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L^p-CONVERGENCE OF LAGRANGE INTERPOLATION ON THE SEMIAXIS

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Abstract. In order to approximate functions defined on $(0,+\infty)$, the authors consider suitable Lagrange polynomials and show their convergence in weighted L^p-spaces.

1. Introduction

This paper deals with the approximation of functions defined on the unbounded interval $\mathbb{R}^+ := (0,+\infty)$, by means of Lagrange polynomials, in weighted L^p spaces. We consider functions having singularities in the origin and increasing exponentially for $x \to +\infty$ and, therefore, we study the convergence of an interpolation process in L^p_u, $1 \leqq p < \infty$, with the weight

$$u(x) = x^\gamma e^{-\frac{x^\beta}{2}}, \qquad \gamma > -\frac{1}{p}, \quad \beta > \frac{1}{2}.$$

At first let us observe that bounded projectors from $L^p_u(0,+\infty)$ onto the space of algebraic polynomials $\mathbb{P}_m$ do not exist for every value of p.

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For instance the Fourier projector $S_m(w,f)$ related to the Laguerre weight $w(x)=x^\alpha e^{-x}$, $\alpha>-1$, is bounded in L^p_u, under suitable conditions on the weights u and w, only for $\frac{4}{3}<p<4$ [1] (see also [12]).

In order to overcome this problem, in [12] the authors, in the special case $\beta=1$, proposed to interpolate a "finite section" of the function estimating the L^p_u norm in a suitable finite interval of $\mathbb{R}^+$.

Here, in a more general context, ($w(x)=x^\alpha e^{-x^\beta}$, $\alpha>-1$, $\beta>\frac{1}{2}$ and $u(x) = x^\gamma e^{-\frac{x^\beta}{2}}$, $\gamma>-\frac{1}{p}$), we consider a Lagrange polynomial based on the zeros of $p_m(w)$ and a special additional point, following an idea of J. Szabados.

To be more precise, let $\{p_m(w)\}_m$ be the sequence of the orthonormal polynomials related to the weight $w(x)$ having positive leading coefficient and

$$\frac{a_m}{m^2}\leqq x_{1m}<x_{2m}<\ldots<x_{mm}\leqq a_m\left(1-\frac{\mathcal{C}}{m^{\frac{2}{3}}}\right)$$

(cf. [6]) be the zeros of $p_m(w)$.

Here and in the sequel $\mathcal{C}$ denotes a positive constant which may assume different values in different formulae. We write $\mathcal{C}\neq\mathcal{C}(a,b,\ldots)$ if $\mathcal{C}$ is independent of the parameters $a,b,\ldots$. If $A,B\geqq 0$ are quantities depending on some parameters, we write $A\sim B$ if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B such that $\frac{B}{\mathcal{C}}\leqq A\leqq \mathcal{C}B$.

If f is a locally continuous function in $\mathbb{R}^+$, let

$$L_m(w,f;x)=\sum_{k=1}^m l_{km}(x)f(x_{km}),$$

with $l_k(x)=l_{km}(x)=\frac{p_m(w,x)}{p'_m(w,x_k)(x-x_k)}$ and $x_k=x_{km}$, and

$$L^*_{m+1}(w,f;x)=\sum_{k=1}^{m+1} l^*_k(x)f(x_k),\quad x_{m+1}=a_m \tag{1.1}$$

with

$$l^*_k(x)=\frac{a_m-x}{a_m-x_k}l_k(x),\quad k=1,\ldots,m,\quad\text{and}\quad l^*_{m+1}(x)=\frac{p_m(w,x)}{p_m(w,a_m)},$$

the fundamental Lagrange polynomials based on the nodes $x_1,x_2,\ldots,x_m$, x_{m+1}, where $a_m=a_m(w)\sim\bar a_{2m}(\tilde w)^2\sim m^{1/\beta}$ and $\bar a_{2m}(\tilde w)$ is the Mhaskar–Rakhmanov–Saff number with respect to the generalized Freud weight $\tilde w(x) = |x|^{2\alpha+1}e^{-x^{2\beta}}$ (cf. [10]). Obviously $\{l_1,\ldots,l_m\}$ is a basis for the set $\mathbb{P}_{m-1}$

of all algebraic polynomials of degree at most $m-1$ and $\left\{l_1^*, \dots, l_m^*, l_{m+1}^*\right\}$ is a basis for $\mathbb{P}_m$.

Now let m be sufficiently large (say $m > m_0$) and the integer $j = j(m)$ be defined by

$$x_j = x_{j(m)} = \min_{1 \leqq k \leqq m} \{x_k : x_k \geqq \theta a_m\}, \tag{1.2}$$

where $\theta \in (0,1)$ is fixed. Define

$$\mathcal{P}_{m-1} = \left\{P \in \mathbb{P}_{m-1} : P(x) = q_{j-1}(x) \prod_{i=j+1}^{m} (x - x_i),\ q_{j-1} \in \mathbb{P}_{j-1}\right\} \subset \mathbb{P}_{m-1}$$

and

$$\mathcal{P}_m^* = \left\{P \in \mathbb{P}_m : P(x) = q_{j-1}(x)(a_m - x) \prod_{i=j+1}^{m} (x - x_i),\ q_{j-1} \in \mathbb{P}_{j-1}\right\} \subset \mathbb{P}_m.$$

Then $\{l_1, \dots, l_j\}$ and $\left\{l_1^*, \dots, l_j^*\right\}$ are bases for $\mathcal{P}_{m-1}$ and $\mathcal{P}_m^*$, respectively, and it is easy to verify that one has the unique representations

$$q_{m-1}(x) = \sum_{i=1}^{j} q_{m-1}(x_i) l_i(x) \quad \forall\, q_{m-1} \in \mathcal{P}_{m-1}$$

and

$$q_m(x) = \sum_{i=1}^{j} q_m(x_i) l_i^*(x) \quad \forall\, q_m \in \mathcal{P}_m^*.$$

Then we can introduce the following interpolation processes:

$$\tilde{L}_m(w, f; x) = \sum_{k=1}^{j} l_k(x) f(x_k), \quad f \in C^0(\mathbb{R}^+)$$

and

$$L_{m+1}^{**}(w, f; x) = \sum_{k=1}^{j} l_k^*(x) f(x_k), \quad f \in C^0(\mathbb{R}^+) \tag{1.3}$$

which define two projectors from $C^0(\mathbb{R}^+)$ into $\mathcal{P}_{m-1}$ and $\mathcal{P}_m^*$, respectively.

Let us consider now the two norms $\left\| \tilde{L}_m(w,f)u \right\|_p$ and $\left\| L^{**}_{m+1}(w,f)u \right\|_p$ (see the definition in Section 2) with

$$u(x) = (1+x)^{\lambda} x^{\gamma} e^{-\frac{x^\beta}{2}}, \quad \lambda > 0, \quad 1 \leqq p \leqq \infty.$$

The first norm does not have a "good" behavior for certain values of p; for instance it is not hard to verify that, for $p=\infty$, it results

$$\sup_{\|fu\|_\infty = 1} \left\| \tilde{L}_m(w,f)u \right\|_\infty \geqq \mathcal{C} m^{\frac{1}{6}}$$

for any α, γ, λ. Let us note that the polynomial $\tilde{L}_m(w,f)$ has a good behavior in $[0, \theta a_m]$, with $0 < \theta < 1$, as it was shown in [12] for $\beta = 1$. For the second norm, it was proved in [7] that $\sup_{\|fu\|_\infty = 1} \left\| L^{**}_{m+1}(w,f)u \right\|_\infty \leqq \mathcal{C} \log m$ under certain assumptions on the parameters α, γ, λ. The order $\log m$ cannot be improved [16].

In this paper we study the behavior of $L^{**}_{m+1}(w,f)$ in $(0,+\infty)$ and we implicitly show that there exist sequences of polynomials of $\mathcal{P}^*_m$ convergent to f in L^p_u, $p \in (1,\infty)$, under suitable conditions on the parameters of the weights u and w. These results can be used in the projection methods in order to approximate the solution of some functional equations.

