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ON GENERALIZED NEIGHBOURHOOD SYSTEMS

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Abstract. It is shown in the paper [1] that every generalized topology can be generated by a generalized neighbourhood system. Following the paper [3], we discuss some questions related to this construction.

0. Introduction

Let X be a non-empty set with power set $\exp X$. According to [1], a *generalized topology* (briefly GT) μ on X is a subset of $\exp X$ such that $\emptyset \in \mu$ and every union of elements of μ belongs to μ. It is described in [1] that a GT μ can always be obtained by a construction starting from a so called generalized neighbourhood system. In [3], some questions concerning generalized neighbourhood systems are discussed in the case when the obtained GT is in fact a topology. The purpose of the present paper is to show that a great part of these questions can be considered in the more general case when they are related with a GT more general than a topology.

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1. Preliminaries

For a given set X, let us consider a subset κ of $\exp X$. We say that κ is *ascending* iff $A \in \kappa$, $A \subset B \subset X$ imply $B \in \kappa$.

LEMMA 1.1. *If* $\kappa \subset \exp X$ *then* $\kappa^+ = \{B \subset X :$ *there is* $A \in \kappa$ *such that* $A \subset B\}$ *is the smallest ascending set containing* κ.

PROOF. If $B \in \kappa^+$ and $B \subset C \subset X$ then there is $A \in \kappa$ such that $A \subset B$. Then $A \subset C$ as well, so that $C \in \kappa^+$ and κ^+ is ascending.

Clearly $\kappa \subset \kappa^+$. Finally if $\lambda \supset \kappa$ is ascending and $B \in \kappa^+$ then there is $A \in \kappa$ such that $A \subset B$. Since $A \in \lambda$ and λ is ascending, clearly $B \in \lambda$. So $\kappa^+ \subset \lambda$. $\square$

For a given $\kappa \subset \exp X$, κ^+ is called the *ascending hull* of κ.

Consider now a map $\psi : X \to \exp(\exp X)$ such that $x \in X$, $V \in \psi(x)$ imply $x \in V$. According to [1], a map ψ satisfying these conditions is called a *generalized neighbourhood system* (briefly GNS) on X. If ψ is a GNS on X, let μ_ψ denote the collection of all subsets $M \subset X$ such that $x \in M$ implies the existence of a set $V \in \psi(x)$ satisfying $V \subset M$. By [1], 1.2, then μ_ψ is a GT on X, *generated* by the GNS ψ.

The generation of a GT by a GNS is a sufficiently general construction since, by [1], 1.3, we have $\mu = \mu_\psi$ for an arbitrary GT μ and the GNS ψ satisfying $\psi(x) = \{M \in \mu : x \in M\}$ $(x \in X)$.

If μ is a GT on X, then the operations i_μ and c_μ are considered in [2]; here $i_\mu A$ is the largest subset of A belonging to μ and $c_\mu A$ is the smallest superset of A having a complement belonging to μ.

Let us now consider a map $\omega : \exp X \to \exp X$. According to [2], such a map is called an *operation* on X. If ω is an operation on X, we write ωA for $\omega(A)$ $(A \subset X)$. An operation ω is said to be *monotonic* iff $A \subset B \subset X$ implies $\omega A \subset \omega B$, *enlarging* iff $A \subset X$ implies $A \subset \omega A$ and *restricting* iff $\omega A \subset A$ for $A \subset X$.

E.g. the above defined operation $^+$ is clearly monotonic and enlarging.

Let us now consider a map $f : X \to Y$, a GT μ on X, a GT ν on Y, a GNS ψ on X and a GNS ϕ on Y. According to [1], the map f is said to be (μ, ν)-*continuous* iff $N \in \nu$ implies $f^{-1}(N) \in \mu$ and (ψ, ϕ)-*continuous* iff $x \in X$, $U \in \phi\big(f(x)\big)$ imply the existence of $V \in \psi(x)$ such that $f(V) \subset U$.

