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HUA'S THEOREM WITH NINE ALMOST EQUAL PRIME VARIABLES

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Abstract. We sharpen Hua's result by proving that each sufficiently large odd integer N can be written as

$$N = p_1^3 + \dots + p_9^3 \quad \text{with} \quad \left| p_j - \sqrt[3]{N/9} \right| \leqq U = N^{\frac{1}{3} - \frac{1}{198} + \varepsilon},$$

where p_j are primes. This result is as good as what was previously derived from the Generalized Riemann Hypothesis.

1. Introduction

In the additive theory of prime numbers, one studies the representation of positive integers by powers of primes. Hua [6] proved the following two classical theorems in this realm.

THEOREM A. *Each sufficiently large integer congruent to* 5 *modulo* 24 *can be written as the sum of five squares of primes.*

THEOREM B. *Each sufficiently large odd integer can be written as the sum of nine cubes of primes.*

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Since then, problems in which the prime variables are restricted in various ways have provided a number of extensions of Hua's theorems (see [13] and [24] for example). Obviously these results give deep insights into Hua's results.

Under the Generalized Riemann Hypothesis (GRH), Liu and Zhan [13] sharpened Theorem A by showing that each sufficiently large integer N congruent to 5 modulo 24 can be written as

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \qquad \left| p_j - \sqrt{\frac{N}{5}} \right| \leqq N^{\frac{1}{2} - \Delta + \varepsilon} \tag{1.1}$$

with $\Delta = \frac{1}{20}$. Later Bauer [1] used the Deuring–Heilbronn phenomenon to deal with the rather large major arcs in the circle method, and showed unconditionally that (1.1) holds with a small constant $\Delta > 0$, which is not determined numerically. An approach to treat the large major arcs without the application of the Deuring–Heilbronn phenomenon was introduced in [14], and the method is cultivated in a series of papers. In this direction, the following results have been achieved:

$$\Delta = \begin{cases} \frac{1}{50}, \frac{1}{46} & \text{in Liu and Zhan [14], [15] respectively;} \\ \frac{19}{850} & \text{in Bauer [2];} \\ \frac{1}{35} & \text{in Lü [16];} \\ \frac{9}{280} & \text{in Bauer and Wang [3];} \\ \frac{1}{28} & \text{in Lü [17];} \\ \frac{1}{20} & \text{in Liu, Lü and Zhan [11].} \end{cases}$$

As a generalization of Theorem B, Meng [19] first proved that under GRH each sufficiently large odd integer can be written as the sum of nine almost equal prime cubes, i.e.,

$$N = p_1^3 + \cdots + p_9^3, \qquad \left| p_j - \sqrt[3]{\frac{N}{9}} \right| \leqq U, \tag{1.2}$$

where $U = N^{\frac{1}{3} - \frac{1}{198} + \varepsilon}$. Later Meng [20] improved this result by showing that (1.2) holds unconditionally for $U = N^{\frac{1}{3} - \frac{1}{495} + \varepsilon}$. Recently Lü [18] further showed that (1.2) holds unconditionally for $U = N^{\frac{1}{3} - \frac{2}{555} + \varepsilon}$.

In this paper, we modify the iterative idea of Liu [9] to treat the equation (1.2) in conjunction with the new result on nonlinear exponential sums over prime variables in Ren [22], and are able to prove unconditionally Meng's previous result under GRH.

THEOREM 1. *Each sufficiently large odd integer N can be written as* (1.2) *with*

$$U = N^{\frac{1}{3}-\frac{1}{198}+\varepsilon}. \tag{1.3}$$

2. Outline of the method

In this section we give an outline of the proof of Theorem 1. In order to apply the circle method, for U as in (1.3) we set

$$\begin{cases} P = N^{64/555}, \qquad P_0 = N^{32/3+48\varepsilon}U^{-32}, \\ Q_0 = U^{37}N^{-34/3-37\varepsilon} = N^{161/198}, \qquad Q = N^{13/15+\varepsilon}. \end{cases} \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1+1/Q)$ may be written in the form

$$\alpha = a/q + \lambda, \qquad |\lambda| \leqq 1/(qQ) \tag{2.2}$$

for some integers a, q with $1 \leqq a \leqq q \leqq Q$ and $(a,q)=1$. Denote by $\mathcal{M}(a,q)$ the set of α satisfying (2.2), and define the major arcs $\mathcal{M}$ as follows

$$\mathcal{M} = \bigcup_{q \leqq P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}(a,q). \tag{2.3}$$

Also by Dirichlet's lemma, each real number $\alpha \in [1/Q, 1+1/Q) \setminus \mathcal{M}$ can be written as $\alpha = a/q + \lambda$, $(a,q)=1$ with $1 \leqq a \leqq q \leqq Q_0$, $|\lambda| \leqq 1/(qQ_0)$.

