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ON WEAK NEIGHBORHOOD SYSTEMS AND SPACES

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Abstract. We introduce and study the concepts of weak neighborhood systems, weak neighborhood spaces, $w(\psi, \psi')$-continuity, w-continuity and w^*-continuity on WNS's.

1. Introduction

In [4], Siwiec introduced the notions of weak neighborhoods and weak base in a topological space. Császár introduced the notions of generalized neighborhood systems and generalized topological spaces in [1]. He also introduced the notions of continuous functions on generalized neighborhood systems and generalized topological spaces. In this paper we introduce the weak neighborhood systems defined by using the notion of weak neighborhoods. They are generalized systems of open neighborhood systems but stronger than generalized neighborhood systems. The weak neighborhood system induces a weak neighborhood space (briefly WNS) which is independent of a neighbor-

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hood space [2]. We introduce and characterize the notions of w-continuity, w^*-continuous, $w(\psi,\psi')$-continuity, and associated interior and closure operators on weak neighborhood spaces.

2. Preliminaries

Let X be a nonempty set and $\psi : X \to \exp\big(\exp(X)\big)$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighborhood* [1] of $x \in X$ and ψ is called a *generalized neighborhood system* (briefly GNS) on X. Let g be a collection of subsets of X. Then g is called a *generalized topology* [1] on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \cup_{i\in I} G_i \in g$. The elements of g are called *g-open* sets and the complements are called *g-closed* sets. Let ψ be a GNS on X and $G \in g_\psi$ iff $G \subseteqq X$ satisfies: If $x \in G$, then there is $V \in \psi(x)$ such that $V \subseteqq G$. Let g and g' be generalized topologies on X and Y, respectively. Then a function $f : X \to Y$ is said to be *(g, g')-continuous* [1] if $G' \in g'$ implies that $f^{-1}(G') \in g$. Let ψ and ψ' be generalized neighborhood systems on X and Y, respectively. Then a function $f : X \to Y$ is said to be *(ψ,ψ')-continuous* [1] if for $x \in X$ and $U \in \psi'\big(f(x)\big)$, there is $V \in \psi(x)$ such that $f(V) \subseteqq U$. f is said to be *gn-continuous* [3] if for every $A \in \psi'\big(f(x)\big)$, $f^{-1}(A)$ is in $\psi(x)$.

A collection $\mathbf{H}$ of subsets of X is called an *m-family* [3] on X if $\cap\mathbf{H} \neq \emptyset$. Let ψ be a GNS in X and let $\mathbf{H}$ be an m-family on X. Then we say that an m-family $\mathbf{H}$ converges to $x \in X$ if $\mathbf{H}$ is finer than $\psi(x)$ i.e., $\psi(x) \subseteqq \mathbf{H}$.

We recall the following concepts defined in [2]. A collection $\mathbf{C}$ of subsets of X is called a *stack* if $A \in \mathbf{C}$ whenever $B \in \mathbf{C}$ and $B \subseteqq A$.

A stack $\mathbf{H}$ on a set X is called a *p-stack* if it satisfies the following condition: $A, B \in \mathbf{H}$ implies $A \cap B \neq \emptyset$.

Let X be a set, $\dot{x} = \big\{ A \subseteqq X : \{x\} \subseteqq A \big\}$, and let $\nu = \big\{ \nu(x) : x \in X \big\}$, where for all $x \in X$, $\nu(x)$ is a p-stack and $\nu(x) \subseteqq \dot{x}$. Then ν is called a *neighborhood structure* on X, $\nu(x)$ is called the *ν-neighborhood stack at x*, and (X,ν) is called a *neighborhood space*.

Let X be a topological space and $x \in X$. A collection T_x of subsets of X is called a collection of *weak neighborhoods* [4] of x if each member of T_x contains x, for any two members of T_x their intersection is also a member of T_x, and the following is true. Letting $\mathbf{B} = \cup\{T_x : x \in X\}$, $\mathbf{B}$ is a *weak base* [4] for X i.e., a set U is open in X iff for every point x in U there exists a $B \in T_x$ such that $B \subseteqq U$.

