, τ, I) is called

(1) R-I-open [15] if $A = \mathrm{Int}\left(\mathrm{Cl}^*(A)\right)$.

(2) R-I-closed [15] if its complement is R-I-open.

THEOREM 9. *For an ideal space (X,τ,I), the following properties are equivalent:*

(1) *X is $\star$-extremally disconnected,*

(2) *Every R-I-open subset of X is $\star$-closed in X,*

(3) *Every R-I-closed subset of X is $\star$-open in X.*

PROOF. (1) $\Rightarrow$ (2). Let X be $\star$-extremally disconnected. Let A be an R-I-open subset of X. Then $A = \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$. Since A is an open set, then $\operatorname{Cl}^*(A)$ is open. Thus, $A = \operatorname{Int}\big(\operatorname{Cl}^*(A)\big) = \operatorname{Cl}^*(A)$ and hence A is $\star$-closed.

(2) $\Rightarrow$ (1). Suppose that every R-I-open subset of X is $\star$-closed in X. Let $A \subset X$ be an open set. Since $\operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$ is R-I-open, then it is $\star$-closed. This implies that $\operatorname{Cl}^*(A) \subset \operatorname{Cl}^*\Big(\operatorname{Int}\big(\operatorname{Cl}^*(A)\big)\Big) = \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$ since $A \subset \operatorname{Int}\big(\operatorname{Cl}^*(A)\big)$. Thus, $\operatorname{Cl}^*(A)$ is open and hence X is $\star$-extremally disconnected.

(2) $\Leftrightarrow$ (3). Obvious. □

REMARK 10. The following example shows that the union of two R-I-open sets need not to be R-I-open.

EXAMPLE 11. Let $X = \{a,b,c,d\}$, $\tau = \big\{X, \emptyset, \{a\}, \{c\}, \{a,c\}\big\}$ and $I = \big\{\emptyset, \{a\}\big\}$. The set $\{a,c\}$ is not R-I-open but the sets $\{a\}$ and $\{c\}$ are R-I-open sets.

THEOREM 12. *If (X,τ,I) is a $\star$-extremally disconnected ideal space, then the following properties hold:*

(1) *$A \cap B$ is R-I-closed for all R-I-closed subsets A and B of X.*

(2) *$A \cup B$ is R-I-open for all R-I-open subsets A and B of X.*

PROOF. (1) Let X be $\star$-extremally disconnected. Let A and B be R-I-closed subsets of X. Since A and B are closed, by Theorem 4, then $\operatorname{Int}^*(A)$ and $\operatorname{Int}^*(B)$ is closed. This implies that

$$A \cap B = \operatorname{Cl}\big(\operatorname{Int}^*(A)\big) \cap \operatorname{Cl}\big(\operatorname{Int}^*(B)\big)$$

$$= \operatorname{Int}^*(A) \cap \operatorname{Int}^*(B) = \operatorname{Int}^*(A \cap B) \subset \operatorname{Cl}\big(\operatorname{Int}^*(A \cap B)\big).$$

On the other hand, we have

$$\operatorname{Cl}\big(\operatorname{Int}^*(A \cap B)\big) = \operatorname{Cl}\big(\operatorname{Int}^*(A) \cap \operatorname{Int}^*(B)\big)$$

$$\subset \operatorname{Cl}\big(\operatorname{Int}^*(A)\big) \cap \operatorname{Cl}\big(\operatorname{Int}^*(B)\big) = A \cap B.$$

Thus, $A \cap B$ is R-I-closed.

(2) It follows from (1). □

Theorem 13. *The following are equivalent for an ideal space (X,τ,I):*
(1) *X is $\star$-extremally disconnected.*
(2) *The $\star$-closure of every semi-I-open subset of X is open.*
(3) *The $\star$-closure of every pre-I-open subset of X is open.*
(4) *The $\star$-closure of every R-I-open subset of X is open.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). Let A be a semi-I-open (pre-I-open) set. Then A is strongly β-I-open and by Theorem 4, $\mathrm{Cl}^*(A)$ is open in X.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4). Let A be any R-I-open set of X. Then A is semi-I-open and pre-I-open and hence, $\mathrm{Cl}^*(A)$ is open in X.