2. Generating of a GT by a GNS

It is easy to see that several GNS's can generate the same GT.

EXAMPLE 2.1. Let $X = \{a, b, c\}$, $\psi(a) = \big\{\{a, b\}\big\}$, $\psi(b) = \big\{\{b, c\}\big\}$, $\psi(c) = \big\{\{a, c\}\big\}$. Then ψ is a GNS on X and clearly $\mu = \mu_\psi = \{\emptyset, X\}$. Similarly if

$\phi(a) = \{\{a,c\}\}$, $\phi(b) = \{\{a,b\}\}$, $\phi(c) = \{\{b,c\}\}$ then $\mu_\phi = \mu$ again. However, $\psi \neq \phi$. □

In fact, it would be sufficient to consider ascending GNS's, where a GNS ψ on X is said to be *ascending* iff $\psi(x)$ is ascending whenever $x \in X$. For an arbitrary GNS ψ on X, let us write ψ^+ for the map ψ^+ such that $\psi^+(x) = \big(\psi(x)\big)^+$ for $x \in X$.

Lemma 2.2. *ψ^+ is a GNS on X whenever ψ is a GNS on X.*

Proof. $V \in \psi^+(x)$ iff there is $U \in \psi(x)$ such that $U \subset V$. Now $x \in U$ implies $x \in V$. □

Proposition 2.3. *If ψ is a GNS on X then $\mu_{\psi^+} = \mu_\psi$.*

Proof. If $M \in \mu_\psi$ and $x \in M$ then there is $V \in \psi(x)$ such that $V \subset M$. $\psi(x) \subset \psi(x)^+$ clearly implies $V \in \psi^+(x) = \psi(x)^+$ so that $M \in \mu_{\psi^+}$.

Conversely if $M \in \mu_{\psi^+}$ and $x \in M$ then there is $V \in \psi^+(x)$ such that $V \subset M$. By $\psi^+(x) = \psi(x)^+$, there is $U \in \psi(x)$ such that $U \subset V$. Then $U \subset M$ for $U \in \psi(x)$ so that $M \in \mu_\psi$. □

E.g. if X, ψ, ϕ are taken from 1.1 then $\psi^+(a) = \{\{a,b\}, X\}$, $\psi^+(b) = \{\{b,c\}, X\}$, $\psi^+(c) = \{\{a,c\}, X\}$. Similarly $\phi^+(x) = \phi(x) \cup \{X\}$ for $x \in X$. Hence $\psi^+ \neq \phi^+$ so that distinct ascending GNS's can generate the same GT.

It is easy to see that there can exist more GNS's ψ on X having the same GNS ψ^+:

Example 2.4. Let $X = \mathbb{R}$ and, for $x \in X$, $V_x = (-\infty, x)$. Let $\psi(x)$ be composed of all sets V_r such that $x \in V_r$ and r is rational. Similarly, let $\phi(x)$ be composed of all sets V_i such that $x \in V_i$ and i is irrational. Clearly both ψ and ϕ are GNS's and $\psi(x) \neq \phi(x)$ whenever $x \in X$. However $\psi^+(x) = \phi^+(x)$ is composed of all sets U such that $V_y \subset U$ for some $y > x$. □

3. The operations ι_ψ and γ_ψ

Now consider a GNS ψ on X. We define two operations ι_ψ and γ_ψ on X by, for $A \subset X$, $x \in \iota_\psi(A)$ iff there is $V \in \psi(x)$ such that $V \subset A$, $x \in \gamma_\psi(A)$ iff $V \in \psi(x)$ implies $V \cap A \neq \emptyset$. (Cf. [1].)