2. Preliminaries

For $1 \leqq p < +\infty$, $X \subseteqq [0,+\infty)$, let $L^p(X)$ be defined in the usual way. With the weight

$$u(x) = x^{\gamma} e^{-\frac{x^\beta}{2}}, \quad \gamma > -\frac{1}{p}, \quad \beta > \frac{1}{2}, \quad x \geqq 0, \tag{2.1}$$

let $L^p_u(X)$ be the collection of all measurable functions f such that $fu \in L^p(X)$. Consequently the norm in $L^p_u(X)$ is defined by

$$\|f\|_{L^p_u(X)} = \left(\int_X |fu|^p(x)\,dx \right)^{\frac{1}{p}}.$$

When $p = +\infty$, we define the spaces

$$B_u := \left\{ f : [0,+\infty) \to \mathbb{R} : \sup_{x \geqq 0} \left| (fu)(x) \right| < +\infty \right\}$$

and

$$L_u^\infty := C_u = \Big\{ f \in C^0\big((0,+\infty)\big) \,:\, \lim_{\substack{x\to 0\\ x\to\infty}} (fu)(x) = 0 \Big\}$$

both equipped with the norm

$$\|f\|_{L_u^\infty} = \|f\|_{C_u} = \sup_{x \geqq 0} \big|(fu)(x)\big|.$$

For $1 \leqq p \leqq +\infty$ and $X = [0,+\infty)$, we shall write $L_u^p\big([0,+\infty)\big) = L_u^p$ and $\|f\|_{L_u^p(X)} = \|f\|_{L_u^p} = \|fu\|_p$. Moreover, for $1 \leqq p \leqq +\infty$, let $W_r^p(u)$ denote the weighted Sobolev-type space of order $r \in \mathbb{N}$, $r \geqq 1$, defined by

$$W_r^p(u) = \Big\{ f \in L_u^p\big([0,+\infty)\big) \,:\, f^{(r-1)} \in AC\big((0,+\infty)\big),\ \big\|f^{(r)}\varphi^r u\big\|_p < +\infty \Big\},$$

with $\varphi(x) = \sqrt{x}$, and equipped with the norm

$$\|f\|_{W_r^p(u)} = \|fu\|_p + \big\|f^{(r)}\varphi^r u\big\|_p.$$

Let us also consider Zygmund-type spaces defined as

$$Z_s^p(u) = \Big\{ f \in L_u^p\big([0,+\infty)\big) \,:\, \|f\|_{Z_{sr}^p(u)} < \infty \Big\}, \quad s \in \mathbb{R},\ s > 0$$

where

$$\|f\|_{Z_{sr}^p(u)} = \|fu\|_p + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}, \quad r > s,$$

and

$$\Omega_\varphi^r(f,t)_{u,p} = \sup_{0<h\leqq t} \big\|(\vec{\Delta}_{h\varphi}^r f)u\big\|_{L^p(I_{rh})},$$

with $\varphi(x) = \sqrt{x}$, $I_{rh} = \big[8(rh)^2, \mathcal{C}h^*\big]$, $\mathcal{C}$ arbitrary fixed constant, $h^* = \frac{1}{h^{\frac{2}{2\beta-1}}}$ and

$$\vec{\Delta}_{h\varphi}^r f(x) = \sum_{i=0}^r (-1)^i \binom{r}{i} f\left(x + \left(\frac{r}{2} - i\right) h\sqrt{x}\right)$$

(cf. [11]). Sometimes we will omit the index r from $\|f\|_{Z_{sr}^p(u)}$, writing simply $\|f\|_{Z_s^p(u)}$. Let

$$E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \big\|(f-P)u\big\|_p$$

the error of L^p-weighted approximation by algebraic polynomials of degree $\leqq m$.

In [11] the following estimates were proved:

$$E_m(f)_{u,p} \leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \left\|f^{(r)}\varphi^r u\right\|_p, \tag{2.2}$$

for all $f \in W_r^p(u)$, and

$$E_m(f)_{u,p} \leqq \mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,dt, \quad \mathcal{C} \neq \mathcal{C}(f,m), \quad r<m \tag{2.3}$$

with $a_m = a_m(u)$ the Mhaskar–Rakhmanov–Saff number with respect to the weight $u(x) = x^\gamma e^{-\frac{x^\beta}{2}}$.

3. Main results

Now we can establish the main results.

Theorem 3.1. *Let $1<p<+\infty$, $v^\sigma(x) = x^\sigma$, $\varphi(x) = \sqrt{x}$ and $q = \frac{p}{p-1}$. If the weights $w(x) = v^\alpha(x)e^{-x^\beta}$ and $u(x) = v^\gamma(x)e^{-\frac{x^\beta}{2}}$ satisfy the conditions*

$$\frac{v^\gamma}{\sqrt{v^\alpha\varphi}} \in L^p(0,1) \quad \text{and} \quad \frac{\sqrt{v^\alpha\varphi}}{v^\gamma} \in L^q(0,1), \tag{3.1}$$

then, for all functions $f \in B_u$, we have

$$\left\|L_{m+1}^{**}(w,f)u\right\|_p \leqq \mathcal{C}\left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k)\right)^{\frac{1}{p}}, \tag{3.2}$$

where $\Delta x_k = x_{k+1} - x_k$, $k = 1,\dots,m$, and $\mathcal{C} \neq \mathcal{C}(m,f)$.

Proposition 3.2. *Let be $1<p<+\infty$. For any function f such that*

$$\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}} \in L^1 \tag{3.3}$$

we have

(3.4)
$$\left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k)\right)^{\frac{1}{p}} \leqq \mathcal{C}\left[\|fu\|_{L^p(0,x_{j+1})} + \left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}}\,dt\right]$$

with $\mathcal{C} \neq \mathcal{C}(m,f)$.

Let us note that (3.3) implies the continuity of f in $(0, +\infty)$.

Theorem 3.1 and Proposition 3.2 are similar to Theorem 2.6 and Lemma 2.7 in [12]. Let us show now some interesting consequences of the previous statements.

THEOREM 3.3. *Under the hypotheses of Theorem* 3.1, *for any polynomial* $P \in \mathcal{P}_m^*$ *the following Marcinkievicz-type equivalence holds:*

$$\|Pu\|_p \sim \left(\sum_{k=1}^{j} \Delta x_k |Pu|^p(x_k) \right)^{\frac{1}{p}} \tag{3.5}$$

where the constant in $\sim$ *is independent of* m *and* P.

We observe also that (3.5) generally is not true for $P \in \mathbb{P}_m$ (see, for instance, [12, pp. 313–314]).

THEOREM 3.4. *Under the hypotheses of Theorem* 3.1, *for any function* f *with* (3.3) *we have*

$$\left\| \left[f - L_{m+1}^{**}(w, f) \right] u \right\|_p \leqq \mathcal{C} \left[\left(\frac{\sqrt{a_m}}{m} \right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f, t)_{u,p}}{t^{1+\frac{1}{p}}}\, dt + e^{-Am} \|fu\|_p \right], \tag{3.6}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$ *and* $A \neq A(m, f)$ ([12, p. 314]).

In particular the following corollary holds true.

COROLLARY 3.5. *Let* $1 < p < +\infty$. *Under the assumptions of Theorem* 3.4, *for any* $f \in Z_s^p(u)$, $s > \frac{1}{p}$, *we have*

$$\left\| \left[f - L_{m+1}^{**}(w, f) \right] u \right\|_p \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^s \|f\|_{Z_s^p(u)} \tag{3.7}$$

and, for any $f \in W_r^p(u)$, $r = 1, 2, \ldots$

$$\left\| \left[f - L_{m+1}^{**}(w, f) \right] u \right\|_p \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \|f\|_{W_r^p(u)} \tag{3.8}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

From (3.7) and (3.8) it follows that, in $Z_s^p(u)$ and $W_r^p(u)$, the polynomials $\left\{ L_{m+1}^{**}(w, f) \right\}$ behave as the polynomials of best approximation.

In the next theorem we estimate the norm of $L_{m+1}^{**}(w, f)$ in the weighted space $L^1_{\frac{w}{\bar{u}}}$ with $\bar{u}(x) = (1 + x)^\lambda x^\gamma e^{-\frac{x^\beta}{2}}$, $\lambda > 0$. Such kind of result can be applied to estimate the norm of some integral operators. The norms involved in the following statement are defined as in Section 2, replacing the weight u by $\bar{u}$.

THEOREM 3.6. *If the weights* $w(x) = x^{\alpha} e^{-x^{\beta}}$ *and* $\bar{u}(x) = (1+x)^{\lambda} x^{\gamma} e^{-\frac{x^{\beta}}{2}}$ *are such that the parameters* α, γ *and* λ *satisfy the conditions*

$$\frac{1}{2} < \lambda \leqq 1 \quad \textit{and} \quad \max\left(0, \frac{\alpha}{2} + \frac{1}{4}\right) < \gamma < \frac{\alpha+1}{2}, \tag{3.9}$$

then, for each $f \in L^{\infty}_{\bar{u}}$, *we have*

$$\left\| L^{**}_{m+1}(w,f) \frac{w}{\bar{u}} \right\|_1 \leqq \mathcal{C} \| f \bar{u} \|_{L^{\infty}([0,x_j])}, \tag{3.10}$$

with $\mathcal{C} \neq \mathcal{C}(m,f)$ *and* j *defined as in* (1.2). *Consequently, under the conditions* (3.9), *for each* $f \in W^{\infty}_r(\bar{u})$ *we have*

$$\left\| \left[f - L^{**}_{m+1}(w,f) \right] \frac{w}{\bar{u}} \right\|_1 \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \| f \|_{W^{\infty}_r(\bar{u})} \tag{3.11}$$

with $\mathcal{C} \neq \mathcal{C}(m,f)$.

4. Proofs

Before proving some results about the interpolating polynomials $L^{**}_{m+1}(w,f;x)$, we recall some basic facts on the orthonormal polynomials $\{p_m(w)\}_m$ which can be deduced from some results in [4, p. 14–17].

