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INEQUALITIES AND PROPERTIES OF SOME GENERALIZED ORLICZ CLASSES AND SPACES

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Abstract. We discuss and complement the knowledge about generalized Orlicz classes $\widetilde{X}_\Phi$ and Orlicz spaces X_Φ obtained by replacing the space L^1 in the classical construction by an arbitrary Banach function space X. Our main aim is to focus on the task to study inequalities in such spaces. We prove a number of new inequalities and also natural generalizations of some classical ones (e.g., Minkowski's, Hölder's and Young's inequalities). Moreover, a number of other basic facts for further study of inequalities and function spaces are included.

1. Introduction

The Orlicz classes $\widetilde{L}_\Phi$ and the Orlicz spaces L_Φ are genuine generalizations of the usual L^p-spaces, see, e.g., [2], [5], [11] and references therein. Another type of generalization of the L^p-spaces, namely the X^p-spaces (X is a Banach function space), has been used and studied, e.g., in [6], [8], [9]. In this paper, we study a unification of these generalizations, namely, the generalized Orlicz classes $\widetilde{X}_\Phi$ and generalized Orlicz spaces X_Φ. This is done by

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replacing the space L^1 in the classical construction by an arbitrary Banach function space X.

Some preliminary studies of generalized Orlicz classes $\widetilde{X}_\Phi$ and generalized Orlicz spaces X_Φ can be found already in the early research report [10] by L. E. Persson. In this paper, we will improve these preliminary results in various ways and put them into a more general frame. Moreover, we will prove a number of results, which are particularly important for the study of inequalities in these spaces. In particular, we will prove both some new inequalities and also some natural generalization of the classical ones (e.g., Minkowski's inequality, Hölder's inequality and Young's inequality).

The paper is organized as follows: In order not to disturb our discussions later on, some preliminaries are collected in Section 2. In Section 3, we discuss the generalized Orlicz class $\widetilde{X}_\Phi$ and prove some of its basic facts and related inequalities. The corresponding questions concerning the generalized Orlicz space X_Φ is handled in Section 4. In Section 5, we equip the space X_Φ with the natural generalization of the famous Luxemburg norm and prove some inequalities and other basic facts of importance for further studies in the theory of function spaces and inequalities. Moreover, in Section 6, we discuss some inequalities we obtained so far and also prove an important comparison result (Proposition 6.1) and a new generalized form of Hölder's inequality (Theorem 6.2).

2. Preliminaries

Let (Ω, Σ, μ) be a complete σ-finite measure space with $\mu(\Omega) > 0$. We denote by $L^0(\Omega)$ the space of all equivalence classes of measurable real valued functions defined and finite a.e. on Ω. A real normed linear space $X = \{u \in L^0(\Omega) : \|u\|_X < \infty\}$ is called a Banach function space (written shortly as BFS) if in addition to the usual norm axioms, $\|u\|_X$ satisfies the following conditions:

(P1) $\|u\|_X$ is defined for every measurable function u on Ω and $u \in X$ if and only if $\|u\|_X < \infty$; $\|u\|_X = 0$ if and only if, $u = 0$ a.e.;

(P2) $0 \leqq u \leqq v$ a.e. $\Rightarrow \|u\|_X \leqq \|v\|_X$;

(P3) $0 < u_n \uparrow u$ a.e. $\Rightarrow \|u_n\|_X \uparrow \|u\|_X$;

(P4) $\mu(E) < \infty \Rightarrow \|\chi_E\|_X < \infty$;

(P5) $\mu(E) < \infty \Rightarrow \int_E u(x)dx \leqq C_E \|u\|_X$,

where $E \subset \Omega$, χ_E denotes the characteristic function of E and C_E is a constant depending only on E. For various properties concerning Banach function spaces see, e.g., [1].

Examples of Banach function spaces are the classical Lebesgue spaces L^p, $1 \leqq p \leqq \infty$, the Orlicz spaces L_Φ, the classical Lorentz spaces $L_{p,q}$, $1 \leqq p, q$

$\leqq \infty$, the generalized Lorentz spaces Λ_ϕ and the Marcinkiewicz spaces M_ϕ (see, e.g. [3]).