(4) $\Rightarrow$ (1). Suppose that the $\star$-closure of every R-I-open subset of X is open. Let $A\subset X$ be an open set. This implies that $\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)$ is an R-I-open set. Then $\mathrm{Cl}^*\left(\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)\right)$ is open. We have

$$\mathrm{Cl}^*(A)\subset \mathrm{Cl}^*\left(\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)\right)=\mathrm{Int}\left(\mathrm{Cl}^*\left(\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)\right)\right)=\mathrm{Int}\left(\mathrm{Cl}^*(A)\right).$$

Thus, $\mathrm{Cl}^*(A)$ is open and hence X is $\star$-extremally disconnected. □

5. The δ-I-closure

Definition 14 [15]. A point x in an ideal space (X,τ,I) is called a δ-I-cluster point of A if $\mathrm{Int}\left(\mathrm{Cl}^*(V)\right)\cap A\neq\emptyset$ for each open neighborhood V of x. The set of all δ-I-cluster points of A is called the δ-I-closure of A and is denoted by δ-I-$\mathrm{Cl}\,(A)$.

Definition 15 [15]. A subset A of an ideal space (X,τ,I) is called
(1) δ-I-closed if δ-I-$\mathrm{Cl}\,(A)=A$.
(2) δ-I-open if its complement is δ-I-closed.

Lemma 16. *If A is a strongly β-I-open set in an ideal space (X,τ,I), then* $\mathrm{Cl}\,(A)=\delta$-I-$\mathrm{Cl}\,(A)$.

Proof. Let A be a strongly β-I-open set. Suppose that $x\notin \mathrm{Cl}\,(A)$. There exists an open set U containing x such that $U\cap A=\emptyset$. We have $U\cap \mathrm{Cl}^*(A)$ $=\emptyset$. This implies that $\mathrm{Int}\left(\mathrm{Cl}^*(U)\right)\cap \mathrm{Cl}^*\left(\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)\right)=\emptyset$. Since A is a strongly β-I-open set, then $\mathrm{Int}\left(\mathrm{Cl}^*(U)\right)\cap A=\emptyset$. Thus, $x\notin\delta$-I-$\mathrm{Cl}\,(A)$ and $\mathrm{Cl}\,(A)\supset\delta$-$I$-$\mathrm{Cl}\,(A)$. On the other hand, we have $\mathrm{Cl}\,(A)\subset\delta$-$I$-$\mathrm{Cl}\,(A)$. Hence, we obtain $\mathrm{Cl}\,(A)=\delta$-I-$\mathrm{Cl}\,(A)$. □

Definition 17. Let (X,τ,I) be an ideal space. An ideal I is called a boundary ideal [9] or a codense ideal [5] if $\tau\cap I=\{\emptyset\}$.

Lemma 18 [3, Corollary 2]. *Let (X,τ,I) be an ideal space and I be codense. Then* $\mathrm{Cl}\,(A)=\mathrm{Cl}^*(A)$ *for every semi-open set A of X.*

Lemma 19. *Let (X,τ,I) be an ideal space and I be codense. If A is a strongly β-I-open set, then* $\mathrm{Cl}^*(A)=\mathrm{Cl}\,(A)=\delta$-$I$-$\mathrm{Cl}\,(A)$.

Proof. Let A be a strongly β-I-open set of X. Then

$$A\subset \mathrm{Cl}^*\big(\mathrm{Int}\,\big(\mathrm{Cl}^*(A)\big)\big) \quad\text{and}\quad \mathrm{Cl}^*(A)=\mathrm{Cl}^*\big(\mathrm{Int}\,\big(\mathrm{Cl}^*(A)\big)\big)$$

and hence, $\mathrm{Cl}^*(A)$ is semi-I-open. Since every semi-I-open set is semi-open, $\mathrm{Cl}^*(A)$ is semi-open and by Lemma 18, $\mathrm{Cl}\,\big(\mathrm{Cl}^*(A)\big)=\mathrm{Cl}^*\big(\mathrm{Cl}^*(A)\big)$. Therefore, $\mathrm{Cl}\,(A)\subset \mathrm{Cl}\,\big(\mathrm{Cl}^*(A)\big)=\mathrm{Cl}^*\big(\mathrm{Cl}^*(A)\big)=\mathrm{Cl}^*(A)$. Since $\tau\subset\tau^*$, $\mathrm{Cl}^*(A)\subset \mathrm{Cl}\,(A)$ and hence, $\mathrm{Cl}^*(A)=\mathrm{Cl}\,(A)$. It follows from Lemma 16 that $\mathrm{Cl}^*(A)=\mathrm{Cl}\,(A)=\delta$-$I$-$\mathrm{Cl}\,(A)$. □

Theorem 20. *The following are equivalent for a codense ideal space (X,τ,I):*

(1) *X is $\star$-extremally disconnected.*
(2) *The δ-I-closure of every semi-I-open subset of X is open.*
(3) *The δ-I-closure of every strongly β-I-open subset of X is open.*
(4) *The δ-I-closure of every pre-I-open subset of X is open.*
(5) *X is extremally disconnected.*

Proof. (1) $\Rightarrow$ (2). Let X be $\star$-extremally disconnected. Let $A\subset X$ be a semi-I-open set. By Lemma 19, we have $\mathrm{Cl}\,(A)=\delta$-I-$\mathrm{Cl}\,(A)$. Since X is $\star$-extremally disconnected, by Theorem 13 and Lemma 19, then $\mathrm{Cl}^*(A)=\mathrm{Cl}\,(A)=\delta$-$I$-$\mathrm{Cl}\,(A)$ is open.