Namely, the zeros $x_1, x_2, \dots, x_m$ of $p_m(w)$ satisfy

$$\Delta x_k = x_{k+1} - x_k \sim \frac{\sqrt{a_m}}{m} \sqrt{x_k} \frac{1}{\sqrt{1 - \frac{x_k}{a_m} + \frac{1}{m^{\frac{2}{3}}}}}, \quad k = 1, 2, \dots, m-1 \tag{4.1}$$

and

$$\frac{1}{\left| p'_m(w,x_k) \sqrt{w(x_k)} \right|} \sim \sqrt[4]{a_m x_k} \Delta x_k \sqrt{1 - \frac{x_k}{a_m} + \frac{1}{m^{\frac{2}{3}}}}, \quad k = 1, \dots, m, \tag{4.2}$$

with constants involved in $\sim$ independent of m and k. Moreover, the following estimate holds:

$$\left| p_m(w,x) \sqrt{w(x)} \right| \leqq \frac{\mathcal{C}}{\sqrt[4]{a_m x} \sqrt[4]{\left| 1 - \frac{x}{a_m} \right| + \frac{1}{m^{\frac{2}{3}}}}}, \tag{4.3}$$

with $\mathcal{C} \frac{a_m}{m^2} \leqq x \leqq \mathcal{C} a_m \left(1 + m^{-\frac{2}{3}}\right)$, $\mathcal{C} \neq \mathcal{C}(m,x)$.

Let us state the following Remez-type inequality which we will use in the following (see [11, p. 107]).

LEMMA 4.1. *Let* $A>0$, $A_m=\left[A\frac{a_m}{m^2}, a_m\left(1-\frac{A}{m^{\frac{2}{3}}}\right)\right]$ *and* $1\leqq p\leqq+\infty$. *Then, for any polynomial* $P_m\in\mathbb{P}_m$, *there exists a constant* $\mathcal{C}=\mathcal{C}(A)$ *independent of* m, p *and* P_m *such that*

$$\left(\int_0^{+\infty}|P_m u|^p(x)\,dx\right)^{\frac{1}{p}}\leqq\mathcal{C}\left(\int_{A_m}|P_m u|^p(x)\,dx\right)^{\frac{1}{p}}. \tag{4.4}$$

Another useful result is the following.

LEMMA 4.2. *If* $x\in\left[(2rh)^2, h^*\right]$ *with* $h>0$, r *a positive integer*, $h^*=\frac{1}{h^{\frac{2}{2\beta-1}}}$, $\beta>\frac{1}{2}$, $y\in\left[x-rh\sqrt{x}, x+rh\sqrt{x}\right]$ *and the weight* u *is defined by* (2.1), *then there exists a constant* $\mathcal{C}$ *independent of* x *and* h *such that*

$$\frac{1}{\mathcal{C}}\leqq\frac{u(y)}{u(x)}\leqq\mathcal{C}$$

holds.

PROOF. Since $h\leqq\frac{\sqrt{x}}{2r}$,

$$x\leqq y+|x-y|\leqq y+rh\sqrt{x}\leqq y+\frac{x}{2}$$

and

$$y\leqq x+|y-x|\leqq x+rh\sqrt{x}\leqq x+\frac{x}{2}$$

hold. Then we get

$$\frac{1}{2}x\leqq y\leqq\frac{3}{2}x. \tag{4.5}$$

Moreover, since $x\leqq h^*$, for $\xi\in I(x,y)$ ($I(x,y)$ being the interval having x and y as endpoints) one has

$$\left|x^\beta-y^\beta\right|=|x-y|\beta\xi^{\beta-1}\leqq\beta rh\sqrt{x}\xi^{\beta-1}\leqq\beta rh(h^*)^{\beta-\frac{1}{2}}=\beta r$$

and, therefore,

$$e^{\frac{\left|x^\beta-y^\beta\right|}{2}}\leqq e^{\frac{\beta r}{2}}. \tag{4.6}$$

Combining (4.5) and (4.6), the lemma follows. $\square$

LEMMA 4.3. *Let* $1 \leqq p < +\infty$, $v^\sigma(x) = x^\sigma \in L^p$, $x_j = \min\{x_k \geqq \theta a_m\}$ *and* $0 < \theta < 1$ *fixed. Then, there exists* $\theta_1 \in (\theta, 1)$ *such that for an arbitrary polynomial* $P \in \mathbb{P}_{lm}$ *(with* l *a fixed integer), we have*

$$\left(\sum_{k=1}^{j} \Delta x_k |Pv^\sigma|^p(x_k)\right)^{\frac{1}{p}} \leqq \mathcal{C}\left(\int_{x_1}^{\theta_1 a_m} |Pv^\sigma|^p(x)\, dx\right)^{\frac{1}{p}}, \tag{4.7}$$

where $\Delta x_k = x_{k+1} - x_k$, $\mathcal{C}$ *is a positive constant independent of* m, P *and* p.

PROOF. Let $P \in \mathbb{P}_{lm}$ (l a fixed integer). The following identities hold true:

$$\Delta x_k P(x_k) = \int_{x_k}^{x_{k+1}} P(x)\, dx - \int_{x_k}^{x_{k+1}} (x_{k+1} - x) P'(x)\, dx$$

for $1 \leqq k < j$ and

$$\Delta x_j P(x_j) = \int_{x_{j-1}}^{x_j} P(x)\, dx + \int_{x_{j-1}}^{x_j} (x - x_{j-1}) P'(x)\, dx.$$

By applying Hölder inequality and

$$v^\sigma(x_{k-1}) \sim v^\sigma(x) \sim v^\sigma(x_k), \quad x \in [x_{k-1}, x_k],$$

(see (4.5)), one can easily obtain

$$\Delta x_k \big| P(x_k) v^\sigma(x_k) \big|^p \leqq \mathcal{C}\Bigg[\int_{x_k}^{x_{k+1}} \big| P(x) v^\sigma(x) \big|^p\, dx + (\Delta x_k)^p \int_{x_k}^{x_{k+1}} \big| P'(x) v^\sigma(x) \big|^p\, dx\Bigg],$$

for $1 \leqq k < j$ and

$$\Delta x_j \big| P(x_j) v^\sigma(x_j) \big|^p \leqq \mathcal{C}\Bigg[\int_{x_{j-1}}^{x_j} \big| P(x) v^\sigma(x) \big|^p\, dx + (\Delta x_j)^p \int_{x_{j-1}}^{x_j} \big| P'(x) v^\sigma(x) \big|^p\, dx\Bigg].$$

Recalling that for $k \leqq j$, $\Delta x_k \sim \frac{\sqrt{a_m}}{m}\sqrt{x_k} \sim \frac{\sqrt{a_m}}{m}\sqrt{x}$, $x \in [x_k, x_{k+1}]$ (see (4.1)), we have

$$\sum_{k=1}^{j} \Delta x_k |Pv^\sigma|^p(x_k)$$

$$\leqq \mathcal{C}\left[\int_{x_1}^{x_j} |Pv^\sigma|^p(x)\,dx + \left(\frac{\sqrt{a_m}}{m}\right)^p \int_{x_1}^{x_j} \left|(P'v^\sigma)(x)\sqrt{x}\right|^p dx\right].$$

Now we estimate the second term on the right hand side. Let us choose $\theta < \theta_1 < 1$ such that $x_j < \theta_1 a_m$. Then, by proceeding as in [12, proof of Lemma 2.5] we can show

$$\int_{x_1}^{x_j} \left|(P'v^\sigma)(x)\sqrt{x}\right|^p dx \leqq \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^p \int_{x_1}^{\theta_1 a_m} \left|P(x)x^\sigma\right|^p dx.$$

In conclusion

$$\left(\sum_{k=1}^{j} \Delta x_k |Pv^\sigma|^p(x_k)\right)^{\frac{1}{p}} \leqq \mathcal{C}\left(\int_{x_1}^{\theta_1 a_m} \left|P(x)x^\sigma\right|^p dx\right)^{\frac{1}{p}}. \quad \square$$

Remark 4.4. Let us observe that, by using Lemma 4.2, a result analogous to Lemma 4.3 can be proved if one replaces the weight function v^σ by the more general one $v^\sigma e^{-\frac{x^\beta}{2}}$.