Define the minor arcs $C(\mathcal{M})$ to be the set of $\alpha \in [1/Q, 1+1/Q) \setminus \mathcal{M}$ satisfying $\alpha = a/q + \lambda$, $(a,q)=1$ such that

$$P_0 \leqq q \leqq Q_0, \qquad |\lambda| \leqq 1/(qQ_0). \tag{2.4}$$

Let the intermediate arcs $\mathcal{R}$ be the complement of $\mathcal{M}$ and $C(\mathcal{M})$ in $[1/Q, 1+1/Q)$. Thus

$$[1/Q, 1+1/Q) = \mathcal{M} \bigcup C(\mathcal{M}) \bigcup \mathcal{R}.$$

For $\alpha \in \mathcal{R}$, we have either

$$q \leqq P, \qquad 1/(qQ) < |\lambda| \leqq 1/(qQ_0) \tag{2.5}$$

or

$$P < q \leqq P_0, \qquad |\lambda| \leqq 1/(qQ_0). \tag{2.6}$$

Denote the sets satisfying (2.5) and (2.6) by $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively.

Let N be a sufficiently large odd integer. Let

$$r(N) = \sum_{\substack{N=p_1^3+\cdots+p_9^3 \\ |p_j - \sqrt[3]{N/9}\,| \leqq U}} (\log p_1)\cdots(\log p_9),$$

where $U = N^{\frac{1}{3}-\frac{1}{198}+\varepsilon}$. For

$$N_1 = \sqrt[3]{N/9} - U, \qquad N_2 = \sqrt[3]{N/9} + U, \tag{2.7}$$

define

$$S(\alpha) = \sum_{N_1 < p \leqq N_2} (\log p) e(p^3\alpha). \tag{2.8}$$

Then we have

$$r(N) = \int_0^1 S^9(\alpha) e(-N\alpha)\, d\alpha = \int_{\mathcal{M}} + \int_{\mathcal{R}} + \int_{C(\mathcal{M})}. \tag{2.9}$$

To estimate the contribution from the minor arcs, we quote the following result:

LEMMA 2.1 (Meng [20]). *Suppose*

$$\max_{\alpha \in C(\mathcal{M})} \bigl|S(\alpha)\bigr| \ll U^{1+\varepsilon}\left(P_0^{-1/16} + \frac{N^{1/96}}{U^{1/16}} + \frac{N^{1/15}}{U^{1/4}} + \frac{Q_0^{1/16} N^{1/24}}{U^{5/16}}\right). \tag{2.10}$$

Then we have for any $A > 0$,

$$\int_{C(\mathcal{M})} \ll \Bigl\{\max_{\alpha \in C(\mathcal{M})} \bigl|S(\alpha)\bigr|\Bigr\} \int_0^1 \bigl|S(\alpha)\bigr|^8 d\alpha \tag{2.11}$$

$$\ll U^3 N^{-2/3-\varepsilon} \int_0^1 \bigl|S(\alpha)\bigr|^8 d\alpha \ll U^8 N^{-2/3} L^{-A}.$$

We can easily estimate the integral on $\mathcal{R}$, if we quote the new estimates of the exponential sums over primes from Ren [22].

LEMMA 2.2. *Let $\alpha = a/q + \lambda$ subject to $(a, q) = 1$. Then*

$$T_3(\alpha) = \sum_{p \leqq N} (\log p) e(p^3\alpha)$$

$$\ll N^{\varepsilon} \left\{ N^{\frac{1}{6}} \sqrt{q\big(1 + N|\lambda|\big)} + N^{\frac{4}{15}} + \frac{N^{\frac{1}{3}}}{\sqrt{q\big(1 + N|\lambda|\big)}} \right\}.$$

On recalling the definition of $\mathcal{R}_1$, it is easy to see that

$$N^{\frac{2}{15} - \varepsilon} < q\big(1 + N|\lambda|\big) \leqq N^{\frac{37}{198}} \quad \text{for} \quad \alpha \in \mathcal{R}_1.$$

We also have

$$N^{\frac{64}{555}} < q\big(1 + N|\lambda|\big) \leqq N^{\frac{37}{198}} \quad \text{for} \quad \alpha \in \mathcal{R}_2.$$

Thus from Lemma 2.2 we have $\max\limits_{\alpha \in \mathcal{R}} \big| S(\alpha) \big| \ll U^3 N^{-2/3 - 2\varepsilon}$. Therefore we obtain

$$\int_{\mathcal{R}} S^9(\alpha) e(-N\alpha)\, d\alpha \ll U^8 N^{-2/3} L^{-A}. \tag{2.12}$$

Now in order to prove Theorem 1, it suffices to prove the following theorem.