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if

$$\Phi(s) = \int_0^s \phi(t)\,dt,$$

where $\phi : [0,\infty) \to [0,\infty]$, $\phi(0) = 0$ is an increasing, left continuous function which is neither identically zero nor identically infinite on $(0,\infty)$. A Young function Φ is continuous, convex, increasing and satisfies

$$\Phi(0) = 0, \qquad \lim_{s\to\infty} \Phi(s) = \infty.$$

Moreover, a Young function Φ satisfies the following useful inequalities: for $s \geqq 0$, we have

$$\begin{cases} \Phi(\alpha s) < \alpha\,\Phi(s), & \text{if } 0 \leqq \alpha < 1 \\ \Phi(\alpha s) \geqq \alpha\,\Phi(s), & \text{if } \alpha \geqq 1. \end{cases} \tag{2.1}$$

We call a Young function an N-function if it satisfies the limit conditions

$$\lim_{s\to\infty} \frac{\Phi(s)}{s} = \infty \qquad \text{and} \qquad \lim_{s\to 0} \frac{\Phi(s)}{s} = 0.$$

Let Φ be a Young function generated by the function ϕ, i.e.,

$$\Phi(s) = \int_0^s \phi(t)\,dt.$$

Then the function Ψ generated by the function ψ, i.e.,

$$\Psi(s) = \int_0^s \psi(t)\,dt,$$

where $\psi(s) = \sup_{\phi(t)\leqq s} t$ is called the complementary function to Φ. It is known that Ψ is a Young function and that Φ is complementary to Ψ. The pair of complementary Young functions Φ, Ψ satisfies the Young inequality

$$u \cdot v \leqq \Phi(u) + \Psi(v), \qquad u, v \in [0,\infty). \tag{2.2}$$

Equality in (2.2) holds if and only if

$$v = \Phi(u) \qquad \text{or} \qquad u = \Psi(v). \tag{2.3}$$

A Young function Φ is said to satisfy the Δ_2-condition, written $\Phi \in \Delta_2$, if there exist $k > 0$ and $T \geqq 0$ such that $\Phi(2t) \leqq k\Phi(t)$ for all $t \geqq T$.

3. The generalized Orlicz classes $\widetilde{X}_\Phi$

Let X be a BFS and denote Φ a non-negative function on $[0,\infty)$. The generalized Orlicz class $\widetilde{X}_\Phi$ consists of all functions $u \in L^0(\Omega)$ such that

$$\rho_X(u;\Phi) = \big\|\Phi\big(|u|\big)\big\|_X < \infty.$$

For the case $\Phi(t) = t^p$, $0 < p < \infty$, $\widetilde{X}_\Phi$ coincides algebraically with the space X^p endowed with the quasi-norm $\|u\|_{X^p} = \big\||u|^p\big\|_X^{\frac{1}{p}}$. The X^p spaces have been studied and used, e.g., in [4], [8] and [9].

We begin with the following inequality:

PROPOSITION 3.1. *Let X be a BFS and Φ be a Young function. Then the inequality*

$$\rho_X(\alpha u + (1-\alpha)v) \leqq \alpha\rho_X(u) + (1-\alpha)\rho_X(v) \tag{3.1}$$

holds for $0 \leqq \alpha \leqq 1$ and all $u, v \in \widetilde{X}_\Phi$, i.e. $\widetilde{X}_\Phi$ is a convex set and $\widetilde{X}_\Phi \subset X$ for $\mu(\Omega) < \infty$.

PROOF. Let $0 \leqq \alpha \leqq 1$. According to the convexity and monotonicity properties of Φ we have

$$\Phi\big(\big|\alpha u + (1-\alpha)v\big|\big) \leqq \alpha\,\Phi\big(|u|\big) + (1-\alpha)\Phi\big(|v|\big).$$

Therefore, by the definition of a BFS, we obtain

$$\begin{aligned}\big\|\Phi\big(\big|\alpha u + (1-\alpha)v\big|\big)\big\|_X &\leqq \big\|\alpha\,\Phi\big(|u|\big) + (1-\alpha)\Phi\big(|v|\big)\big\|_X\\ &\leqq \alpha\big\|\Phi\big(|u|\big)\big\|_X + (1-\alpha)\big\|\Phi\big(|v|\big)\big\|_X,\end{aligned}$$

i.e., (3.1) holds.