Proof of Theorem 3.1. By (4.4) we have

$$\left\| L^{**}_{m+1}(w,f)u\right\|_p \leqq \mathcal{C}\left\| L^{**}_{m+1}(w,f)u\right\|_{L^p(x_1,a_m)}$$

$$= \mathcal{C} \sup_{\|g\|_q=1} \left| \int_{x_1}^{a_m} L^{**}_{m+1}(w,f,x)u(x)g(x)\,dx\right|$$

with $\frac{1}{p}+\frac{1}{q}=1$. Set $A(g) = \int_{x_1}^{a_m} L^{**}_{m+1}(w,f,x)g(x)u(x)\,dx$, then

$$A(g) = \sum_{k=1}^{j} \frac{f(x_k)u(x_k)}{p'_m(w,x_k)(a_m-x_k)u(x_k)} \int_{x_1}^{a_m} \frac{(a_m-x)p_m(w,x)}{x-x_k} g(x)u(x)\,dx.$$

Since

$$\frac{1}{\left|p'_m(w,x_k)u(x_k)\right|} \sim x_k^{\frac{\alpha}{2}+\frac{1}{4}-\gamma}\sqrt[4]{a_m}\,\Delta x_k$$

(see (4.2)) and $\frac{1}{a_m - x_k} \leqq \frac{\mathcal{C}}{a_m}$ one has

$$A(g) \leqq \mathcal{C} a_m^{-\frac{3}{4}} \sum_{k=1}^{j} f(x_k) u(x_k) \Delta x_k x_k^{\frac{\alpha}{2}+\frac{1}{4}-\gamma}$$

$$\times \left\{ \int_{x_1}^{\bar{\theta} a_m} + \int_{\bar{\theta} a_m}^{a_m} \right\} \frac{(a_m - x) p_m(w,x)}{x - x_k} g(x) u(x)\, dx =: \mathcal{C} \big[A_1(g) + A_2(g) \big]$$

where $\bar{\theta} \in (0,1)$. We shall specify the choice of $\bar{\theta}$ later. At this moment we assume that $\bar{\theta} a_m > x_j$. Let us estimate $A_1(g)$. To this end, for a fixed $l \in \mathbb{N}$ we consider the polynomial $\pi(t) \in \mathbb{P}_{(l+1)m}$ defined by

$$\pi(t) = \int_{x_1}^{\bar{\theta} a_m} \frac{(a_m - x) p_m(w,x) q(x) - (a_m - t) p_m(w,t) q(t)}{x - t} \, \frac{g(x) u(x)}{q(x)}\, dx$$

where $q(x)$ is an arbitrary polynomial of degree lm. Then

$$\big| A_1(g) \big| \leqq a_m^{-\frac{3}{4}} \sum_{k=1}^{j} \big| f(x_k) \big| u(x_k) \Delta x_k x_k^{\frac{\alpha}{2}+\frac{1}{4}-\gamma} \big| \pi(x_k) \big|$$

$$\leqq a_m^{-\frac{3}{4}} \left[\sum_{k=1}^{j} \Delta x_k \big| (fu)(x_k) \big|^p \right]^{\frac{1}{p}} \left[\sum_{k=1}^{j} \Delta x_k \Big| x_k^{\frac{\alpha}{2}+\frac{1}{4}-\gamma} \pi(x_k) \Big|^q \right]^{\frac{1}{q}}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. By using Lemma 4.3,

$$B := a_m^{-\frac{3}{4}} \left[\sum_{k=1}^{j} \Delta x_k \Big| x_k^{\frac{\alpha}{2}+\frac{1}{4}-\gamma} \pi(x_k) \Big|^q \right]^{\frac{1}{q}} \leqq \mathcal{C} a_m^{-\frac{3}{4}} \left\| v^{\frac{\alpha}{2}+\frac{1}{4}-\gamma} \pi \right\|_{L^q(x_1, \theta_1 a_m)}$$

with $\theta < \theta_1 < 1$ and $v^{\rho}(x) = x^{\rho}$.

Now let $\theta < \theta_1 < \bar{\theta}$. Hence

$$B \leqq \mathcal{C} a_m^{-\frac{3}{4}} \Bigg\{ \left[\int_{x_1}^{\bar{\theta} a_m} \left| t^{\frac{\alpha}{2}+\frac{1}{4}-\gamma} \int_{x_1}^{\bar{\theta} a_m} \frac{(a_m - x) p_m(w,x) g(x) u(x)}{x - t}\, dx \right|^q dt \right]^{\frac{1}{q}}$$

$$+ \left[\int_{x_1}^{\bar{\theta} a_m} \left| t^{\frac{\alpha}{2}+\frac{1}{4}-\gamma} (a_m - t) p_m(w,t) q(t) \int_{x_1}^{\bar{\theta} a_m} \frac{g(x) u(x)}{q(x)} \frac{dx}{x - t} \right|^q dt \right]^{\frac{1}{q}} \Bigg\}.$$

By

$$\left|p_m(w,x)\sqrt{w(x)}x^{\frac{1}{4}}\right| \leqq \frac{\mathcal{C}}{\sqrt[4]{a_m - x}}$$

(see (4.3)), we may choose a polynomial $q(x)$ such that $q(x) \sim u(x)$ in $[x_1, a_m]$. It follows

$$B \leqq \mathcal{C}a_m^{-\frac{3}{4}}\left[\int_{x_1}^{\bar{\theta}a_m}\left|t^{\frac{\alpha}{2}+\frac{1}{4}-\gamma}\int_{x_1}^{\bar{\theta}a_m}\frac{(a_m-x)p_m(w,x)g(x)u(x)}{x-t}\,dx\right|^q dt\right]^{\frac{1}{q}}$$

$$+\left[\int_{x_1}^{\bar{\theta}a_m}\left|\int_{x_1}^{\bar{\theta}a_m}\frac{g(x)u(x)}{q(x)}\frac{dx}{x-t}\right|^q dt\right]^{\frac{1}{q}}$$

$$\leqq \mathcal{C}a_m^{-\frac{3}{4}}\left[\int_{x_1}^{\bar{\theta}a_m}\left|t^{\frac{\alpha}{2}+\frac{1}{4}-\gamma}(a_m-t)p_m(w,t)g(t)u(t)\right|^q dt\right]^{\frac{1}{q}}$$

$$+\left[\int_{x_1}^{\bar{\theta}a_m}\left|\frac{g(t)u(t)}{q(t)}\right|^q dt\right]^{\frac{1}{q}} \leqq \mathcal{C}\|g\|_q = \mathcal{C}$$

where we used the boundedness of the Hilbert transform in $L^q_{\frac{\sqrt{v^\alpha\varphi}}{v^\gamma}}(0,1)$ (cf. [14]) and $\|g\|_q = 1$. It remains to estimate

$$A_2(g) = a_m^{-\frac{3}{4}}\sum_{k=1}^{j} f(x_k)u(x_k)\Delta x_k x_k^{\frac{\alpha}{2}+\frac{1}{4}-\gamma}\int_{\bar{\theta}a_m}^{a_m}\frac{(a_m-x)p_m(w,x)}{x-x_k}g(x)u(x)\,dx.$$

Now, since $x - x_k \geqq (\bar{\theta}-\theta_1)a_m$ and

$$\left|(a_m-x)p_m(w,x)u(x)\right| \leqq \mathcal{C}(a_m-x)^{\frac{3}{4}}x^{\gamma-\frac{\alpha}{2}-\frac{1}{4}} \leqq \mathcal{C}a_m^{\frac{3}{4}}x^{\gamma-\frac{\alpha}{2}-\frac{1}{4}},$$

we get, by applying Hölder inequality,

$$\left|A_2(g)\right| \leqq \mathcal{C}\left[\sum_{k=1}^{j}\Delta x_k\left|(fu)(x_k)\right|^p\right]^{\frac{1}{p}}\left[\sum_{k=1}^{j}\Delta x_k x_k^{(\frac{\alpha}{2}+\frac{1}{4}-\gamma)q}\right]^{\frac{1}{q}}$$

$$\times\frac{1}{a_m}\int_0^{a_m}x^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\left|g(x)\right|dx.$$

The last two factors of this product are dominated by (see Lemma 4.3)

$$\frac{1}{a_m}\left[\int_{x_1}^{\theta_1 a_m}\left|\frac{\sqrt{v^\alpha\varphi}}{v^\gamma}\right|^q(x)\,dx\right]^{\frac{1}{q}}\left[\int_0^{a_m}\left|\frac{v^\gamma}{\sqrt{v^\alpha\varphi}}\right|^p(x)\,dx\right]^{\frac{1}{p}}\|g\|_q \leqq \mathcal{C},$$

under our assumptions. In conclusion one has (3.2). □

PROOF OF PROPOSITION 3.2. We start by recalling the following inequality (see [3]):

$$\delta^{\frac{1}{p}} \max_{x\in X} \left| f(x) \right| \leqq \mathcal{C}\left(\|f\|_{L^p(X)} + \delta^{\frac{1}{p}} \int_0^{\delta} \frac{\omega^r(f,t)_{L^p(X)}}{t^{1+\frac{1}{p}}}\, dt \right), \tag{4.8}$$

that holds for any continuous function f in $X=[a,a+\delta]$, with $1<p<+\infty$, $\mathcal{C}\neq\mathcal{C}(f,\delta)$ and ω^r the r-th ordinary modulus of continuity. By applying (4.8) with $X=I_k=[x_k,x_{k+1}]$, $\delta=\Delta x_k$ and recalling that $u(x)\sim u(x_k)$ for $x\in I_k$ (see Lemma 4.2), we can write

$$(\Delta x_k)^{\frac{1}{p}} \left| f(x_k) \right| u(x_k) \leqq \mathcal{C}\left(\|fu\|_{L^p(I_k)} + (\Delta x_k)^{\frac{1}{p}} \int_0^{\Delta x_k} \frac{\omega^r(f,t)_{L^p_u(I_k)}}{t^{1+\frac{1}{p}}}\, dt \right)$$

where $\omega^r(f,t)_{L^p_u(I_k)} = \sup_{h\leqq t} \left(\int_0^{\Delta x_k - rh} \left| \Delta_h^r f(x)u(x) \right|^p dx \right)^{\frac{1}{p}}$, with Δ_h^r the r-th forward difference. Making the change of variable $t\to\sqrt{x_k}\,\tau$ and taking into account relation (4.1), we have

$$(\Delta x_k)^{\frac{1}{p}} \left| f(x_k) \right| u(x_k)$$
$$\leqq \mathcal{C}\left(\|fu\|_{L^p(I_k)} + \left(\frac{\Delta x_k}{\sqrt{x_k}}\right)^{\frac{1}{p}} \int_0^{\frac{\Delta x_k}{\sqrt{x_k}}} \frac{\omega^r\left(f,\sqrt{x_k}\,\tau\right)_{L^p_u(I_k)}}{\tau^{1+\frac{1}{p}}}\, d\tau \right)$$
$$\leqq \mathcal{C}\left(\|fu\|_{L^p(I_k)} + \left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\omega^r\left(f,\sqrt{x_k}\,\tau\right)_{L^p_u(I_k)}}{\tau^{1+\frac{1}{p}}}\, d\tau \right).$$