Consider a function $I : 2^X \to 2^X$ satisfying these axioms:

(C1) $I(A) \subseteqq A$, for all $A \subseteqq X$;
(C2) $I(A \cap B) = I(A) \cap I(B)$, for all $A, B \in 2^X$;
(C3) $I(X) = X$.

Throughout this paper, we use the term *interior operator* on X to mean a set function $I : 2^X \to 2^X$ which satisfies (C1), (C2), and (C3).

3. Weak neighborhood systems and spaces

DEFINITION 3.1. Let $\psi : X \to \exp\big(\exp(X)\big)$. Then ψ is called a *weak neighborhood system* on X if it satisfies the following:
(1) For $x \in X$ and $V \in \psi(x)$, $x \in V$.
(2) For $U, V \in \psi(x)$, $V \cap U \in \psi(x)$.
(3) For $x \in X$, $\psi(x) \neq \emptyset$.

Then the pair (X, ψ) is called a *weak neighborhood space* (briefly WNS) on X. $V \in \psi(x)$ is called a *weak neighborhood* of $x \in X$.

DEFINITION 3.2. Let (X, ψ) be a WNS on X and $A \subseteqq X$. The interior and closure of A on ψ (denoted by $\iota_\psi(A)$, $\gamma_\psi(A)$, respectively) are defined as follows:

$$\iota_\psi(A) = \big\{x \in A : \text{ there exists } V \in \psi(x) \text{ such that } V \subset A\big\};$$

$$\gamma_\psi(A) = \big\{x \in X : V \cap A \neq \emptyset \text{ for all } V \in \psi(x)\big\}.$$

THEOREM 3.3. *Let (X, ψ) be a WNS on X. Then the following hold:*
(1) $\iota_\psi(A) \subseteqq A$*, for all* $A \subseteqq X$.
(2) $\iota_\psi(A \cap B) = \iota_\psi(A) \cap \iota_\psi(B)$*, for all* $A, B \in 2^X$.
(3) $\iota_\psi(X) = X$.

PROOF. (1) and (3) are obvious.

(2) Let $x \in \iota_\psi(A) \cap \iota_\psi(B)$; then there are U, V in $\psi(x)$ such that $U \subseteqq A$, $V \subseteqq B$. By Definition 3.1, $U \cap V \in \psi(x)$. It follows that $x \in \iota_\psi(A \cap B)$.

The converse inclusion is obvious.

EXAMPLE 3.4. Let $X = \{a, b, c, d\}$ and $\psi : X \to \exp\big(\exp(X)\big)$ be a weak neighborhood system defined as follows: $\psi(a) = \big\{\{a, c\}\big\}$, $\psi(b) = \big\{\{b, c\}\big\}$, $\psi(c) = \psi(d) = \{X\}$. Then for $A = \{a, b, c\} \subseteqq X$, $\iota_\psi(A) = \{a, b\}$ but $\iota_\psi\big(\iota_\psi(A)\big) = \emptyset$, and so $\iota_\psi(A) \neq \iota_\psi\big(\iota_\psi(A)\big)$.

THEOREM 3.5. *Let (X, ψ) be a WNS on X. Then the following hold:*
(1) $\emptyset = \gamma_\psi(\emptyset)$.
(2) $A \subseteqq \gamma_\psi(A)$*, for all* $A \subseteqq X$.
(3) $\gamma_\psi(A \cup B) = \gamma_\psi(A) \cup \gamma_\psi(B)$*, for all* $A, B \in 2^X$.
(4) $\gamma_\psi(A) = X - \iota_\psi(X - A)$, $\iota_\psi(A) = X - \gamma_\psi(X - A)$.

PROOF. By Definition 3.2, it is obvious.

From Theorem 3.3 and the definition of interior operator, we get the following results:

THEOREM 3.6. (1) *Let* (X,ψ) *be a WNS on* X *and* $I: 2^X \to 2^X$ *be defined as* $I(A) = \iota_\psi(A)$ *for each* $A \subseteqq X$. *Then* I *is an interior operator.*

(2) *Let* $I: 2^X \to 2^X$ *be an interior operator and* $\phi_I: X \to \exp\big(\exp(X)\big)$ *be defined as* $\phi_I(x) = \big\{ I(A) : x \in I(A) \text{ for } A \subseteqq X \big\}$. *Then* ϕ_I *is a weak neighborhood system induced by* I.