(2) $\Rightarrow$ (1). Suppose that the δ-I-closure of every semi-I-open subset of X is open. Let $A\subset X$ be an open set. By Lemma 19, $\mathrm{Cl}\,(A)=\mathrm{Cl}^*(A)=\delta$-$I$-$\mathrm{Cl}\,(A)$. Thus, $\mathrm{Cl}^*(A)$ is open and hence X is $\star$-extremally disconnected.

(1) $\Rightarrow$ (3). Let A be a strongly β-I-open set. By Theorem 4, $\mathrm{Cl}^*(A)$ is open and hence, by Lemma 19 δ-I-$\mathrm{Cl}\,(A)$ is open.

(3) $\Rightarrow$ (2) $\big((3)\Rightarrow(4)\big)$. Let A be a semi-I-open (resp. pre-I-open) set of X. Since every semi-I-open (resp. pre-I-open) set is strongly β-I-open, by (3) δ-I-$\mathrm{Cl}\,(A)$ is open.

(2) $\Rightarrow$ (5) $\big((4)\Rightarrow(5)\big)$. Let A be an open set of X. Every open set is semi-I-open and pre-I-open. By (2) (resp. (4)), δ-I-$\mathrm{Cl}\,(A)$ is open and hence, by Lemma 19, $\mathrm{Cl}\,(A)$ is open. Therefore, X is extremally disconnected.

(5) $\Rightarrow$ (1). Let A be an open set of X. By (5) and Lemma 19, $\mathrm{Cl}^*(A)=\mathrm{Cl}\,(A)$ is open in X and X is $\star$-extremally disconnected. □

6. Semi*-I-open sets

Definition 21 [6]. A subset A of an ideal space (X,τ,I) is called

(1) semi*-I-open if $A \subset \mathrm{Cl}\left(\mathrm{Int}^*(A)\right)$.

(2) semi*-I-closed if its complement is semi*-I-open.

LEMMA 22. *A subset A of an ideal space (X,τ,I) is semi*-I-open if and only if* $\mathrm{Cl}(A)=\mathrm{Cl}\left(\mathrm{Int}^*(A)\right)$.

PROOF. Let A be semi*-I-open. We have $A \subset \mathrm{Cl}\left(\mathrm{Int}^*(A)\right)$ and hence $\mathrm{Cl}(A) \subset \mathrm{Cl}\left(\mathrm{Int}^*(A)\right)$. Since $\mathrm{Cl}\left(\mathrm{Int}^*(A)\right) \subset \mathrm{Cl}(A)$, then

$$\mathrm{Cl}(A)=\mathrm{Cl}\left(\mathrm{Int}^*(A)\right).$$

Conversely, since $\mathrm{Cl}(A)=\mathrm{Cl}\left(\mathrm{Int}^*(A)\right)$, then $A \subset \mathrm{Cl}(A)=\mathrm{Cl}\left(\mathrm{Int}^*(A)\right)$. Thus, A is semi*-I-open. □

THEOREM 23. *The following are equivalent for an ideal space (X,τ,I):*

(1) *X is $\star$-extremally disconnected.*

(2) *If A is strongly β-I-open and B is semi*-I-open, then* $\mathrm{Cl}^*(A)\cap\mathrm{Cl}(B) \subset \mathrm{Cl}(A\cap B)$.

(3) *If A is semi-I-open and B is semi*-I-open, then* $\mathrm{Cl}^*(A)\cap\mathrm{Cl}(B) \subset \mathrm{Cl}(A\cap B)$.

(4) $\mathrm{Cl}^*(A)\cap\mathrm{Cl}(B)=\emptyset$ *for every semi-I-open set A and every semi*-I-open set B with $A\cap B=\emptyset$.*

(5) *If A is pre-I-open and B is semi*-I-open, then* $\mathrm{Cl}^*(A)\cap\mathrm{Cl}(B) \subset \mathrm{Cl}(A\cap B)$.