Now, let g be an absolutely continuous function in $\mathbb{R}^+$ ($g\in AC(\mathbb{R}^+)$) such that $\left\|g^{(r)}\varphi^r u\right\|_p<\infty$, $\varphi(x)=\sqrt{x}$. Proceeding as in [12, Proof of Lemma 2.7], since $\sqrt{x_k}\sim\sqrt{x}$, for $x\in I_k$, we can write

$$\omega^r\left(f,\sqrt{x_k}\,\tau\right)_{L^p_u(I_k)} \leqq \mathcal{C}\left(\left\|(f-g)u\right\|_{L^p(I_k)} + \tau^r \left\|g^{(r)}\varphi^r u\right\|_{L^p(I_k)} \right) =: A_k(\tau),$$

from which it follows

$$\Delta x_k \left| f(x_k)u(x_k) \right|^p \leqq 2^p \mathcal{C}\left[\|fu\|^p_{L^p(I_k)} + \frac{\sqrt{a_m}}{m} \left(\int_0^{\frac{\sqrt{a_m}}{m}} \frac{A_k(\tau)}{\tau^{1+\frac{1}{p}}}\, d\tau \right)^p \right].$$

Then, by Minkowski inequality,

$$\left(\sum_{k=1}^{j}\Delta x_k\left|f(x_k)u(x_k)\right|^p\right)^{\frac{1}{p}}$$
$$\leqq \mathcal{C}\left\{\|fu\|_{L^p(x_1,x_{j+1})}+\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\left(\sum_{k=1}^{j}\left(A_k(\tau)\right)^p\right)^{\frac{1}{p}}}{\tau^{1+\frac{1}{p}}}d\tau\right\}.$$

Now we have

$$\left(\sum_{k=1}^{j}\left(A_k(\tau)\right)^p\right)^{\frac{1}{p}}\leqq \mathcal{C}\Big[\left\|(f-g)u\right\|_{L^p(x_1,x_{j+1})}+\tau^r\left\|g^{(r)}\varphi^r u\right\|_{L^p(x_1,x_{j+1})}\Big],$$

and for $0<h\leqq\tau$, $(x_1,x_{j+1})\subseteqq I_{rh}=\left[\mathcal{C}(2rh)^2,\mathcal{C}h^*\right]$, with $h^*=\frac{1}{h^{\frac{2}{2\beta-1}}}$, for some constant $\mathcal{C}$. Therefore we have

$$\left(\sum_{k=1}^{j}\left(A_k(\tau)\right)^p\right)^{\frac{1}{p}}\leqq \mathcal{C}\sup_{0<h\leqq\tau}\left\{\left\|(f-g)u\right\|_{L^p(I_{rh})}+h^r\left\|g^{(r)}\varphi^r u\right\|_{L^p(I_{rh})}\right\}$$

and, finally, taking the infimum on $g^{(r-1)}\in AC(\mathbb{R}^+)$,

$$\left(\sum_{k=1}^{j}\left(A_k(\tau)\right)^p\right)^{\frac{1}{p}}\leqq \mathcal{C}\sup_{0<h\leqq\tau}\inf_{g^{(r-1)}\in AC(\mathbb{R}^+)}\Big\{\left\|(f-g)u\right]_{L^p(I_{rh})}$$
$$+h^r\left\|g^{(r)}\varphi^r u\right\|_{L^p(I_{rh})}\Big\}\sim\Omega_\varphi^r(f,t)_{u,p}$$

where we used a result proved in [11, Theorem 3.1]. □

Proof of Theorem 3.3. The inequality

$$\|Pu\|_p\leqq\mathcal{C}\left(\sum_{k=1}^{j}\Delta x_k|Pu|^p(x_k)\right)^{\frac{1}{p}}$$

trivially follows from (3.2), since $L_{m+1}^{**}(w)$, as we have already observed, is a projector into $\mathcal{P}_m^*$. To prove the inverse inequality, taking into account Remark 4.4, we can write

$$\left(\sum_{k=1}^{j}\Delta x_k|Pu|^p(x_k)\right)^{\frac{1}{p}}\leqq\mathcal{C}\left(\int_{x_1}^{\theta_1 a_m}|Pu|^p(x)\,dx\right)^{\frac{1}{p}}$$
$$\leqq\mathcal{C}\left(\int_0^{\infty}|Pu|^p(x)\,dx\right)^{\frac{1}{p}}.\quad\square$$

Let us introduce and recall some notations useful for our aims. Let $\theta \in (0,1)$ fixed and m a positive sufficiently large integer. Let $j=j(m)$ be the corresponding integer defined as in Section 1 and $\Phi_j(x)$ be the characteristic function of the interval $[0,x_j]$. For any continuous function f on $[0,+\infty)$, define the function $f_j(x)=f(x)\Phi_j(x)$. Since, by definitions, $f_j=f$ in $[0,x_j]$ and $f_j=0$ in $(x_j,+\infty)$, one has

$$L^{**}_{m+1}(w,f)=L^{*}_{m+1}(w,f_j),$$

according to (1.1) and (1.3). Finally, let $M=\left[\left(\frac{\theta}{\theta+1}\right)^{\beta}m\right]$. Obviously, $M\sim m$.

Now we recall a lemma (see [11]) which we will use later.

Lemma 4.5. *Let $1\leqq p\leqq+\infty$ and $f\in L^p_u$. Then, for any sufficiently large m, we have*

$$\left\|(f-f_j)u\right\|_p\leqq \mathcal{C}\left[E_M(f)_{u,p}+e^{-Am}\|fu\|_p\right],$$

with constants $\mathcal{C}$ and A positive and independent of m and f.

Proof of Theorem 3.4. We use the following decomposition with $(P)_j = P\Phi_j$:

$$f-L^{**}_{m+1}(w,f)=f-L^{*}_{m+1}(w,f_j)$$
$$=(f-f_j)+\left(f_j-(P)_j\right)+\left((P)_j-P\right)+\left(P-L^{*}_{m+1}(w,f_j)\right)$$

where, for a fixed $0<\theta<1$, $M=\left[\left(\frac{\theta}{\theta+1}\right)^{\beta}m\right]$, $P=P_M\in\mathbb{P}_M$ is a polynomial of "quasi best approximation", i.e. $\left\|(f-P)u\right\|_p\leqq \mathcal{C}E_M(f)_{u,p}$. By Minkowski's inequality we have

$$\left\|\left[f-L^{**}_{m+1}(w,f)\right]u\right\|_p\leqq\left\|[f-f_j]u\right\|_p+\left\|\left[f_j-(P)_j\right]u\right\|_p$$
$$+\left\|\left[(P)_j-P\right]u\right\|_p+\left\|\left[L^{*}_{m+1}(w,P-f_j)\right]u\right\|_p.$$

From the previous lemma it follows

$$\left\|[f-f_j]u\right\|_p\leqq \mathcal{C}\left[E_M(f)_{u,p}+e^{-Am}\|fu\|_p\right],$$
$$\left\|\left[(P)_j-P\right]u\right\|_p\leqq \mathcal{C}e^{-Am}\|Pu\|_p\leqq \mathcal{C}e^{-Am}\|fu\|_p,$$

being $\mathcal{C}$ and A positive constants independent of f and m. Moreover, by definitions,

$$\left\|\left[f_j-(P)_j\right]u\right\|_p\leqq\left\|[f-P]u\right\|_p\leqq \mathcal{C}E_M(f)_{u,p}.$$

It remains to estimate $\left\| \left[L^*_{m+1}(w, P - f_j) \right] u \right\|_p$. We have

$$\left\| \left[L^*_{m+1}(w, P - f_j) \right] u \right\|_p$$
$$\leqq \left\| \left[L^*_{m+1}\big(w, (P - f)_j\big) \right] u \right\|_p + \left\| \left[L^*_{m+1}\big(w, P - (P)_j\big) \right] u \right\|_p .$$