Let (X,ψ) be a WNS on X and $I: 2^X \to 2^X$ be defined as $I(A) = \iota_\psi(A)$ for each $A \subseteqq X$. Then for a weak neighborhood system ϕ_{ι_ψ} induced by $I = \iota_\psi$, there is no relation between ϕ_{ι_ψ} and ψ as the following example shows.

EXAMPLE 3.7. Let $X = \{a,b,c\}$ and let $\psi: X \to \exp\big(\exp(X)\big)$ be a weak neighborhood system defined as follows: $\psi(a) = \big\{ \{a\}, \{a,c\} \big\}$, $\psi(b) = \big\{ \{b\}, \{b,c\} \big\}$, $\psi(c) = \{X\}$. Then $\phi_{\iota_\psi}: X \to \exp\big(\exp(X)\big)$ is a weak neighborhood system induced by ι_ψ as follows: $\phi_{\iota_\psi}(a) = \big\{ \{a\}, \{a,b\}, X \big\}$, $\phi_{\iota_\psi}(b) = \big\{ \{b\}, \{a,b\}, X \big\}$, $\phi_{\iota_\psi}(c) = \{X\}$.

DEFINITION 3.8. Let (X,ψ) and (Y,ϕ) be two WNS's. Then $f: X \to Y$ is said to be *$w(\psi,\phi)$-continuous* if for $x \in X$ and $V \in \phi\big(f(x)\big)$, there is $U \in \psi(x)$ such that $f(U) \subseteqq V$.

THEOREM 3.9. *Let* $f: X \to Y$ *be a function on two WNS's* (X,ψ) *and* (Y,ϕ). *Then the following statements are equivalent:*

(1) f *is* $w(\psi,\phi)$*-continuous.*

(2) $f\big(\gamma_\psi(A)\big) \subseteqq \gamma_\phi f(A)$ *for* $A \subseteqq X$.

(3) $\gamma_\psi f^{-1}(B) \subseteqq f^{-1}\big(\gamma_\phi(B)\big)$ *for* $B \subseteqq Y$.

(4) $f^{-1}\big(\iota_\phi(B)\big) \subseteqq \iota_\psi f^{-1}(B)$ *for* $B \subseteqq Y$

PROOF. (1) $\Rightarrow$ (2). Let $x \in \gamma_\psi(A)$. If $f(x)$ is not in $\gamma_\phi f(A)$, then there exists $V \in \phi\big(f(x)\big)$ such that $V \cap \gamma_\phi f(A) = \emptyset$. By the (ψ,ϕ)-continuity, there is $U \in \psi(x)$ such that $f(U) \subseteqq V$, and so $f(U) \cap \gamma_\phi f(A) = \emptyset$. Consequently, $U \cap A = \emptyset$; a contradiction.

(2) $\Rightarrow$ (3). Let $A = f^{-1}(B)$ for $B \subseteqq Y$; then by (2), $f\big(\gamma_\psi(A)\big) \subseteqq \gamma_\phi f(A) = \gamma_\phi f\big(f^{-1}(B)\big) \subseteqq \gamma_\phi(B)$. Thus $\gamma_\psi f^{-1}(B) \subseteqq f^{-1}\big(\gamma_\phi(B)\big)$.

(3) $\Rightarrow$ (4). By Theorem 3.5, it is obvious.

(4) $\Rightarrow$ (1). For $x \in X$, let $V \in \phi\big(f(x)\big)$; then $f(x) \in \iota_\phi(V)$. By (4), $x \in f^{-1}\big(\iota_\phi(V)\big) \subseteqq \iota_\psi f^{-1}(V)$. By the definition of interior, there exists $U \in \psi(x)$ such that $U \subseteqq f^{-1}(V)$. Thus we get the result.

4. w-open sets and w-continuity

DEFINITION 4.1. Let (X,ψ) be a WNS on X and $G \subseteqq X$. Then G is called a w_ψ-*open* set if for each $x \in G$, there is $V \in \psi(x)$ such that $V \subseteqq G$.

Let $W_\psi(X)$ denote the collection of all w_ψ-*open* sets on a WNS (X,ψ). The complements of w_ψ-*open* sets are called w_ψ-*closed* sets.