Further, property (2.1) of a Young function Φ implies that there exists $c > 0$ such that if $u = u(t)$, $\big|u(t)\big| > c$, then

$$\frac{\Phi\big(|u|\big)}{|u|} > 1, \quad \text{i.e.,} \quad \Phi\big(|u|\big) > |u|.$$

Let $u = u(t) \in \widetilde{X}_\Phi$ and $E = \big\{t \in \Omega : \big|u(t)\big| > c\big\}$. Then the last estimate gives

$$\big\||u|\chi_E\big\|_X \leqq \big\|\Phi\big(|u|\big)\big\|_X < \infty.$$

Moreover, $\big\||u|\chi_{\Omega\setminus E}\big\|_X \leqq c\|\chi_\Omega\|_X < \infty$ and, consequently,

$$\big\||u|\big\|_X = \big\||u|\chi_{\Omega\setminus e} + |u|\chi_e\big\| \leqq c\|\chi_\Omega\|_X + \big\|\Phi\big(|u|\big)\big\|_X < \infty,$$

i.e., $u \in X$, and also the inclusion $\widetilde{X}_\Phi \subset X$ is proved. □

We say that the BFS X has the *L-property* if there exists a positive constant a such that the inequality

$$\left\| \sum_{n=1}^{\infty} u_n \right\|_X \geqq a \sum_{n=1}^{\infty} \|u_n\|_X$$

holds for every sequence $\{u_n\}$, $u_n = u_n(t) > 0$ with

$$\operatorname{supp} u_n \cap \operatorname{supp} u_m = \emptyset, \qquad n \neq m.$$

PROPOSITION 3.2. *Let $\mu(\Omega) < \infty$ and assume that X is a BFS satisfying the L-property. Then, for $u \in X$, there exists a Young function Φ such that $u \in \widetilde{X}_\Phi(\Omega)$.*

PROOF. Denote

$$\Omega_n = \left\{ x \in \Omega : n - 1 \leqq \left| u(x) \right| < n \right\}.$$

Let $u \in X$. Then

$$\infty > \left\| \left| u(x) \right| \right\|_X = \left\| \sum_{n=1}^{\infty} \left| u(x) \right| \chi_{\Omega_n} \right\|_X \geqq a \sum_{n=1}^{\infty} \left\| \left(\left| u(x) \right| \right) \chi_{\Omega_n} \right\|_X$$

$$\geqq a \sum_{n=1}^{\infty} (n-1) \|\chi_{\Omega_n}\|_X = a \sum_{n=1}^{\infty} n \|\chi_{\Omega_n}\|_X - a \sum_{n=1}^{\infty} \|\chi_{\Omega_n}\|_X$$

$$\geqq a \sum_{n=1}^{\infty} n \|\chi_{\Omega_n}\|_X - \|\chi_\Omega\|_X.$$

Since $\mu(\Omega) < \infty$, we have that $\|\chi_\Omega\|_X < \infty$ and conclude that $\sum_{n=1}^{\infty} n \|\chi_{\Omega_n}\|_X$ converges. Now choose a non-decreasing sequence $\{\alpha_n\}$ such that

$$\alpha_n > 1, \quad \lim_{n\to\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n\alpha_n \|\chi_{\Omega_n}\| < \infty. \tag{3.2}$$

Moreover, define

$$\phi(t) = \begin{cases} t, & \text{for } t \in [0,1) \\ \alpha_n, & \text{for } t \in [n, n+1),\ n = 1, 2, 3, \dots, \end{cases}$$

then

$$\Phi(t) = \int_0^t \phi(s) ds \tag{3.3}$$

is a Young function and $\Phi(n) \leqq n\alpha_n$ for $n = 1, 2, \dots$. Hence we have

$$\left\|\Phi\big(\left|u(x)\right|\big)\right\|_X = \left\|\sum_{n=1}^{\infty}\Phi\big(\left|u(x)\right|\big)\chi_{\Omega_n}\right\|_X$$

$$\leqq \sum_{n=1}^{\infty}\Phi(n)\|\chi_{\Omega_n}\|_X \leqq \sum_{n=1}^{\infty} n\alpha_n\|\chi_{\Omega_n}\|_X < \infty$$

and also the assertion that $u \in \widetilde{X}_\Phi$ is proved with Φ defined by (3.3). □

REMARK 3.3. In fact, our proof shows that the function Φ defined by (3.3) satisfies the inequality

$$\rho_X(u;\Phi) \leqq \sum_{n=1}^{\infty} n\alpha_n\|\chi_{\Omega_n}\|_X$$

with $\{\alpha_n\}$ satisfying (3.2).