By Theorem 3.1 and Proposition 3.2 we deduce

$$\left\| \left[L^*_{m+1}\big(w, (P - f)_j\big) \right] u \right\|_p$$
$$\leqq \mathcal{C} \left[E_M(f)_{u,p} + \left(\frac{\sqrt{a_m}}{m} \right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f - P, t)_{u,p}}{t^{1+\frac{1}{p}}} \, dt \right] .$$

Since

$$\Omega^r_\varphi(f - P, t)_{u,p} \leqq \Omega^r_\varphi(f, t)_{u,p} + \Omega^r_\varphi(P, t)_{u,p}$$

and [11, Theorem 4.5]

$$\Omega^r_\varphi \left(P, \frac{\sqrt{a_m}}{m} \right)_{u,p} \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \left\| P^{(r)} \varphi^r u \right\|_p ,$$

we get

$$\left\| \left[L^*_{m+1}\big(w, (P - f)_j\big) \right] u \right\|_p \leqq \mathcal{C} \Bigg[E_M(f)_{u,p} + \left(\frac{\sqrt{a_m}}{m} \right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+\frac{1}{p}}} \, dt$$
$$+ \left(\frac{\sqrt{a_m}}{m} \right)^r \left\| P^{(r)} \varphi^r u \right\|_p \Bigg] .$$

Furthermore, following, with some small changes, the proof in [2], Theorem 8.3.1, pp. 98–100, we can also get

$$\left(\frac{\sqrt{a_m}}{m} \right)^r \left\| P^{(r)} \varphi^r u \right\|_p \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+\frac{1}{p}}} \, dt ,$$

from which we finally deduce

$$\left\| \left[L^*_{m+1}\big(w, (P - f)_j\big) \right] u \right\|_p$$
$$\leqq \mathcal{C} \left[E_M(f)_{u,p} + \left(\frac{\sqrt{a_m}}{m} \right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+\frac{1}{p}}} \, dt \right]$$

$$\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}}\,dt$$

obtained by applying also (2.3).

At the end we have to estimate $\left\|\left[L_{m+1}^*\big(w, P-(P)_j\big)\right]u\right\|_p$. Applying the Remez-type inequality (4.4) we have

$$\left\|\left[L_{m+1}^*\big(w, P-(P)_j\big)\right]u\right\|_p \leqq \mathcal{C}\left(\int_{x_1}^{a_m} \left|L_{m+1}^*\big(w, P-(P)_j; x\big)\,u(x)\right|^p dx\right)^{\frac{1}{p}}$$

$$\leqq \mathcal{C}(a_m - x_1)^{\frac{1}{p}} \sup_{x\in[x_1,a_m]} \left|L_{m+1}^*\big(w, P-(P)_j; x\big)\,u(x)\right|$$

$$\leqq \mathcal{C} a_m^{\frac{1}{p}} m^{\frac{2}{p}} \sup_{x\in[x_1,a_m]} \sum_{k=j+1}^{m+1} \left|\frac{l_k^*(x)u(x)}{u(x_k)} P(x_k)u(x_k)\right| \left(\frac{x}{x_k}\right)^{\frac{1}{p}}$$

$$\leqq \mathcal{C}(a_m m^2)^{\frac{1}{p}} \|Pu\|_{L^\infty([\theta a_m,\infty))} \sup_{x\in[x_1,a_m]} \left[\sum_{k=1}^{m+1} \frac{\left|l_k^*(x)\right| \tilde{u}(x)}{\tilde{u}(x_k)}\right]$$

$$= \mathcal{C}(a_m m^2)^{\frac{1}{p}} \|Pu\|_{L^\infty([\theta a_m,\infty))} \left\| L_{m+1}^*(w)\right\|_{C_{\tilde{u}}\to C_{\tilde{u}}}$$

where $\tilde{u}(x) = u(x)x^{\frac{1}{p}} = x^{\gamma+\frac{1}{p}}e^{-\frac{\beta}{2}}$. Now in [7] it was proved the following

Proposition 4.6. *Let* $w(x) = x^\alpha e^{-x^\beta}$, $\bar{u}(x) = x^\gamma(1+x)^\lambda e^{-\frac{x^\beta}{2}}$ *and assume*

$$\max\left\{0, \frac{\alpha}{2}+\frac{1}{4}\right\} \leqq \gamma \leqq \frac{\alpha}{2}+\frac{5}{4}, \quad 0 \leqq \lambda \leqq 1.$$

Then, for every $f \in C_{\bar{u}}$, *we have*

$$\left\| L_{m+1}^*(w,f)\bar{u}\right\|_\infty \leqq \mathcal{C}\log m\|fu\|_\infty$$

with $\mathcal{C} \neq \mathcal{C}(f,m)$.

By recalling the hypotheses on α and γ, we can apply Proposition 4.6 with $\lambda = 0$ and $\gamma + \frac{1}{p}$ in place of γ to get

$$\left\|\left[L_{m+1}^*\big(w, P-(P)_j\big)\right]u\right\|_p \leqq \mathcal{C}(a_m m^2)^{\frac{1}{p}} (\log m)\|Pu\|_{L^\infty([\theta a_m,\infty))}.$$

Combining the previous inequalities, using Lemma (4.5) and the Nikolski inequality (see [11])

$$\|q_m u\|_\infty \leqq \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^{\frac{2}{p}} \|q_m u\|_p,$$

with $1 \leqq p < +\infty$, $q_m \in \mathbb{P}_m$ and $\mathcal{C} \neq \mathcal{C}(m,p,q_m)$, we can write

$$\left\| \left[L_{m+1}^{*}\left(w, P-(P)_j\right) \right] u \right\|_p \leqq \mathcal{C} a_m^{\frac{1}{p}} m^{\frac{2}{p}} (\log m) \|Pu\|_{L^\infty([\theta a_m,\infty))}$$

$$\leqq \mathcal{C} a_m^{\frac{1}{p}} m^{\frac{2}{p}} (\log m) e^{-Am} \|Pu\|_\infty \leqq \mathcal{C} m^{\frac{3}{p}} (\log m) e^{-Am} \|Pu\|_p$$

$$\leqq \mathcal{C} e^{-\bar{A}m} \left[E_M(f)_{u,p} + \|fu\|_p \right] \leqq \mathcal{C} e^{-\bar{A}m} \|fu\|_p. \quad \square$$

Proof of Corollary 3.5. By using (3.6) we get

$$\left\| \left[f - L_{m+1}^{**}(w,f) \right] u \right\|_p \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} t^{s-\frac{1}{p}-1}\, dt$$

$$\leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{s} \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{s} |f\|_{Z_s^p(u)}$$

and (3.7) is proved. If $f \in W_r^p(u)$, since (see [11])

$$\Omega_\varphi^r(f,t)_{u,p} \leqq \mathcal{C} t^r \left\| f^{(r)} \varphi^r u \right\|_p,$$

one has

$$\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}}\, dt \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{r-\frac{1}{p}} \left\| f^{(r)} \varphi^r u \right\|_p,$$

and, finally, using (3.6) again, we get

$$\left\| \left[f - L_{m+1}^{**}(w,f) \right] u \right\|_p \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{r} \left\| f^{(r)} \varphi^r u \right\|_p \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{r} \|f\|_{W_r^p(u)}$$

i.e. (3.8). $\square$

Proof of Theorem 3.6. Set $\sigma(x) = \frac{w(x)}{\bar{u}(x)} = \frac{x^{\alpha-\gamma} e^{-\frac{x^\beta}{2}}}{(1+x)^\lambda}$. By the Remez-type inequality (4.4), we can write

$$\left\| L_{m+1}^{**}(w,f)\sigma \right\|_1 = \left\| L_{m+1}^{*}(w,f_j)\sigma \right\|_1 \leqq \mathcal{C} \left\| L_{m+1}^{*}(w,f_j)\sigma \right\|_{L^1_{(x_1,a_m)}}$$

$$= \mathcal{C} \sum_{k=1}^{j} \frac{f(x_k)}{p_m'(w,x_k)(a_m-x_k)} \int_{x_1}^{a_m} \frac{(a_m-x)p_m(w,x)}{x-x_k} \tilde{f}(x)\sigma(x)\, dx$$

$$= \mathcal{C} \sum_{k=1}^{j} \frac{f(x_k)}{p_m'(w,x_k)(a_m - x_k)}$$

$$\times \left\{ \int_{x_1}^{\bar\theta a_m} + \int_{\bar\theta a_m}^{a_m} \right\} \frac{(a_m - x) p_m(w,x)}{x - x_k} \tilde f(x)\sigma(x)\,dx =: \mathcal{C}\left[A_1(f) + A_2(f) \right]$$

with $\tilde f = \operatorname{sgn}\left(L^*_{m+1}(w, f_j)\right)$ and $\bar\theta \in (0,1)$, where $\theta a_m < x_j < \bar\theta a_m$.

