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HOW TO DERIVE FINITE SEMIMODULAR LATTICES FROM DISTRIBUTIVE LATTICES?

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Abstract. It is proved that the class of finite semimodular lattices is the same as the class of cover-preserving join-homomorphic images of direct products of finitely many finite chains.

There is a trivial "representation theorem" for finite lattices: each of them is a join-homomorphic image of a finite distributive lattice. This follows from the fact that the finite free join semilattices (with zero) are the finite Boolean lattices. The goal of the present paper is to give two analogous but stronger representation theorems for finite *semimodular* (also called upper semimodular) lattices. Both theorems state that these lattices are very special join-homomorphic images of appropriate finite distributive lattices. This way we generalize the main results of G. Grätzer and E. Knapp [4] and [5]. Since even the above-mentioned trivial representation theorem was useful in [1], there is a hope that the new achievements will be of some interest in the future.

To formulate our results we need the following notions. A sublattice $\{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$ of a lattice is called a *covering square* if $a_1 \wedge a_2 \prec a_i \prec a_1 \vee a_2$ for $i = 1, 2$. A planar lattice is called *slim* if every covering square is an interval. Now let L and K be finite lattices. A join-homomorphism

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$\varphi: L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$. Similarly, a join-congruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving. As usual, $\mathrm{J}(L)$ stands for the poset of all nonzero join-irreducible elements of L. For a poset P, $\mathrm{H}(P)$ denotes the lattice of all hereditary subsets (order ideals) of P. The *width* $\mathrm{w}(P)$ of a (finite) poset P is defined to be $\max\{n : P$ has an n-element antichain$\}$.

First we prove:

LEMMA. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then $a \equiv a \vee b$ (Φ) implies $b \equiv a \vee b$ (Φ).*

PROOF. To verify the "only if" part suppose that some S fails the described property. Then $[b]\Phi \neq [a \vee b]\Phi$ in M/Φ gives $[b]\Phi < [a \vee b]\Phi = [a]\Phi$ in M/Φ. Similarly, $[a \wedge b]\Phi < [b]\Phi$. Hence $[a \wedge b]\Phi \not\preceq [a]\Phi$ shows that Φ is not cover-preserving.

Conversely, to prove the "if" part, assume that Φ is not cover-preserving. We have to find a covering square S for which the described property fails. By the assumption on Φ there are $a \prec b \in M$ such that, with the notations $A = [a]\Phi \in M/\Phi$ and $B = [b]\Phi \in M/\Phi$, $A \preceq B$ fails. Hence there is a $C \in M/\Phi$ with $A < C < B$. With some $c_0 \in C$ let $c = a \vee c_0 \in A \vee C = C$ and let $d = b \vee c \in B \vee C = B$. Let $a = x_0 \prec x_1 \prec \cdots \prec x_{n-1} \prec x_n = c$ be a maximal chain in the interval $[a, c]$, and let i be the smallest subscript with $x_i \notin A$. The situation for $(n, i) = (6, 4)$ is depicted in Fig. 1; notice however that $x_i \in C$ would also be possible.

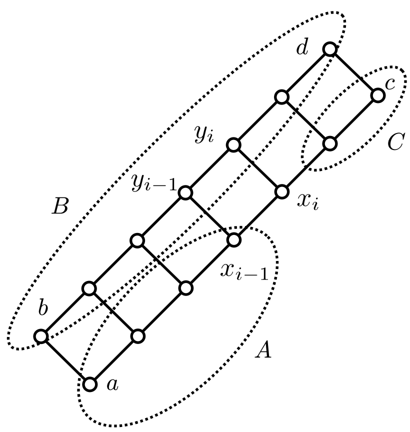

Fig. 1

Let $y_j := b \vee x_j$, $0 \leqq j \leqq i$. Using semimodularity, it follows via induction that $x_j \prec y_j$ and $y_{j-1} \prec y_j$ for all $1 \leqq j \leqq n$. In other words, each edge of Fig. 1 represents a pair of covering elements. Hence $S = \{x_{i-1}, x_i, y_{i-1}, y_i\}$ is a covering square we were looking for. □

THEOREM 1. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of* $\mathrm{w}\big(\mathrm{J}(L)\big)$ *finite chains.*

PROOF. Let $k = \mathrm{w}\big(\mathrm{J}(L)\big)$. In virtue of Dilworth [2], $\mathrm{J}(L)$ is the union of k appropriate chains. Let us extend these chains of $\mathrm{J}(L)$ to *maximal* chains $C_1, \dots, C_k$ of L. Then $\mathrm{J}(L) \subseteqq C_1 \cup \dots \cup C_k$. We may assume that $k \geqq 2$. Denote $C_1 \times \dots \times C_k$ by C and define a join-homomorphism

$$\varphi \colon C \to L, \quad (x_1, \dots, x_k) \mapsto x_1 \vee \dots \vee x_k.$$