THEOREM 4.2. *Let (X,ψ) be a WNS on X. Then the collection $W_\psi(X)$ of all w_ψ-*open *subsets of X is a topology on X.*

PROOF. (1) Clearly both $\emptyset$ and X are w_ψ-open.

(2) Let A,B be two w_ψ-open sets; then for each $x \in A\cap B$ there exist $U,V \in \psi(x)$ such that $U \subseteqq A$, $V \subseteqq B$. Since $\psi(x)$ is a weak neighborhood, $U\cap V \in \psi(x)$. Thus $A\cap B$ is also w_ψ-open.

(3) Let $\mathbf{C} \subseteqq W_\psi(X)$; then for each $x \in \cup\mathbf{C}$, there exists $A \in \mathbf{C}$ such that $x \in A$, and so there is $V \in \psi(x)$ such that $V \subseteqq A \subseteqq \cup\mathbf{C}$. Thus $\cup\mathbf{C}$ is w_ψ-open.

REMARK. In a topological space (X,τ), let us define $\psi : X \to \exp\big(\exp(X)\big)$ as $\psi(x) = \{G \in \tau : x \in G\}$ for $x \in X$. Then ψ is a weak neighborhood system, and so $W_\psi(X) = \tau$.

DEFINITION 4.3. Let (X,ψ) be a WNS on X and $A \subseteqq X$. The w_ψ-*interior* of A (denoted by $i_\psi(A)$) is the union of all $G \subseteqq A$, $G \in W_\psi(X)$, and the w_ψ-*closure* of A (denoted by $c_\psi(A)$) is the intersection of all w_ψ-closed sets containing A.

THEOREM 4.4. *Let (X,ψ) be a WNS on X and $A \subseteqq X$. Then the following hold:*

(1) *$i_\psi(A) \subseteqq \iota_\psi(A)$;*

(2) *$\gamma_\psi(A) \subseteqq c_\psi(A)$;*

(3) *A is w_ψ-open iff $i_\psi(A) = A$;*

(4) *A is w_ψ-closed iff $c_\psi(A) = A$.*

PROOF. (1) For $x \in i_\psi(A)$, there exists a w_ψ-open set G such that $x \in G \subseteqq A$. By the definition of w_ψ-open sets, there exists $V \in \psi(x)$ such that $x \in V \subseteqq G \subseteqq A$. Thus $x \in \iota_\psi(A)$.

(2) It is similar to (1).

(3) and (4) are obvious.

EXAMPLE 4.5. Let $X = \{a,b,c,d\}$ and $\psi : X \to \exp\big(\exp(X)\big)$ a weak neighborhood system defined as follows: $\psi(a) = \big\{\{a,c\}\big\}$, $\psi(b) = \big\{\{b\}\big\}$, $\psi(c) = \psi(d) = \{X\}$. Then for $A = \{a,b,c\} \subseteqq X$, $\iota_\psi(A) = \{a,b\}$ and $i_\psi(A) = \{b\}$, and so $\iota_\psi(A) \neq i_\psi(A)$.

THEOREM 4.6. *Let (X,ψ) be a WNS on X and $A \subseteqq X$. Then $\iota_\psi(A) = A$ iff A is w_ψ-open.*

Proof. Suppose that $\iota_\psi(A) = A$. For each $x \in \iota_\psi(A)$, there exists $V_x \in \psi(x)$ such that $x \in V_x \subseteqq A = \iota_\psi(A)$, and so $\iota_\psi(A)$ is a w_ψ-open set.

For the converse, let A be w_ψ-open; then from Theorem 4.4(3), it follows $A = i_\psi(A) \subseteqq \iota_\psi(A) \subseteqq A$, i.e., $\iota_\psi(A) = A$.

Theorem 4.7. *Let $I : 2^X \to 2^X$ be an interior operator. Then there exists a weak neighborhood system ψ induced by I such that A is w_ψ-open iff $I(A) = A$.*

Proof. Let us define $\psi : X \to \exp\big(\exp(X)\big)$ as follows: for $x \in X$, $\psi(x) = \big\{V : I(V) = V \text{ and } x \in V\big\}$. Then ψ is a WNS. Now A is w_ψ-open iff $A = \cup_{x\in A} V$ where $V \in \psi(x)$. Since $I(V) = V$ for $V \in \psi(x)$, $A = \cup_{x\in A} V = \cup_{x\in A} I(V) \subseteqq I(\cup_{x\in A} V) = I(A)$. Consequently, we can say A is w_ψ-open iff $I(A) = A$.