Proposition 3.1. *The operation ι_ψ is monotonic and restricting.*

Proof. If $A \subset B \subset X$ and $x \in \iota_\psi A$ then there is $V \in \psi(x)$ such that $V \subset A$. Then $V \subset B$ shows $x \in \iota_\psi B$. Thus ι_ψ is monotonic. If $x \in \iota_\psi A$ then there is $V \in \psi(x)$ satisfying $V \subset A$, hence $x \in V \subset A$ showing that $\iota_\psi A \subset A$ and ι_ψ is restricting. □

Proposition 3.2. *γ_ψ is monotonic and enlarging.*

PROOF. If $x \in \gamma_\psi A$ and $A \subset B \subset X$ then $V \in \psi(x)$ implies $V \cap A \neq \emptyset$, consequently $V \cap B \neq \emptyset$. If $x \in A$ and $V \in \psi(x)$ then $x \in V \cap A \neq \emptyset$. □

PROPOSITION 3.3. *$\iota_\psi(X - A) = X - \gamma_\psi A$ for each $A \subset X$.*

PROOF. If $x \in \iota_\psi(X - A)$ then there is $V \in \psi(x)$ such that $V \subset X - A$, hence $V \cap A = \emptyset$ and $x \notin \gamma_\psi A$, $x \in X - \gamma_\psi A$. Conversely if $x \in X - \gamma_\psi A$ then there is $V \in \psi(x)$ such that $V \cap A = \emptyset$, so $V \subset X - A$ and $x \in \iota_\psi(X - A)$. □

Cf. [1], 1.4.

As a converse of 3.1, we can say:

PROPOSITION 3.4. *If the operation ω on X is monotonic and restricting, then there is a GNS ψ on X satisfying $\omega = \iota_\psi$.*

PROOF. Assume $\omega : \exp X \to \exp X$ is monotonic and restricting. Define, for $x \in X$ and $V \subset X$, $V \in \psi(x)$ iff $x \in \omega V$.

ψ is a GNS as $x \in X$, $V \in \psi(x)$ imply $x \in V$ because $x \in \omega V \subset V$.

If $x \in \iota_\psi A$ for some $A \subset X$ then there is $V \in \psi(x)$ such that $V \subset A$. Then $x \in \omega V$ by the definition of $\psi(x)$ so that $x \in \omega A$ as ω is monotonic. Hence $\iota_\psi A \subset \omega A$.

If $x \in \omega A$ then $A \in \psi(x)$ so that (taking $V = A$), there is $V \in \psi(x)$ satisfying $V \subset A$, so $x \in \iota_\psi A$. Consequently $\omega A \subset \iota_\psi A$. Thus $\omega = \iota_\psi$. □

Analogously:

PROPOSITION 3.5. *If the operation ω' is monotonic and enlarging then there is a GNS ψ such that $\omega' = \gamma_\psi$.*

PROOF. Define $\omega A = X - \omega'(X - A)$. Then $A \subset B \subset X$ implies $X - B \subset X - A$, hence $\omega'(X - B) \subset \omega'(X - A)$, $X - \omega'(X - A) \subset X - \omega'(X - B)$, $\omega A \subset \omega B$. Therefore ω is monotonic. Moreover, since $X - A \subset \omega'(X - A)$, necessarily $\omega A = X - \omega'(X - A) \subset X - (X - A) = A$ and ω is restricting.

By 3.4, there is a GNS ψ on X satisfying $\omega = \iota_\psi$. Hence, for $A \subset X$, $\gamma_\psi A = X - \iota_\psi(X - A) = X - \omega(X - A) = \omega' A$ (cf. 3.3). □

The definition of ι_ψ can be formulated with the help of ψ^+:

THEOREM 3.6. *If ψ is a GNS on X, $x \in X$ and $A \subset X$, then $x \in \iota_\psi A$ iff $A \in \psi^+(x)$.*

PROOF. $x \in \iota_\psi A$ iff there is $V \in \psi(x)$ such that $V \subset A$ iff $A \in \psi(x)^+$ iff $A \in \psi^+(x)$. □

We can use the operation ι_ψ in the construction of the GT μ_ψ:

THEOREM 3.7. *For a GNS ψ on X and $M \subset X$, we have $M \in \mu_\psi$ iff $\iota_\psi M = M$.*

PROOF. If $M \in \mu_\psi$ then, for each point $x \in M$, there is a set $V \in \psi(x)$ such that $V \subset M$ and then $x \in \iota_\psi M$. Hence $M \subset \iota_\psi M$. Also $\iota_\psi M \subset M$ by 3.1, hence $M \in \mu_\psi$ implies $\iota_\psi M = M$.