REMARK 3.4. In view of Propositions 3.1 and 3.2 we see that if $\mu(\Omega) < \infty$ and the BFS X satisfies the L-property, then the representation formula

$$X = \bigcup_{\Phi} \widetilde{X}_\Phi$$

holds, where the union is taken over all Young functions Φ.

Next, we study the inclusion between two generalized Orlicz classes. Note that for two Young functions Φ_1 and Φ_2 satisfying

$$\Phi_1(t) \leqq \Phi_2(t) \quad \text{for all} \quad t \geqq 0, \tag{3.4}$$

we clearly have

$$\widetilde{X}_{\Phi_2} \subset \widetilde{X}_{\Phi_1}. \tag{3.5}$$

However, if $\mu(\Omega) < \infty$, then the condition (3.4) can be weakened. More precisely, we prove the following:

PROPOSITION 3.5. *Let $\mu(\Omega) < \infty$ and Φ_1, Φ_2 be two Young functions. If there exists $c > 0$ and $T \geqq 0$ such that*

$$\Phi_1(t) \leqq c\,\Phi_2(t) \quad \textit{for all} \quad t \geqq T \tag{3.6}$$

then the inclusion (3.5) *holds.*

PROOF. By (P4) since $\mu(\Omega) < \infty$, we have $\|X_\Omega\|_X < \infty$. Let $u \in \widetilde{X}_2(\Omega)$ and put $\Omega_1 = \{x \in \Omega : |u(x)| \leqq T\}$. Then (3.6) and the properties of the BFS X ensure that

$$\|\Phi_1(|u(x)|)\|_X \leqq \|\chi_{\Omega_1}\Phi_1(|u(x)|)\|_X + \|\chi_{\Omega\setminus\Omega_1}\Phi_1(|u(x)|)\|_X$$

$$\leqq \Phi_1(T)\|\chi_\Omega\|_X + c\,\rho_X(u;\Phi_2) < \infty$$

and the assertion follows. □

In general, $\widetilde{X}_\Phi$ need not be a linear set. However, the following holds:

PROPOSITION 3.6. *Let Φ be a Young function satisfying the Δ_2-condition (with $T = 0$ if $\mu(\Omega) = \infty$). Then $\widetilde{X}_\Phi$ is a linear set.*

PROOF. Assume first that $\mu(\Omega) < \infty$. For any $\alpha \geqq 0$, there exists $n \in \mathbb{N}$ such that $\alpha \leqq 2^n$. The monotonicity of Φ together with the fact that $\Phi \in \Delta_2$ gives that there exist $k > 0$ and $T \geqq 0$ such that

$$\Phi(\alpha t) \leqq \Phi(2^n t) \leqq k^n \Phi(t), \qquad t \geqq T.$$

Taking $\Phi_1(t) = \Phi(\alpha t)$, $\Phi_2(t) = \Phi(t)$ and applying Proposition 3.5, we get $\widetilde{X}_{\Phi_2} \subset \widetilde{X}_{\Phi_1}$. Hence, if $u \in \widetilde{X}_\Phi \equiv \widetilde{X}_{\Phi_2}$, then $u \in \widetilde{X}_{\Phi_1}$ which immediately yields that $\alpha u \in \widetilde{X}_\Phi$.

Further, let $u, v \in \widetilde{X}_\Phi$. Then $2u, 2v \in \widetilde{X}_\Phi$. Moreover, the convexity and monotonicity properties of Φ give

$$\Phi(|u(x) + v(x)|) \leqq \Phi\left(\frac{1}{2}|2u(x)| + \frac{1}{2}|2v(x)|\right)$$

$$\leqq \frac{1}{2}\Phi(|2u(x)|) + \frac{1}{2}\Phi(|2v(x)|),$$

which implies

$$\|\Phi(|u(x) + v(x)|)\|_X \leqq \frac{1}{2}\|\Phi(|2u(x)|)\|_X + \frac{1}{2}\|\Phi(|2v(x)|)\|_X < \infty,$$

i.e., $u + v \in \widetilde{X}_\Phi$ and consequently $\widetilde{X}_\Phi$ is a linear set.