Definition 4.8. Let $f : X \to Y$ be a function on two WNS's (X, ψ) and (Y, ϕ). Then f is said to be *w-continuous* if for every $A \in W_\phi(Y)$, $f^{-1}(A)$ is in $W_\psi(X)$.

Theorem 4.9. *Let $f : X \to Y$ be a function on two WNS's (X, ψ) and (Y, ϕ). Then the following are equivalent:*

(1) *f is w-continuous.*
(2) *For each w_ϕ-closed set F in Y, $f^{-1}(F)$ is w_ψ-closed in X.*
(3) *$f\big(c_\psi(A)\big) \subseteqq c_\phi\big(f(A)\big)$ for all $A \subseteqq X$.*
(4) *$c_\psi\big(f^{-1}(B)\big) \subseteqq f^{-1}\big(c_\phi(B)\big)$ for all $B \subseteqq Y$.*
(5) *$f^{-1}\big(i_\phi(B)\big) \subseteqq i_\psi\big(f^{-1}(B)\big)$ for all $B \subseteqq Y$.*
(6) *$f : (X, W_\psi(X)) \to \big(Y, W_\phi(Y)\big)$ is continuous.*

Proof. Obvious.

Theorem 4.10. *Let $f : X \to Y$ be a function on two WNS's (X, ψ) and (Y, ϕ). If f is (ψ, ϕ)-continuous, then it is also w-continuous.*

Proof. Let $A \in W_\phi(Y)$; then from Theorem 4.6, it follows $\iota_\phi(A) = A$. By Theorems 4.4 and 4.9(5), we get $f^{-1}(A) = f^{-1}\big(\iota_\phi(A)\big) \subseteqq \iota_\psi f^{-1}(A)$. Thus $f^{-1}(A)$ is w_ψ-open.

From the following example, we can say that not every w-continuous function is $w(\psi, \phi)$-continuous.

Example 4.11. Let $X = \{a, b, c, d\}$ and let $\psi : X \to \exp\big(\exp(X)\big)$ be a weak neighborhood system defined as in Example 4.5. Let us define the function $f : (X, \psi) \to (X, \psi)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$, $f(d) = c$. Then f is w-continuous, but it is not $w(\psi, \psi)$-continuous because for $a \in X$ and $V = \{a, c\} \in \psi\big(f(a)\big)$, there is no $U \in \psi(a)$ such that $f(U) \subseteqq V$.

5. New interior and closure operators on a WNS

DEFINITION 5.1. Let (X,ψ) be a WNS and $A \subseteqq X$.

(1) $I^*_\psi(A) = \{x \in A : A \in \psi(x)\}$.

(2) $\mathrm{cl}^*_\psi(A) = \{x \in X : X - A \notin \psi(x)\}$.

THEOREM 5.2. *Let (X,ψ) be a WNS and $A, B \subset X$. Then the following hold:*

(1) $I^*_\psi(A) \subseteqq A$.

(2) $I^*_\psi(A) \cap I^*_\psi(B) \subseteqq I^*_\psi(A \cap B)$.

(3) $I^*_\psi(A) \subseteqq \iota_\psi(A)$.

(4) $I^*_\psi(A) = X - \mathrm{cl}^*_\psi(X - A)$.

PROOF. (1) Obvious.

(2) For $A, B \subseteqq X$, let $x \in I^*_\psi(A) \cap I^*_\psi(B)$; then $A, B \in \psi(x)$. From the property of weak neighborhood, it follows that $A \cap B \in \psi(x)$. Hence $x \in I^*_\psi(A \cap B)$.

(3) Obvious.

(4) Let $x \in I^*_\psi(A)$ for $A \subseteqq X$; then $A = X - (X - A) \in \psi(x)$ and by Definition 5.1, $x \notin \mathrm{cl}^*_\psi(X - A)$. Thus we have $x \in X - \mathrm{cl}^*_\psi(X - A)$.

The converse is obvious.