Now we estimate $A_1(f)$. By using (4.2), we have

$$\left| A_1(f) \right| \leqq \mathcal{C} \| f \bar u \|_{L^\infty[x_1,x_j]} a_m^{-1} \sum_{k=1}^{j} \frac{\sqrt[4]{a_m}\Delta x_k x_k^{\frac{\alpha}{2}+\frac14-\gamma}}{(1+x_k)^\lambda} \left| \pi(x_k) \right|,$$

where

$$\pi(t) = \int_{x_1}^{\bar\theta a_m} \frac{(a_m - x)p_m(w,x)q(x) - (a_m - t)p_m(w,t)q(t)}{x - t} \frac{\tilde f(x)\sigma(x)}{q(x)}\,dx$$

and q is an arbitrary polynomial of degree lm (l fixed). Then $\pi \in \mathbb{P}_{lm+m}$ and, by using a Marcinkiewicz-type inequality (see [12] and Lemma (4.3)), we have

$$\left| A_1(f) \right| \leqq \mathcal{C} \| f_j \bar u \|_\infty a_m^{-1} \int_{x_1}^{\theta_1 a_m} \frac{\sqrt[4]{a_m} t^{\frac{\alpha}{2}+\frac14-\gamma}}{(1+t)^\lambda} \left| \pi(t) \right| dt$$

where $0 < \theta_1 < 1$ is such that $\theta_1 a_m > x_j$. Now we can specify the parameter $\bar\theta$ by setting $\theta_1 < \bar\theta < 1$ and then $x_j < \theta_1 a_m < \bar\theta a_m < a_m$.

With $F(x) := (a_m - x)\sqrt[4]{a_m} p_m(w,x)\tilde f(x)\sigma(x)$, we have

$$I := a_m^{-1} \int_{x_1}^{\theta_1 a_m} \frac{\sqrt[4]{a_m} t^{\frac{\alpha}{2}+\frac14-\gamma}}{(1+t)^\lambda} \left| \pi(t) \right| dt = a_m^{-1} \int_{x_1}^{\theta_1 a_m} \frac{t^{\frac{\alpha}{2}+\frac14-\gamma}}{(1+t)^\lambda} \left| \int_{x_1}^{\bar\theta a_m} \frac{F(x)}{x-t}\,dx \right.$$

$$\left. - \sqrt[4]{a_m}(a_m - t)p_m(w,t)q(t) \int_{x_1}^{\bar\theta a_m} \frac{\sigma(x)\tilde f(x)}{q(x)(x-t)}\,dx \right| dt,$$

from which

$$I \leqq a_m^{-1} \int_{x_1}^{\bar\theta a_m} \left(\frac{t}{1+t} \right)^{\frac{\alpha}{2}+\frac14-\gamma} (1+t)^{-\lambda+\frac{\alpha}{2}+\frac14-\gamma} \left| \int_{x_1}^{\bar\theta a_m} \frac{F(x)}{x-t}\,dx \right| dt$$

$$+ a_m^{-1} \int_{x_1}^{\bar\theta a_m} \frac{t^{\frac{\alpha}{2}+\frac14-\gamma}}{(1+t)^\lambda} \sqrt[4]{a_m}(a_m - t) \left| p_m(w,t)q(t) \right| \left| \int_{x_1}^{\bar\theta a_m} \frac{\sigma(x)\tilde f(x)}{q(x)(x-t)}\,dx \right| dt$$

$$=: I_1 + I_2.$$

The integrals are in the sense of principal value.

Now we recall the following inequality (cf. [14, p. 440]):

$$\int_0^\infty \left(\frac{t}{1+t}\right)^r (1+t)^s \left| \int_0^\infty \frac{F^*(x)}{x-t}\,dx \right| dt \tag{4.9}$$

$$\leqq \mathcal{C}\left[1+\int_0^\infty \left(\frac{t}{1+t}\right)^R (1+t)^S \left|F^*(t)\right| \left(1+\log^+\left|F^*(t)\right| + \log^+ t\right) dt\right].$$

Here $r>-1$, $R \leqq 0$, $r \geqq R$, $s<0$, $S \geqq -1$, $s \leqq S$, $\log^+ x = \log\max(1,x)$ and $\mathcal{C} \neq \mathcal{C}(f)$. In order to apply (4.9) to I_1, we observe that, by hypothesis, $\frac{\alpha}{2}+\frac{1}{4} < \gamma < \frac{\alpha}{2}+\frac{1}{2}$, from which $-\frac{1}{4} < \frac{\alpha}{2}+\frac{1}{4}-\gamma < 0$ and, obviously, $0 > -\lambda + \frac{\alpha}{2} + \frac{1}{4} - \gamma$. Then by using (4.9) with $R = \frac{\alpha}{2}+\frac{1}{4}-\gamma$ and $S=-\lambda$ since $-\lambda \geqq -1$, we obtain

$$I_1 \leqq \mathcal{C} a_m^{-1}\left[1+\int_{x_1}^{\bar\theta a_m} \left(\frac{t}{1+t}\right)^{\frac{\alpha}{2}+\frac{1}{4}-\gamma} \frac{1}{(1+t)^\lambda}\left|F(t)\right| \right.$$

$$\left. \times \left(1+\log^+\left|F(t)\right| + \log^+ t\right) dt\right].$$

By (4.3)

$$\left|F(t)\right| \leqq \mathcal{C} a_m \frac{t^{\frac{\alpha}{2}-\frac{1}{4}-\gamma}}{(1+t)^\lambda}$$

whence, by the assumptions on α, γ, λ, we have

$$I_1 \leqq \mathcal{C}\left[a_m^{-1} + \int_0^\infty \frac{t^{\alpha-2\gamma}}{(1+t)^{2\lambda+\frac{\alpha}{2}+\frac{1}{4}-\gamma}} \left(1+\log^+ \frac{t^{\frac{\alpha}{2}+\frac{3}{4}-\gamma}}{(1+t)^\lambda}\right) dt\right] \leqq \mathcal{C}.$$

Now we estimate I_2. To this end we choose $q(x) \sim x^\gamma e^{-\frac{x^\beta}{2}}$ (see [5, pp. 485–493], [13, pp. 200–208], [15, p. 335]). We have

$$\left|(a_m - t) p_m(w,t) q(t) \sqrt[4]{a_m}\right| \leqq \mathcal{C} a_m t^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}$$

whence

$$I_2 \leqq \mathcal{C}\int_{x_1}^{\bar\theta a_m} \frac{1}{(1+t)^\lambda} \left| \int_{x_1}^{\bar\theta a_m} \frac{\sigma(x)\tilde f(x)}{q(x)(x-t)}\,dx \right| dt.$$

By using (4.9) with $r=R=0$ and $s=S=-\lambda(\geqq -1)$,

$$I_2 \leqq \mathcal{C}\left[1+\int_{x_1}^{\bar\theta a_m} \frac{1}{(1+t)^\lambda}\,\frac{t^{\alpha-\gamma}}{(1+t)^\lambda}\,\frac{e^{-\frac{t^\beta}{2}}}{q(t)}\left(1+\log^+\frac{\sigma(t)}{q(t)}+\log^+ t\right)dt\right]$$

$$\leqq \mathcal{C}\left[1+\int_0^\infty \frac{t^{\alpha-2\gamma}}{(1+t)^{2\lambda}}\left(1+\log^+\frac{t^{\alpha+1-\gamma}}{(1+t)^\lambda}\right)dt\right] \leqq \mathcal{C},$$

recalling that $q(x)\sim x^\gamma e^{-\frac{x^\beta}{2}}$, $\gamma<\frac{\alpha+1}{2}$ and $2\lambda+2\gamma-\alpha>1$. Then by using the previous inequalities we obtain

$$A_1(f)\leqq \mathcal{C}\|f_j\bar u\|_\infty.$$

To complete the proof we have to estimate $A_2(f)$. By estimate (4.2) we have

$$\left|A_2(f)\right|$$

$$\leqq \mathcal{C}\|f_j\bar u\|_\infty \sqrt[4]{a_m}\sum_{k=1}^{j}\frac{x_k^{\frac{\alpha}{2}+\frac14-\gamma}}{(1+x_k)^\lambda}\Delta x_k\int_{\bar\theta a_m}^{a_m}\frac{a_m-x}{a_m-x_k}\left|p_m(w,x)\right|\frac{\sigma(x)}{x-x_k}\,dx.$$

Now recall that we chose $\bar\theta$ such that $\theta a_m\leqq x_j<\theta_1 a_m<\bar\theta a_m<a_m$. Then, for $x>x_j\geqq x_k$, we have $\frac{a_m-x}{a_m-x_k}<1$, $x-x_k\geqq x-x_j\geqq(\bar\theta-\theta)a_m$, whence by (4.3)

$$\left|p_m(w,x)\sigma(x)\right|\leqq \mathcal{C}\frac{x^{\frac{\alpha}{2}-\frac14-\gamma}}{(1+x)^\lambda\sqrt[4]{a_m-x}}\leqq \mathcal{C}\frac{x_k^{\frac{\alpha}{2}-\frac14-\gamma}}{(1+x_k)^\lambda\sqrt[4]{a_m-x}}.$$

Hence we obtain

$$\int_{\bar\theta a_m}^{a_m}\frac{a_m-x}{a_m-x_k}\left|p_m(w,x)\right|\frac{\sigma(x)}{x-x_k}\,dx\leqq \mathcal{C}\frac{x_k^{\frac{\alpha}{2}-\frac14-\gamma}}{(1+x_k)^\lambda a_m(\bar\theta-\theta)}\int_{\bar\theta a_m}^{a_m}\frac{dx}{\sqrt[4]{a_m-x}}$$

$$\sim\frac{x_k^{\frac{\alpha}{2}-\frac14-\gamma}}{(1+x_k)^\lambda\sqrt[4]{a_m}}$$

from which it follows

$$\left|A_2(f)\right|\leqq \mathcal{C}\|f_j\bar u\|_\infty\sum_{k=1}^{j}\frac{x_k^{\alpha-2\gamma}}{(1+x_k)^{2\lambda}}\Delta x_k\leqq \mathcal{C}\|f_j\bar u\|_\infty.$$

The proof of estimate (3.10) is complete.