For $\mu(\Omega) = \infty$, we have $\Phi(T) = \Phi(0) = 0$ as well as $\Phi(\alpha T) = \Phi(0) = 0$. Now repeating the arguments above, we get the assertion also in this case. □

4. The generalized Orlicz spaces X_Φ

We begin by defining the generalized Orlicz space X_Φ.

Definition 4.1. Let X be a BFS and Φ, Ψ be a pair of complementary Young functions. The generalized Orlicz space, denoted by X_Φ, is the set of all $u \in L^0(\Omega)$ such that

$$\|u\|_\Phi := \sup_v \big\| |u \cdot v| \big\|_X, \tag{4.1}$$

where the supremum is taken over all $v \in \widetilde{X}_\Psi$ for which $\rho_X(v;\Psi) \leqq 1$.

It can be seen that for a given Young function Φ, the generalized Orlicz space X_Φ contains the generalized Orlicz class $\widetilde{X}_\Phi$. Indeed, if Ψ is the Young function complementary to Φ, then, in view of the Young's inequality (2.2), we obtain that for $u \in \widetilde{X}_\Phi$, $v \in \widetilde{X}_\Psi$

$$\big\| |u \cdot v| \big\|_X \leqq \big\|\Phi\big(|u|\big)\big\|_X + \big\|\Psi\big(|v|\big)\big\|_X = \rho_X(u;\Phi) + \rho_X(v;\Psi), \tag{4.2}$$

which implies that $\|u\|_\Phi \leqq \rho_X(u;\Phi) + 1 < \infty$ whenever $\rho_X(v;\Psi) \leqq 1$ and, hence, $\widetilde{X}_\Phi \subset X_\Phi$.

Remark 4.2. The inequality (4.2) may obviously be regarded as a natural generalization of the Young inequality in an Orlicz frame. In particular, we find that if Φ, Ψ is a pair of complementary Young functions and $u \in \widetilde{X}_\Phi$, $v \in \widetilde{X}_\Psi$, then $u \cdot v \in X$.

Theorem 4.3. *Let X be a BFS and Φ be a Young function. Then the space X_Φ is a BFS, with the norm defined by* (4.1).

Proof. Let $u, v \in X_\Phi(\Omega)$ and $w \in \widetilde{X}_\Psi$, $\rho_X(w,\Psi) \leqq 1$. By (3.1) and the fact that X is a BFS, we have

$$\begin{aligned} \|u+v\|_\Phi &= \sup_w \big\| |u+v| \cdot |w| \big\|_X \leqq \sup_w \big\| |u \cdot w| + |v \cdot w| \big\|_X \\ &\leqq \sup_w \big\| |u \cdot w| \big\|_X + \sup_w \big\| |v \cdot w| \big\|_X = \|u\|_\Phi + \|v\|_\Phi. \end{aligned} \tag{4.3}$$

Moreover, Definition 4.1 implies that $\|au\|_\Phi = |a|\,\|u\|_\Phi$ for any complex number a. Obviously, $u = 0$ a.e. implies that $\|u\|_\Phi = 0$. Further, let $E \subset \Omega$ be an arbitrary set such that $0 < \mu(E) < \infty$. Since $\lim_{t\to 0} \Psi(t) = 0$, for $k = \frac{1}{\|\chi_E\|_X}$, there exists k_1, $0 < k_1 < \infty$ such that $\Psi(k_1) \leqq \frac{1}{\|\chi_E\|_X}$. Define

$$v(x) = \begin{cases} k_1 & \text{for } x \in E \\ 0 & \text{for } x \in \Omega \setminus E \end{cases}$$

and note that

$$\rho_X(v;\Psi) = \big\|\Psi\big(\big|v(x)\big|\big)\big\|_X = \big\|\Psi(k_1)\chi_E\big\|_X = \Psi(k_1)\|\chi_E\|_X \leqq 1.$$

Consequently,

$$\|u\|_\Phi \geqq \|u\cdot v\|_X = \|u\cdot k_1\chi_E\|_X = k_1\|u\cdot\chi_E\|_X.$$

We conclude that if $\|u\|_\Phi = 0$, then $u = 0$ a.e. in E. Since $E \subset \Omega$ was arbitrary, $u = 0$ for almost all $x \in \Omega$ and (P1) is proved.