EXAMPLE 5.3. Let $X = \{a,b,c\}$ and $A = \{a,b\}$. Consider a weak neighborhood system $\psi : X \to \exp\bigl(\exp(X)\bigr)$ defined as follows: $\psi(a) = \bigl\{\{a,b\}\bigr\}$, $\psi(b) = \bigl\{\{b\}\bigr\}$, $\psi(c) = \{X\}$. Then we get the following results:

(1) Since $I^*_\psi(A) = \{a\}$ and $\iota_\psi(A) = \{a,b\}$, so $I^*_\psi(A) \neq \iota_\psi(A)$.

(2) Let $B = X$; then $A \subseteqq B$ but since $I^*_\psi(B) = \{c\}$, $I^*_\psi(A) \nsubseteqq I^*_\psi(B)$.

(3) Since $I^*_\psi(A \cap B) = \{a\}$ and $I^*_\psi(A) \cap I^*_\psi(B) = \emptyset$, $I^*_\psi(A \cap B) \nsubseteqq I^*_\psi(A) \cap I^*_\psi(B)$.

(4) Since $I^*_\psi(A) = \{a\}$ and $I^*_\psi\bigl(I^*_\psi(A)\bigr) = \emptyset$, $I^*_\psi\bigl(I^*_\psi(A)\bigr) \neq I^*_\psi(A)$.

THEOREM 5.4. *Let (X,ψ) be a WNS and $A \subseteqq X$. Then the following hold:*

(1) $A \subseteqq \mathrm{cl}^*_\psi(A)$.

(2) $\mathrm{cl}^*_\psi(A \cup B) \subseteqq \mathrm{cl}^*_\psi(A) \cup \mathrm{cl}^*_\psi(B)$.

(3) $\mathrm{cl}^*_\psi(A) = X - I^*_\psi(X - A)$.

(4) $\gamma_\psi(A) \subseteqq \mathrm{cl}^*_\psi(A)$.

PROOF. (1), (3) and (4) are obvious.

(2) Let $x \notin \mathrm{cl}^*_\psi(A) \cup \mathrm{cl}^*_\psi(B)$; $X - A \in \psi(x)$ and $X - B \in \psi(x)$. From the property of weak neighborhood, it follows that $(X - A) \cap (X - B) \in \psi(x)$. Hence $x \notin \mathrm{cl}^*_\psi(A \cup B)$.

REMARK 5.5. From Example 5.3 and Theorem 5.4, we can deduce that the following statements are not always true:

(1) $\mathrm{cl}^*_\psi(A) = \gamma_\psi(A)$ for $A \subseteqq X$.

(2) For $A, B \subseteqq X$ if $A \subseteqq B$, then $\mathrm{cl}^*_\psi(A) \subseteqq \mathrm{cl}^*_\psi(B)$.

(3) $\mathrm{cl}^*_\psi(A) \cup \mathrm{cl}^*_\psi(B) \subseteqq \mathrm{cl}^*_\psi(A \cup B)$ for $A, B \subseteqq X$.

(4) $\mathrm{cl}^*_\psi\big(\mathrm{cl}^*_\psi(A)\big) = \mathrm{cl}^*_\psi(A)$ for $A \subseteqq X$.

THEOREM 5.6. *Let (X, ψ) be a WNS and $A \subseteqq X$. Then the following hold:*

(1) *If $I^*_\psi(A) = A$, then A is w_ψ-open.*

(2) *If $\mathrm{cl}^*_\psi(A) = A$, then A is w_ψ-closed.*

PROOF. (1) For $A \subseteqq X$, if $I^*_\psi(A) = A$, then by Theorem 5.2(3), $\iota_\psi(A) = A$. From Theorem 4.6, A is w_ψ-open.

(2) From Theorem 5.4(3), it is obvious.

EXAMPLE 5.7. Let $X = \{a, b, c, d\}$. Consider a weak neighborhood system $\psi : X \to \exp\big(\exp(X)\big)$ defined as follows: $\psi(a) = \big\{\{a, b\}\big\}$, $\psi(b) = \big\{\{b, c\}\big\}$, $\psi(c) = \big\{\{b, c\}\big\}$, $\psi(d) = \{X\}$. Let $A = \{a, b, c\}$; then A is w_ψ-open but $I^*_\psi(A) = \emptyset$, that is, $I^*_\psi(A) \neq A$.