The proof of (3.11) is similar to that of (3.6). At first let us observe that for $f \in C_{\bar u}$, taking into account the assumptions on the parameters γ and λ, one has

$$\left\| f\frac{w}{\bar u}\right\|_1 \leqq \|f\bar u\|_\infty \int_0^\infty \frac{w(x)}{\left[\bar u(x)\right]^2}\,dx \leqq \mathcal{C}\|f\bar u\|_\infty. \tag{4.10}$$

Now let $M = \left[\left(\frac{\theta}{\theta+1}\right)^\beta m\right]$, $P = P_M \in \mathbb{P}_M$ the best approximation polynomial of f in $C_{\bar u}$ and $(P)_j = P\Phi_j$, with Φ_j the characteristic function of the interval $[0, x_j]$. By Minkowski's inequality we have

$$\begin{aligned}\left\|\left[f - L^{**}_{m+1}(w,f)\right]\frac{w}{\bar u}\right\|_1 &= \left\|\left[f - L^{*}_{m+1}(w,f_j)\right]\frac{w}{\bar u}\right\|_1\\ &\leqq \left\|[f - f_j]\frac{w}{\bar u}\right\|_1 + \left\|\left[f_j - (P)_j\right]\frac{w}{\bar u}\right\|_1\\ &+ \left\|\left[(P)_j - P\right]\frac{w}{\bar u}\right\|_1 + \left\|\left[L^{*}_{m+1}(w,P - f_j)\right]\frac{w}{\bar u}\right\|_1.\end{aligned}$$

In virtue of (4.10), Lemma 4.5 and (2.3), we have

$$\begin{aligned}\left\|[f - f_j]\frac{w}{\bar u}\right\|_1 &\leqq \mathcal{C}\left\|[f - f_j]\bar u\right\|_\infty \leqq \mathcal{C}\left[E_M(f)_{\bar u,\infty} + e^{-Am}\|f\bar u\|_\infty\right]\\ &\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^\infty(\bar u)},\end{aligned}$$

$$\begin{aligned}\left\|\left[f_j - (P)_j\right]\frac{w}{\bar u}\right\|_1 &\leqq \left\|[f - P]\frac{w}{\bar u}\right\|_1 \leqq \mathcal{C}\left\|[f - P]\bar u\right\|_\infty\\ &\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^\infty(\bar u)}\end{aligned}$$

and

$$\left\|\left[(P)_j - P\right]\frac{w}{\bar u}\right\|_1 \leqq \mathcal{C}\left\|\left[(P)_j - P\right]\bar u\right\|_\infty \leqq \mathcal{C}e^{-Am}\|P\bar u\|_\infty \leqq \mathcal{C}e^{-Am}\|f\bar u\|_\infty.$$

It remains to estimate the last term $\left\|\left[L^*_{m+1}(w,P-f_j)\right]\frac{w}{\bar u}\right\|_1$. At first we apply Minkowski's inequality and write

$$\begin{aligned}&\left\|\left[L^*_{m+1}(w,P-f_j)\right]\frac{w}{\bar u}\right\|_1\\ &\leqq \left\|\left[L^*_{m+1}\big(w,(P-f)_j\big)\right]\frac{w}{\bar u}\right\|_1 + \left\|\left[L^*_{m+1}\big(w,P-(P)_j\big)\right]\frac{w}{\bar u}\right\|_1,\end{aligned}$$

then estimate both terms on the right hand side.

For the first one we use (3.10) and get

$$\Big\| \big[L^*_{m+1}(w, P-f)_j \big] \frac{w}{\bar{u}} \Big\|_1 \leqq \mathcal{C} \big\| (P-f)_j \bar{u} \big\|_\infty \leqq \mathcal{C} E_M(f)_{\bar{u},\infty}$$

$$\leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \|f\|_{W_r^\infty(\bar{u})}.$$

To estimate the second term, proceeding as in the proof of (3.6), we obtain

$$\Big\| \big[L^*_{m+1}\big(w, P-(P)_j\big) \big] \frac{w}{\bar{u}} \Big\|_1 \leqq \mathcal{C} \int_{x_1}^{a_m} \big| L^*_{m+1}\big(w, P-(P)_j; x\big) \big| \frac{w(x)}{\bar{u}(x)}\, dx$$

$$\leqq \mathcal{C} \sup_{x \in [x_1, a_m]} \big| L^*_{m+1}\big(w, P-(P)_j; x\big) \bar{u}(x) \big| \int_{x_1}^{a_m} \frac{w(x)}{\big[\bar{u}(x)\big]^2}\, dx$$

$$\leqq \mathcal{C} \|P\bar{u}\|_{L^\infty([\theta a_m, \infty))} \sup_{x \in [x_1, a_m]} \sum_{k=j+1}^{m+1} \frac{\big| l^*_k(x) \big| \bar{u}(x)}{\bar{u}(x_k)}$$

$$\leqq \mathcal{C} \log m e^{-Am} \|P\bar{u}\|_\infty \leqq \mathcal{C} \log m e^{-Am} \|f\bar{u}\|_\infty \leqq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \|f\|_{W_r^\infty(\bar{u})}. \qquad \square$$

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References

[1] R. Askey and S. Wainger, Mean convergence of expansion in Laguerre and Hermite series, *Amer. J. Math.,* **87** (1965), 695–708.

[2] Z. Ditzian and V. Totik, *Moduli of Smoothness,* SCMG Springer-Verlag, (New York, 1987).

[3] K. G. Ivanov, On the behaviour of two moduli of functions, II *Serdica,* **12**, 196–203.

[4] T. Kasuga and R. Sakai, Orthonormal polynomials with generalized Freud-type weights, *J. Approx. Theory,* **121** (2003), 13–53.

[5] A. L. Levin and D. S. Lubinsky, Christoffel functions, orthogonal polynomials and Nevai's conjecture for Freud weights, *Constr. Approx.,* **8** (1992), 463–535.

[6] A. L. Levin and D. S., Lubinski, Orthogonal polynomials for weights $x^{2\rho}e^{-2Q(x)}$ on $[0, d)$, *J. Approx. Theory,* **134** (2005), 199–256.

[7] G. Mastroianni and G. V. Milovanović, Some numerical methods for second kind Fredholm integral equation on the real semiaxis, to appear.

[8] G. Mastroianni and G. V. Milovanović, *Interpolation Processes: Basic Theory and Applications,* in progress.

[9] G. Mastroianni and J. Szabados, Polynomial approximation on infinite intervals with weights having inner zeros, *Acta Math. Hungar.,* **96** (2002), 221–258.

[10] G. Mastroianni and J. Szabados, Direct and converse polynomial approximation theorems on the real line withh weights having zeros, in: *Frontiers in Interpolation and Approximation,* Dedicated to the memory of A. Sharma, eds. N. K. Govil, H. N. Mhaskar, R. N. Mohapatra, Z. Nashed and J. Szabados, Taylor & Francis Books, Boca Raton, Florida (2006), pp. 287–306.

[11] G. Mastroianni and J. Szabados, Polynomial approximation on the real semiaxis with generalized Laguerre weights, *Studia Univ. Babeş–Bolyai, Mathematica,* **52** (2007), 105–128.

[12] G. Mastroianni and P. Vértesi, Fourier sums and Lagrange interpolation on $(0,+\infty)$ and $(-\infty,+\infty)$, in: *Frontiers in Interpolation and Approximation,* Dedicated to the memory of A. Sharma, eds. N. K. Govil, H. N. Mhaskar, R. N. Mohapatra, Z. Nashed and J. Szabados, Taylor & Francis Books, Boca Raton, Florida (2006), pp. 307–344.

[13] H. N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation,* World Scientific (Singapore, New Jersey, London, Hong Kong, 1996).

[14] B. Muckenhoupt, Mean convergence of Hermite and Laguerre series II, *Trans. Amer. Math. Soc.,* **147** (1970), 433–460.

[15] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields,* Springer (Berlin, 1997).

[16] P. Vértesi, Oral communication.