The properties (P2) and (P3) are obvious. As regards (P4), let $\mu(E) < \infty$, and note that we have by the generalized Young's inequality (4.2)

$$\big\|\chi_E\cdot|w|\big\|_X \leqq \big\|\phi(\chi_E)\big\|_X + \big\|\Psi\big(|w|\big)\big\|_X = \Phi(1)\|\chi_E\|_X + \rho_X(w;\Psi).$$

Now, since X is a BFS, we have that $\|\chi_E\|_X < \infty$. Consequently, taking the supremum in the last estimate over all w for which $\rho_X(w;\Psi) \leqq 1$, we get $\|\chi_E\|_\Phi < \infty$, and (P4) is proved.

Finally, let $E \subset \Omega$ be such that $0 < \mu(E) < \infty$. Then according to (4.2), we have for $u \in X_\Phi$, that $u\cdot\chi_E \in X$. Moreover, by using (P5) applied on the BFS X we get that

$$\int_E u\cdot\chi_E d\mu \leqq C_E\|u\cdot\chi_E\|_X$$

and, since $\rho_X(\chi_E;\Psi) \leqq C'_E$,

$$\int_E u d\mu \leqq C_E\|u\cdot\chi_E\|_X \leqq C''_E\|u\|_\Phi.$$

The last estimate proves (P5) for the space X_Φ and, hence, X_Φ is a BFS. □

5. The generalized Luxemburg norm

We generalize the definition of the famous Luxemburg norm (see e.g. [7]) in the following way:

$$\|u\|'_\Phi = \inf\left\{k > 0 : \rho_X\left(\frac{|u|}{k},\Phi\right) \leqq 1\right\}. \tag{5.1}$$

THEOREM 5.1. *The quantity defined by* (5.1) *is a norm in* X_Φ *and*

$$\|u\|_\Phi \leqq 2\|u\|'_\Phi, \quad u \in X_\Phi. \tag{5.2}$$

Proof. In view of (5.1), we find that

$$\left\|\Phi\left(\frac{|u|}{k}\right)\right\|_X \leqq 1 \quad \text{if} \quad k = \|u\|'_\Phi.$$

Now, let $u, v \in X_\Phi$, $\|u\|'_\Phi + \|v\|'_\Phi = a \neq 0$, $\lambda_0 = \frac{\|u\|'_\Phi}{a}$ and $\lambda_1 = 1 - \lambda_0$. We use the monotonicity and convexity properties of Φ and obtain

$$\Phi\left(\frac{|u+v|}{a}\right) \leqq \Phi\left(\frac{|u|+|v|}{a}\right) \leqq \lambda_0 \Phi\left(\frac{|u|}{a\lambda_0}\right) + \lambda_1 \Phi\left(\frac{|v|}{a\lambda_1}\right)$$

which, by using (5.2), gives that

$$\rho_X\left(\frac{|u+v|}{a}, \Phi\right) = \left\|\Phi\left(\frac{|u+v|}{a}\right)\right\|_X \leqq \lambda_0 \cdot 1 + \lambda_1 \cdot 1 = 1$$

and we conclude that $\|u+v\|'_\Phi \leqq a$, i.e.,

$$\|u+v\|'_\Phi \leqq \|u\|'_\Phi + \|v\|'_\Phi. \tag{5.3}$$

The other norm properties of $\|\cdot\|'_\Phi$ are obvious.

According to (4.2), for the pair of complementary Young functions Φ, Ψ, we have

$$\|u.v\|_X \leqq \rho_X(u;\Phi) + \rho_X(v;\Psi), \qquad u \in X_\Phi,\ v \in X_\Psi$$

which, in view of (4.1), implies that

$$\|u\|_\Phi \leqq \rho_X(u;\Phi) + 1. \tag{5.4}$$

If we consider $w = \frac{|u|}{\|u\|'_\Phi}$, we note from (4.2) that $\rho_X(w;\Phi) \leqq 1$ and hence, according to (5.3), $\|w\|_\Phi \leqq 2$ and the assertion follows. $\square$

We have defined two norms on the generalized Orlicz space X_Φ, namely (4.1) and (5.1). With respect to the norm (4.1), X_Φ is a BFS (see Theorem 4.3). We will now prove a similar result for the generalized Luxemburg norm yielding, in particular, that the two norms are equivalent.