THEOREM 5.8. *Let (X, ψ) be a WNS, $A \subseteqq X$ and let $\mathbf{B} = \big\{A \subseteqq X : I^*_\psi(A) = A\big\}$. Then $\Psi_{I^*} = \{\cup\sigma : \sigma \subseteqq \mathbf{B}\}$ is contained in $W_\psi(X)$.*

PROOF. From Theorem 5.6, we get the statement.

THEOREM 5.9. *Let (X, ψ) be a WNS and $A \subseteqq X$. Then the following hold:*

(1) $I^*_\psi(A) = \{x \in A : A \in \mathbf{H}$, *for every m-family* $\mathbf{H}$ *converging to* $x\}$.

(2) $\mathrm{cl}^*_\psi(A) = \{x \in X :$ *there exists an m-family* $\mathbf{H}$ *such that* $\mathbf{H}$ *converges to* x *and* $X - A \notin \mathbf{H}\}$.

PROOF. (1) Let $x \in I^*_\psi(A)$ and let an m-family $\mathbf{H}$ converge to x. Then it is obvious that $A \in \psi(x) \subseteqq \mathbf{H}$.

Suppose that for every m-family $\mathbf{H}$ converging to x, $A \in \mathbf{H}$. Then since clearly $\psi(x)$ converges to x, by hypothesis, $A \in \psi(x)$, so that $x \in I^*_\psi(A)$.

(2) Let $x \in \mathrm{cl}^*_\psi(A)$; then $X - A \notin \psi(x)$. Let $\mathbf{H} = \psi(x)$; then $\mathbf{H}$ satisfies the condition.

For the converse, let $\mathbf{H}$ be an m-family converging to x and $X - A \notin \mathbf{H}$. Then, since $\psi(x)$ is contained in $\mathbf{H}$, $X - A \notin \psi(x)$ so that $x \in \mathrm{cl}^*_\psi(A)$.

DEFINITION 5.10. Let $f : (X, \psi) \to (Y, \phi)$ be a function between WNS's (X, ψ) and (Y, ϕ). Then f is called *w^*-continuous* if for every $A \in \phi\big(f(x)\big)$, $f^{-1}(A)$ is in $\psi(x)$.

Every w^*-continuous function is $w(\psi, \phi)$-continuous but the converse may not be true as the following example shows.

EXAMPLE 5.11. Let $X=\{a,b,c\}$. Consider two weak neighborhood systems $\psi,\phi: X\to\exp\big(\exp(X)\big)$ defined as follows: $\psi(a)=\big\{\{a\},\{a,b\}\big\}$, $\psi(b)=\big\{\{b\}\big\}$, $\psi(c)=\{X\}$, $\phi(a)=\big\{\{a\},\{a,b\}\big\}$, $\phi(b)=\big\{\{a,b\}\big\}$, $\phi(c)=\{X\}$.

Let $f:(X,\psi)\to(X,\phi)$ be a function defined by $f(x)=x$, for $x\in X$. Then f is $w(\psi,\phi)$-continuous, but not w^*-continuous.

We get the following implications:

$$\begin{array}{ccccc}
w^*\text{-continuous} & \Longrightarrow & w(\psi,\phi)\text{-continuous} & \Longrightarrow & w\text{-continuous}\\
\Downarrow & & \Downarrow & & \Downarrow\\
gn\text{-continuous} & \Longrightarrow & (\psi,\phi)\text{-continuous} & \Longrightarrow & (g_\psi,g_\phi)\text{-continuous}
\end{array}$$

THEOREM 5.12. *Let $f:(X,\psi)\to(Y,\phi)$ be a function between the WNS's (X,ψ) and (Y,ϕ). Then the following are equivalent:*

(1) *f is w^*-continuous.*

(2) $f^{-1}\big(I^*_\phi(B)\big)\subseteqq I^*_\psi\big(f^{-1}(B)\big)$ *for* $B\subseteqq Y$.

(3) $\mathrm{cl}^*_\psi\big(f^{-1}(B)\big)\subseteqq f^{-1}\big(\mathrm{cl}^*_\phi(B)\big)$ *for* $B\subseteqq Y$.