PROOF. (1) ⇒ (2). Let A be strongly β-I-open and B be semi*-I-open. By Theorem 4, $\mathrm{Cl}^*(A)$ is open. We have

$$\mathrm{Cl}^*(A)\cap\mathrm{Cl}(B)=\mathrm{Cl}^*(A)\cap\mathrm{Cl}\left(\mathrm{Int}^*(B)\right) \subset \mathrm{Cl}\left(\mathrm{Cl}^*(A)\cap\mathrm{Int}^*(B)\right)$$

$$\subset \mathrm{Cl}\left(\mathrm{Cl}^*\left(A\cap\mathrm{Int}^*(B)\right) \subset \mathrm{Cl}\left(\mathrm{Cl}\left(A\cap\mathrm{Int}^*(B)\right)\right.\right.$$

$$=\mathrm{Cl}\left(A\cap\mathrm{Int}^*(B)\right) \subset \mathrm{Cl}(A\cap B).$$

Thus, $\mathrm{Cl}^*(A)\cap\mathrm{Cl}(B)=\mathrm{Cl}(A\cap B)$.

(2) ⇒ (3). It follows from the fact that every semi-I-open set is strongly β-I-open.

(3) ⇒ (4). Obvious.

(4) ⇒ (1). Let A be a semi-I-open set. Since A and $X\setminus\mathrm{Cl}^*(A)$ are disjoint semi-I-open and semi*-I-open, respectively, by (4), we have $\mathrm{Cl}^*(A) \cap \mathrm{Cl}\left(X\setminus\mathrm{Cl}^*(A)\right)=\emptyset$. This implies that $\mathrm{Cl}^*(A)\subset\mathrm{Int}\left(\mathrm{Cl}^*(A)\right)$. Thus, $\mathrm{Cl}^*(A)$ is open. Hence, by Theorem 13, X is $\star$-extremally disconnected.

(2) ⇒ (5). It follows from the fact that every pre-I-open set is strongly β-I-open.

(5) $\Rightarrow$ (1). Let A and B be open and $\star$-open, respectively, with $A \cap B = \emptyset$. Since A and B are pre-I-open and semi*-I-open, respectively, by (5) $\mathrm{Cl}^*(A) \cap \mathrm{Cl}(B) \subset \mathrm{Cl}(A \cap B) = \emptyset$. Thus, $\mathrm{Cl}^*(A) \cap \mathrm{Cl}(B) = \emptyset$. By Theorem 5, X is $\star$-extremally disconnected. □

LEMMA 24. *If A is a semi*-I-open set in an ideal space (X, τ, I), then $\mathrm{Cl}(A) = \delta$-I-$\mathrm{Cl}(A)$.*

PROOF. Let A be a semi*-I-open set. We have $\mathrm{Cl}(A) \subset \delta$-$I$-$\mathrm{Cl}(A)$. We shall show that $\mathrm{Cl}(A) \supset \delta$-$I$-$\mathrm{Cl}(A)$. Suppose that $x \notin \mathrm{Cl}(A)$. There exists an open set U containing x such that $U \cap A = \emptyset$. We have $U \cap \mathrm{Int}^*(A) = \emptyset$ and hence $\mathrm{Int}\left(\mathrm{Cl}^*(U)\right) \cap \mathrm{Cl}\left(\mathrm{Int}^*(A)\right) = \emptyset$. Since A is a semi*-I-open set, then $\mathrm{Int}\left(\mathrm{Cl}^*(U)\right) \cap A = \emptyset$. Thus, $x \notin \delta$-I-$\mathrm{Cl}(A)$ and $\mathrm{Cl}(A) \supset \delta$-$I$-$\mathrm{Cl}(A)$. Hence, $\mathrm{Cl}(A) = \delta$-I-$\mathrm{Cl}(A)$. □

COROLLARY 25. *The following are equivalent for an ideal space (X, τ, I):*

(1) *X is $\star$-extremally disconnected.*

(2) *If A is strongly β-I-open and B is semi*-I-open, then $\mathrm{Cl}^*(A) \cap \delta$-$I$-$\mathrm{Cl}(B) \subset \mathrm{Cl}(A \cap B)$.*

(3) *If A is semi-I-open and B is semi*-I-open, then $\mathrm{Cl}^*(A) \cap \delta$-$I$-$\mathrm{Cl}(B) \subset \mathrm{Cl}(A \cap B)$.*

(4) *$\mathrm{Cl}^*(A) \cap \delta$-$I$-$\mathrm{Cl}(B) = \emptyset$ for every semi-I-open set A and every semi*-I-open set B with $A \cap B = \emptyset$.*

(5) *If A is pre-I-open and B is semi*-I-open, then $\mathrm{Cl}^*(A) \cap \delta$-$I$-$\mathrm{Cl}(B) \subset \mathrm{Cl}(A \cap B)$.*

PROOF. It follows from Theorem 23 and Lemma 24. □

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