Conversely if $\iota_\psi M = M$ then $x \in M$ implies the existence of $V \in \psi(x)$ such that $V \subset M$, hence $M \in \mu_\psi$. $\square$

The operations ι_ψ and γ_ψ have some relations with i_{μ_ψ} and c_{μ_ψ}:

PROPOSITION 3.8. *If ψ is a GNS on X and $A \subset X$ then $i_{\mu_\psi} A \subset \iota_\psi A$, $\gamma_\psi A \subset c_{\mu_\psi} A$.*

PROOF. If $x \in i_{\mu_\psi} A$ then $M = i_{\mu_\psi} A \in \mu_\psi$ so that there is $V \in \psi(x)$ such that $V \subset M$, hence $V \subset A$ since $M \subset A$. Therefore $x \in \iota_\psi A$.

The other inclusion can be deduced from the former one using $\gamma_\psi A = X - \iota_\psi(X - A)$ (cf. 3.3) and $c_{\mu_\psi} A = X - i_{\mu_\psi}(X - A)$ (cf. [2], (1.6)). $\square$

Cf. [1], 1.4.

The following corollary is similar to 3.4:

COROLLARY 3.9. *If the operation ω is monotonic and restricting then there exists a GT μ such that $A \in \mu$ iff $\omega A = A$.*

PROOF. By 3.4 there exists a GNS ψ such that $\omega = \iota_\psi$. Let $\mu = \mu_\psi$. Then $A \in \mu$ iff $A \in \mu_\psi$ iff $\iota_\psi A = A$ (cf. 3.7) iff $\omega A = A$. $\square$

4. Questions on continuity

Let ψ be a GNS on X, ϕ a GNS on Y and $f : X \to Y$. We show that the (ψ, ϕ)-continuity of f can be characterized by properties of ι_ψ and ι_ϕ:

PROPOSITION 4.1. *The mapping f is (ψ, ϕ)-continuous iff $B \subset Y$ implies*

$$f^{-1}(\iota_\phi B) \subset \iota_\psi\big(f^{-1}(B)\big).$$

PROOF. First let f be (ψ, ϕ)-continuous. If $x \in f^{-1}(\iota_\phi(B)$ i.e. $f(x) \in \iota_\phi(B)$ then there is $U \in \phi\big(f(x)\big)$ such that $U \subset B$. We have some $V \in \psi(x)$ satisfying $f(V) \subset U$, $V \subset f^{-1}(U) \subset f^{-1}(B)$ implying $x \in \iota_\psi\big(f^{-1}(B)\big)$.

Now suppose $f^{-1}(\iota_\phi(B) \subset \iota_\psi\big(f^{-1}(B)\big)$ for $B \subset Y$. If $x \in X$ and $U \in \phi\big(f(x)\big)$ then $f(x) \in \iota_\phi U$ and $x \in f^{-1}\big(\iota_\phi(U)\big)$ so that $x \in \iota_\psi\big(f^{-1}(U)\big)$. Therefore there is $V \in \psi(x)$ satisfying $V \subset f^{-1}(U)$, i.e. $f(V) \subset U$. Hence f is (ψ, ϕ)-continuous. $\square$

This is similar to [1], 2.5. By [1], 2.1, the (ψ, ϕ)-continuity of f implies its (μ_ψ, μ_ϕ)-continuity. The example [1], 2.2 shows that the converse is not true in general.

References

[1] Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (2002), 351–357.
[2] Á. Császár, Generalized open sets in generalized topologies, *Acta Math. Hungar.*, **105** (2005), 53–66.
[3] W. K. Min, On weak neighborhood systems and spaces, *Acta Math. Hungar.* (to appear).