Clearly, φ is surjective. Let Φ denote the kernel of φ, and let $\{a \wedge b, a, b, a \vee b\}$ be a covering square in C. Apart from indexing we may suppose that $a \wedge b = (x, y, z_3, \dots, z_k)$, $a = (x^+, y, z_3, \dots, z_k)$ and $b = (x, y^+, z_3, \dots, z_k)$ where x^+ resp. y^+ denotes the unique cover of x resp. y in C_1 resp. in C_2. Under the assumption $a \equiv a \vee b\ (\Phi)$, $a \wedge b \not\equiv a\ (\Phi)$ and $a \wedge b \not\equiv b\ (\Phi)$ we have to show that $b \equiv a \vee b\ (\Phi)$. With the notation $u = \varphi(a \wedge b) = x \vee y \vee z_3 \vee \dots \vee z_k$ our assumption means

$$x^+ \not\leqq u, \qquad y^+ \not\leqq u, \qquad y^+ \leqq x^+ \vee u.$$

Using semimodularity we obtain from $x \prec x^+$ that $u = x \vee u \prec x^+ \vee u$, and we conclude $u \prec y^+ \vee u$ similarly. Hence $u \prec y^+ \vee u \leqq x^+ \vee u$ yields $x^+ \vee u = y^+ \vee u$, implying $\varphi(b) = y^+ \vee u = x^+ \vee y^+ \vee u = \varphi(a \vee b)$. This shows $b \equiv a \vee b\ (\Phi)$. □

The representation defined in the proof of Theorem 1 is illustrated by Fig. 2.

COROLLARY 1. *The cover-preserving join-homomorphic images of finite distributive lattices are exactly the finite semimodular lattices.*

PROOF. This follows from Theorem 1 since cover-preserving join-homomorphisms preserve semimodularity in virtue of Lemma 16 of Grätzer and Knapp [4]. □

Let us recall the main result from Grätzer and Knapp [5]:

THEOREM 2 (Grätzer and Knapp [5]). *Each finite planar semimodular lattice can be obtained from a cover-preserving join-homomorphic image of the direct product of two finite chains via adding doubly irreducible elements to the interiors of covering squares.*

Now we formulate a corollary of Theorem 1, which easily implies Theorem 2.

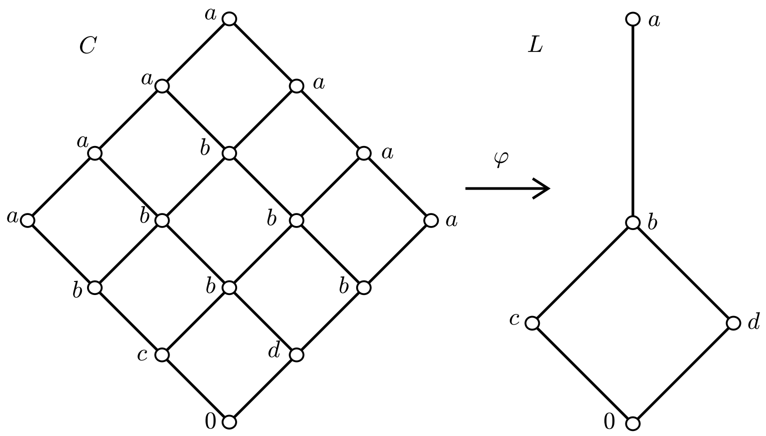

Fig. 2

Corollary 2. *Each finite planar* slim *semimodular lattice is a cover-preserving join-homomorphic image of the direct product of two finite chains.*

Proof. In virtue of Theorem 1, it is sufficient to show that $\mathrm{w}\big(\mathrm{J}(L)\big) = 2$ when L is a planar slim semimodular lattice. Let C_1 and C_2 be the left and right boundary chain in L, resp. We intend to show that every $p \in \mathrm{J}(L)$ is in $C_1 \cup C_2$. Suppose this is not the case and choose a $p \in \mathrm{J}(L) \setminus (C_1 \cup C_2)$. Let q stand for the unique lower cover of p. Let u resp. v denote the greatest element of C_1 resp. C_2 such that $u \leqq p$ resp. $v \leqq p$. Let u^+ resp. v^+ be the upper cover of u in C_1 resp. v in C_2. Then $u \leqq q$ and $v \leqq q$ but $u^+ \nleqq q$ and $v^+ \nleqq q$. From $u \prec u^+$ we conclude $q \prec q \vee u^+$ by semimodularity, and $q \prec q \vee v^+$ follows similarly.

Now p, $q \vee u^+$ and $q \vee v^+$ are all upper covers of q and, since $u^+ \nleqq p$, $p \notin \{q \vee u^+, q \vee v^+\}$. We distinguish two cases.

First assume that $x = q \vee u^+ = q \vee v^+$. Let x be leftward from p in the (fixed planar) Hasse diagram of L, and remember that v and v^+ belong to the right boundary C_2. Consider the paths A and B in the diagram witnessing $v < p \prec p \vee x$ and $v^+ < x$, resp. Since $v^+ \nleqq p$, these two paths intersect in the plane but not at a vertex, which contradicts planarity. When x is rightward from p, the situation is similar. This shows that $q \vee u^+ = q \vee v^+$ is impossible.