THEOREM 2. *Let $\mathcal{M}$ be as above with P, Q determined by (2.1). Then we have for any $A > 0$,*

$$\int_{\mathcal{M}} S^9(\alpha) e(-N\alpha)\, d\alpha = \frac{1}{3^9} M_0 \sum_{q \leqq P} A(N, q) + O\big(U^8 N^{-2/3} L^{-A}\big), \tag{2.13}$$

where

$$U^8 N^{-2/3} \ll M_0 = \sum_{\substack{m_1 + \cdots + m_9 = N \\ N_1^3 < m_j \leqq N_2^3}} (m_1 m_2 \cdots m_9)^{-\frac{2}{3}} \ll U^8 N^{-2/3} \tag{2.14}$$

and the value of the singular series $\sum_{q \leqq P} A(N, q)$ is larger than a certain positive constant c.

3. Preliminaries for Theorem 2

For χ mod q, define

$$C(\chi,a)=\sum_{h=1}^{q}\bar{\chi}(h)e\left(\frac{ah^3}{q}\right),\qquad C(q,a)=C(\chi^0,a). \tag{3.1}$$

If $\chi_1,\chi_2,\dots,\chi_9$ are characters mod q, then write

$$B(N,q,\chi_1,\dots,\chi_9)=\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aN}{q}\right)C(\chi_1,a)C(\chi_2,a)\cdots C(\chi_9,a), \tag{3.2}$$

and

$$B(N,q)=B(N,q,\chi^0,\dots,\chi^0),\qquad A(N,q)=\frac{B(N,q)}{\varphi^9(q)}. \tag{3.3}$$

The following lemma is important for proving Theorem 2.

LEMMA 3.1. *Let* χ_j mod r_j *with* $j=1,\dots,9$ *be primitive characters,* $r_0=[r_1,\dots,r_9]$*, and* χ^0 *the principal character* mod q*. Then*

$$\sum_{\substack{q\le x\\ r_0\mid q}}\frac{1}{\varphi^9(q)}\left|B(n,q,\chi_1\chi^0,\chi_2\chi^0,\dots,\chi_9\chi^0)\right|\ll r_0^{-7/2+\varepsilon}\log^c x.$$

PROOF. It is similar to that of Lemma 7 in [8], so we omit the details.

Recall N_1, N_2 as in (2.7), and define

$$V(\lambda)=\sum_{N_1<m\le N_2}e(m^3\lambda), \tag{3.4}$$

$$W(\chi,\lambda)=\sum_{N_1<p\le N_2}(\log p)\chi(p)e(p^3\lambda)-\delta_\chi\sum_{N_1<m\le N_2}e(m^3\lambda),$$

where $\delta_\chi=1$ or 0 according as χ is principal or not. Define further

$$J(g)=\sum_{r\le P}[g,r]^{-7/2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\max_{|\lambda|\le 1/(rQ_0)}\left|W(\chi,\lambda)\right|, \tag{3.5}$$

and

$$K(g)=\sum_{r\le P}[g,r]^{-7/2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\left(\int_{-1/(rQ)}^{1/(rQ)}\left|W(\chi,\lambda)\right|^2 d\lambda\right)^{1/2}, \tag{3.6}$$

where the sum $\sum_{\chi \bmod r}{}^{*}$ denotes summation for all primitive characters mod r. Our Theorem 2 depends on the following three lemmas, which will be proved in Sections 5 and 6.

LEMMA 3.2. *For P, Q satisfying* (2.1), *we have*

$$J(g) \ll g^{-7/2+\varepsilon} U L^c. \tag{3.7}$$

LEMMA 3.3. *Let P, Q be as in* (2.1). *For $g = 1$, Lemma 3.2 can be improved to*

$$J(1) \ll U L^{-A}, \tag{3.8}$$

where $A > 0$ is arbitrary.

LEMMA 3.4. *For P, Q as in* (2.1), *we have*

$$K(g) \ll g^{-7/2+\varepsilon} U^{1/2} N^{-1/3} L^c. \tag{3.9}$$

4. Proof of Theorem 2

With Lemmas 3.2.–3.4 known, we can use the iterative idea to prove Theorem 2.