Theorem 5.2. *If X is a BFS, then X_Φ is a BFS with the generalized Luxemburg norm defined by* (5.1).

PROOF. Properties (P1) and (P5) follow in view of Theorem 5.1 while (P2) and (P3) are obvious. We only prove (P4).

First we note that the equality

$$\left\| \Phi \left(\frac{|u|}{k_0} \right) \right\|_X = 1$$

always implies that $k_0 = \|u\|'_\Phi$. Let $E \subset \Omega$ be such that $0 < \mu(E) < \infty$. Since X is a BFS, we have that $0 < \|\chi_E\|_X < \infty$. We see that

$$\left\| \Phi \left[\Phi^{-1} \left(\frac{1}{\|\chi_E\|_X} \right) \cdot \chi_E \right] \right\|_X = \left\| \frac{1}{\|\chi_E\|_X} \cdot \chi_E \right\|_X = 1$$

and consequently (5.4) gives

$$\|\chi_E\|'_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\|\chi_E\|_X} \right)},$$

which proves (P4). $\square$

REMARK 5.3. The space $\left(X_\Phi, \|\cdot\|'_\Phi\right)$ can also be regarded as a Calderon space $\varphi(X)$. More exactly, $\varphi(X) \equiv \left(X_\Phi, \|\cdot\|'_\Phi\right)$, where $\varphi(t)$ is the inverse of $\Phi(t)$. Indeed, if $u \in \left(X_\Phi; \|\cdot\|'_\Phi\right)$, then, according to (2.1) and the definition (5.1) of the Luxemburg norm, we have

$$\left\| \Phi \left(\frac{|u|}{\| |u| \|_\Phi} \right) \right\|_X \leqq 1$$

and, thus, the representation

$$|u| = \|u\|'_\Phi \varphi \left(\Phi \left(\frac{|u|}{\|u\|'_\Phi} \right) \right)$$

gives us that $u \in \varphi(X)$ and $\|u\|_{\varphi(X)} \leqq \|u\|'_\Phi$. The proof in the opposite direction is similar.

6. More on inequalities

In particular, we have so far pointed out a number of inequalities in this connection. In this final section, we will focus our interest more on this aspect by deriving some new inequalities, summing up what we have done so far and also raise a question for further research (see Remark 6.5).

First, we prove the following elementary but useful inequalities for comparing the quantities $\rho_X(u,\Phi)$ and $\|u\|'_\Phi$:

PROPOSITION 6.1. *Let $u \in X_\Phi$. Then*

$$\rho_X(u;\Phi) \leqq \|u\|'_\Phi \quad \text{if} \quad \|u\|'_\Phi \leqq 1 \tag{6.1}$$

and

$$\rho_X(u;\Phi) \geqq \|u\|'_\Phi \quad \text{if} \quad \|u\|'_\Phi > 1. \tag{6.2}$$

PROOF. By property (P2), (5.2) and (2.1) with $\alpha = \|u\|'_\Phi$ and u replaced by $\frac{u(t)}{\alpha}$, we obtain that if $\|u\|'_\Phi \leqq 1$, then

$$\begin{aligned}\rho_X(u;\Phi) = \big\| \Phi(u) \big\|_X = \left\| \Phi\left(\alpha \frac{u}{\alpha}\right) \right\|_X \leqq \alpha \left\| \Phi\left(\frac{u}{\alpha}\right) \right\|_X \\ = \|u\|'_\Phi \left\| \Phi\left(\frac{|u|}{\|u\|'_\Phi}\right) \right\|_X \leqq \|u\|'_\Phi,\end{aligned}$$

and (6.1) is proved.