PROOF. (1) $\Rightarrow$ (2). Suppose f is w^*-continuous and $x\in f^{-1}\big(I^*_\phi(A)\big)$; then $A\in\phi\big(f(x)\big)$. $f^{-1}(A)\in\psi(x)$ follows from the w^*-continuity, so that $x\in I^*_\psi\big(f^{-1}(A)\big)$.

(2) $\Rightarrow$ (1). It is obtained by Definition 5.1.

(2) $\Leftrightarrow$ (3). It is obvious by Theorem 5.2 and Theorem 5.4.

THEOREM 5.13. *Let $f:(X,\psi)\to(Y,\phi)$ be a bijective function between the WNS's (X,ψ) and (Y,ϕ). Then f is w^*-continuous iff $f\big(\mathrm{cl}^*_\psi(A)\big)\subseteqq\mathrm{cl}^*_\phi\big(f(A)\big)$ for $A\subseteqq X$.*

PROOF. Suppose f is w^*-continuous and $A\subseteqq X$. From Theorem 5.12, it follows $\mathrm{cl}^*_\psi\big(f^{-1}(f(A))\big)\subseteqq f^{-1}\big(\mathrm{cl}^*_\phi\big(f(A)\big)\big)$. Since f is injective, $\mathrm{cl}^*_\psi(A)\subseteqq f^{-1}\big(\mathrm{cl}^*_\phi\big(f(A)\big)\big)$.

Suppose $f\big(\mathrm{cl}^*_\psi(A)\big)\subseteqq\mathrm{cl}^*_\phi\big(f(A)\big)$ for $A\subseteqq X$. For $B\subseteqq Y$, by hypothesis and surjectivity, $f\big(\mathrm{cl}^*_\psi\big(f^{-1}(B)\big)\big)\subseteqq\mathrm{cl}^*_\phi\big(f\big(f^{-1}(B)\big)\big)=\mathrm{cl}^*_\phi(B)$. Hence from Theorem 5.12, it follows f is w^*-continuous.

THEOREM 5.14. *Let $f:(X,\psi)\to(Y,\phi)$ be a bijective function between the WNS's (X,ψ) and (Y,ϕ). Then f is w^*-continuous iff for an m-family $\mathbf{H}$ converging to $x\in X$, $f(\mathbf{H})$ converges to $f(x)$.*

PROOF. Suppose f is w^*-continuous and $\mathbf{H}$ is an m-family converging to $x\in X$. It is obvious that $f(\mathbf{H})=\big\{f(F): F\in\mathbf{H}\big\}$ is an m-family on Y. By hypothesis and surjectivity, we get $\phi\big(f(x)\big)\subseteqq f\big(\psi(x)\big)\subseteqq f(\mathbf{H})$, so that $f(\mathbf{H})$ converges to $f(x)$.

For the converse, let $G \in \phi\big(f(x)\big)$ for $G \subseteqq Y$. Clearly since $\psi(x)$ converges to x, by hypothesis, we get $\phi\big(f(x)\big) \subseteqq f\big(\psi(x)\big)$ for $x \in X$. Since f is injective, $f^{-1}(G) \in \psi(x)$, so that f is w^*-continuous.

THEOREM 5.15. *Let $f : (X,\psi) \to (Y,\phi)$ be a function between two WNS's (X,ψ) and (Y,ϕ). Then the following are equivalent:*

(1) *For $x \in X$ and for every $A \in \psi(x)$, $f(A) \in \phi\big(f(x)\big)$.*

(2) *$f\big(I^*_\psi(A)\big) \subseteqq I^*_\phi\big(f(A)\big)$ for $A \subseteqq X$.*

PROOF. (1) $\Rightarrow$ (2). Let $y \in f\big(I^*_\psi(A)\big)$; then there exists $x \in I^*_\psi(A)$ such that $f(x) = y$, and so $A \in \psi(x)$. From (1), it follows $f(A) \in \phi\big(f(x)\big)$, so that we have $y \in I^*_\phi\big(f(A)\big)$.

(2) $\Rightarrow$ (1). For the proof, let $A \in \psi(x)$; then by hypothesis, $f(x) \in f\big(I^*_\psi(A)\big) \subseteqq I^*_\phi\big(f(A)\big)$. Hence we have $f(A) \in \phi\big(f(x)\big)$.

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