Therefore q has three distinct covers. For brevity, let $\{a, b, c\} = \{p, q \vee u^+, q \vee v^+\}$. Since each of a, b, c covers q, semimodularity yields that each of $a \vee b$, $a \vee c$ and $b \vee c$ covers its two joinands. If two of these joins, say $a \vee b$ and $b \vee c$, coincide then $a < a \vee c \leqq a \vee b \vee c = a \vee b$ and $a \prec a \vee b$ give $a \vee c = a \vee b$, whence $\{a, b, c\}$ generates an M_3 sublattice, which contradicts the assumption that L is slim. Hence $\{a \vee b, a \vee c, b \vee c\}$ is a three element antichain. It is well-known, cf. Lemma I.5.9 in Grätzer [3], that this antichain generates a sublattice isomorphic to the eight element boolean lattice. This is a contradiction, for this boolean lattice is not planar. □

Roughly saying, the join-homomorphism acts identically on the poset of all join-irreducible elements in our second representation theorem.

THEOREM 3. *Every finite semimodular lattice L is a cover-preserving join-homomorphic image of the unique distributive lattice D determined by $\mathrm{J}(D) \cong \mathrm{J}(L)$. Moreover, the restriction of an appropriate cover-preserving join-homomorphism from D onto L is a $\mathrm{J}(D) \to \mathrm{J}(L)$ order isomorphism.*

PROOF. The argument, based on the Lemma, is similar to the proof of the previous theorem. Let L be a finite semimodular lattice and let D be the distributive lattice $\mathrm{H}\big(\mathrm{J}(L)\big)$. Then, clearly, $\varphi\colon \mathrm{H}\big(\mathrm{J}(L)\big) \to L$, $I \mapsto \bigvee_L I$ is a join-homomorphism from $D = \mathrm{H}\big(\mathrm{J}(L)\big)$ onto L. This is illustrated in Fig. 3. Let Φ denote the kernel of φ.

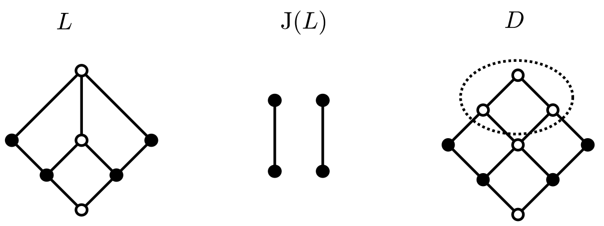

Fig. 3: L, $\mathrm{J}(L)$ and the corresponding distributive lattice $D = \mathrm{H}\big(\mathrm{J}(L)\big)$

Now we prove that φ is cover-preserving. Take a covering square $S = \{a \wedge b, a, b, a \vee b\}$ in $D = \mathrm{H}\big(\mathrm{J}(L)\big)$. Denote $a \wedge b$ by I; it is a hereditary subset of $\mathrm{J}(L)$. The condition $a \wedge b \prec a$ means that there exists an element $p \in \mathrm{J}(L) \setminus I$ such that the hereditary subset (p, I) generated by p and I does not contain any other join-irreducible element, i.e. $a = (p, I) = \{p\} \cup I$. Similarly, $b = (q, I) = \{q\} \cup I$ for some $q \in \mathrm{J}(L)$, $q \notin I$.

We assume in L that $p \nleq \bigvee_L I$ (this is equivalent to the assumption $a \wedge b \not\equiv a \ (\Phi)$ formulated in D), $q \nleq \bigvee_L I$ (equivalent to $a \wedge b \not\equiv b \ (\Phi)$) and $p \vee \bigvee_L I \geqq q$ in L (equivalent to $a \equiv a \vee b \ (\Phi)$). Obviously,

$$p \vee \bigvee\nolimits_L I \geqq q \vee \bigvee\nolimits_L I > \bigvee\nolimits_L I.$$

From $a = (p, I) = \{p\} \cup I$ it follows that there is no $r \in \mathrm{J}(L) \setminus I$ such that $r < p$, which gives $p \wedge \bigvee_L I \prec p$ in L. But L is a semimodular lattice and therefore $\bigvee_L I = (p \wedge \bigvee_L I) \vee \bigvee_L I \prec p \vee \bigvee_L I$. This implies $p \vee \bigvee_L I = q \vee \bigvee_L I$, or equivalently $b \equiv a \vee b \ (\Phi)$. By the Lemma this proves that φ is cover-preserving.

For any poset P, $\mathrm{J}\big(\mathrm{H}(P)\big) \cong P$. This gives $\mathrm{J}(D) = \mathrm{J}(\mathrm{H}\big(\mathrm{J}(L)\big) \cong \mathrm{J}(L)$, whence the restriction of φ to $J(D)$ is an order isomorphism onto $\mathrm{J}(L)$. □

Finally, let L be a finite geometric lattice. Then $\mathrm{J}(L)$ is the set of atoms, this is an unordered set and therefore $\mathrm{H}\big(\mathrm{J}(L)\big)$ is a boolean lattice. We conclude:

COROLLARY 3. *The cover-preserving join-homomorphic images of finite boolean lattices are exactly the finite geometric lattices.*

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