Now, let $\|u\|'_\Phi > 1$. Then, clearly, for every ε, $0 < \varepsilon < \|u\|'_\Phi$,

$$\left\| \Phi\left(\frac{|u|}{k}\right) \right\|_X > 1 \quad \text{if} \quad k = \|u\|'_\Phi - \varepsilon. \tag{6.3}$$

By using (P2), (6.1) and (2.1) with $\alpha = \frac{1}{\|u\|'_\Phi - \varepsilon}$, we find as above that

$$\rho_X(u;\Phi) \geqq \big(\|u\|'_\Phi - \varepsilon \big) \left\| \Phi\left(\frac{|u|}{\|u\|' - \varepsilon}\right) \right\|_X \geqq \|u\|'_\Phi - \varepsilon.$$

The proof of (6.2) follows if we let $\varepsilon \to 0$. □

We are now ready to prove the following generalized version of Hölder's inequality:

THEOREM 6.2. *Let $u \in X_\Phi$ and $v \in X_\Psi$, where Φ and Ψ are Young functions complementary to each other. Thenu $\cdot v \in X$ and $\|u \cdot v\|_X \leqq \|u\|_\Phi \|v\|'_\Psi$.*

PROOF. First we note that Proposition 6.1 guarantees that $\rho_X(w;\Psi) \leqq 1$ if and only if $\|w\|'_\Psi \leqq 1$. Therefore $\|u\|_\Phi = \sup_w \|u \cdot w\|_X$, where the supremum now is taken over all functions w satisfying $\|w\|'_\Psi \leqq 1$. In particular, if $w = \frac{v}{\|v\|'_\Psi}$, $\|v\|'_\Psi \neq 0$, then $\|w\|'_\Phi < 1$ and thus

$$\|u\|_\Phi \geqq \left\| u \frac{v}{\|v\|'_\Psi} \right\|_\Phi = \frac{1}{\|v\|'_\Psi} \|u \cdot v\|$$

and the statement follows. □

We now summarize some generalized forms of classical inequalities we have obtained (as usual Φ and Ψ denote Young functions complementary to each other):

(H) *Hölder's inequality* (see Theorem 6.2):

$$\|u \cdot v\|_X \leqq \|u\|_\Phi \|v\|'_\Psi.$$

Another Hölder type inequality is the following: If $\rho_X(v,\Psi) \geqq 1$, then

$$\|u \cdot v\|_X \leqq \|u\|_\Phi \, \rho_X(v,\Psi).$$

This inequality follows directly from the definition (3.1) because (2.1) yields that $\rho_X\left(\frac{v}{\rho_X(v,\Psi)}\right) \leqq 1$. (for the case $\rho_X(v,\Psi) \leqq 1$, we only have that $\|u \cdot v\|_X \leqq \|u\|_\Phi$.)

(M) *Minkowski's inequality* (two forms, see (4.3) and (5.3)):

$$\|u+v\|_\Phi \leqq \|u\|_\Phi + \|v\|_\Phi, \quad \|u+v\|'_\Phi \leqq \|u\|'_\Phi + \|v\|'_\Phi.$$

(Y) *Young's inequality* (see (4.2)):

$$\|u \cdot v\|_X \leqq \rho_X(v,\Phi) + \rho_X(v,\Psi).$$

REMARK 6.3. For the case when $\Phi(u) = |u|^p$, $p \geqq 1$ the following special cases of (H), (M) and (Y) in X^p-spaces hold, respectively:

$$\|u \cdot v\|_X \leqq \|u\|_{X^p} \|v\|_{X^q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

$$\|u+v\|_{X^p} \leqq \|u\|_{X^p} + \|v\|_{X^p}, \qquad p > 1$$

and

$$\|u \cdot v\|_X \leqq \frac{\|u\|^p_{X^p}}{p} + \frac{\|v\|^q_{X^q}}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1.$$

REMARK 6.4. We have also found a number of other inequalities during this presentation, e.g., the important estimate (5.2) in Theorem 5.1 and (6.1)–(6.2) in Proposition 6.2.

REMARK 6.5. In the classical case $X = L^1$, it is well-known that by using (H), (M) and (Y) it is possible to derive a number of other inequalities. In a similar way our versions of (H), (M) and (Y) together with Remarks 6.3–6.4 can be a good starting point to derive a theory for inequalities for the generalized Orlicz classes and spaces discussed in this paper.

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