PROOF OF THEOREM 2. For $q \leqq P$ and $N_1 < p \leqq N_2$, we have $(q,p) = 1$. Therefore we can rewrite the exponential sum $S(\alpha)$ as

$$S\left(\frac{a}{q} + \lambda\right) = \frac{C(q,a)}{\varphi(q)} V(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C(\chi, a) W(\chi, \lambda),$$

where $V(\lambda)$ and $W(\chi,\lambda)$ are as in (3.4). Thus,

$$\int_{\mathcal{M}} S^9(\alpha) e(-N\alpha)\, d\alpha = \sum_{j=0}^{9} C_9^j I_j, \tag{4.1}$$

where

$$I_j = \sum_{q \leqq P} \frac{1}{\varphi^9(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^{9-j}(q,a) e\left(-\frac{aN}{q}\right) \int_{-1/(qQ)}^{1/(qQ)} V^{9-j}(\lambda)$$
$$\times \left\{ \sum_{\chi \bmod q} C(\chi, a) W(\chi, \lambda) \right\}^j e(-N\lambda)\, d\lambda.$$

We will prove that I_0 gives the main term, and $I_1, I_2, \ldots, I_9$ the error terms.

The computation of I_0 is standard, and therefore we give the result directly

$$I_0 = \frac{1}{3^9} M_0 \sum_{q \leqq P} \frac{B(N,q)}{\varphi^9(q)} + O(U^8 N^{-2/3} L^{-A}). \tag{4.2}$$

A similar computation can be found in [10].

To bound the contributions of the other terms, we begin with I_9, the most complicated one. Reducing the characters in I_9 into primitive characters, we have by Lemma 3.1

$$\begin{aligned}
|I_9| \leqq {} & \sum_{r_1 \leqq P} \cdots \sum_{r_9 \leqq P} \ \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_9 \bmod r_9} \int_{-1/(r_0 Q)}^{1/(r_0 Q)} \left|W(\chi_1,\lambda)\right| \cdots \left|W(\chi_9,\lambda)\right| d\lambda \\
& \times \sum_{\substack{q \leqq P \\ r_0 \mid q}} \frac{\left|B(N,q,\chi_1\chi^0,\ldots,\chi_9\chi^0)\right|}{\varphi^9(q)} \\
& \ll L^c \sum_{r_1 \leqq P} \cdots \sum_{r_9 \leqq P} r_0^{-7/2+\varepsilon} \ \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_9 \bmod r_9} \\
& \times \int_{-1/(r_0 Q)}^{1/(r_0 Q)} \left|W(\chi_1,\lambda)\right| \cdots \left|W(\chi_9,\lambda)\right| d\lambda.
\end{aligned}$$

In the last integral, we take it for χ_i with $i = 1,\ldots,7$, and then use Cauchy's inequality, to get

$$\begin{aligned}
|I_9| \ll L^c & \sum_{r_1 \leqq P} \ \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda| \leqq 1/(r_1 Q)} \left|W(\chi_1,\lambda)\right| \cdots \\
& \times \sum_{r_7 \leqq P} \ \sideset{}{^*}\sum_{\chi_3 \bmod r_7} \max_{|\lambda| \leqq 1/(r_7 Q)} \left|W(\chi_7,\lambda)\right| \\
& \times \sum_{r_8 \leqq P} \ \sideset{}{^*}\sum_{\chi_8 \bmod r_8} \left(\int_{-1/(r_8 Q)}^{1/(r_8 Q)} \left|W(\chi_8,\lambda)\right|^2 d\lambda \right)^{1/2} \\
& \times \sum_{r_9 \leqq P} r_0^{-7/2+\varepsilon} \ \sideset{}{^*}\sum_{\chi_9 \bmod r_9} \left(\int_{-1/(r_9 Q)}^{1/(r_9 Q)} \left|W(\chi_9,\lambda)\right|^2 d\lambda \right)^{1/2}.
\end{aligned} \tag{4.3}$$

Now we introduce an iterative procedure to bound the above sums over $r_9,\ldots,r_1$ consecutively. We first estimate the above sum over r_9 in (4.3) via Lemma 3.4. Since $r_0 = [r_1, \dots, r_9] = \big[[r_1, \dots, r_8], r_9\big]$, the sum over r_9 in (4.3) is

$$= \sum_{r_9 \leqq P} \big[[r_1, \dots, r_8], r_9\big]^{-7/2+\varepsilon} \sideset{}{^*}\sum_{\chi_9 \bmod r_9} \left(\int_{-1/(r_9Q)}^{1/(r_9Q)} \big|W(\chi_9, \lambda)\big|^2 \, d\lambda \right)^{1/2}$$

$$= K\big([r_1, \dots, r_8]\big) \ll [r_1, \dots, r_8]^{-7/2+\varepsilon} U^{1/2} N^{-1/3} L^c.$$

This contributes to the sum over r_8 of (4.3) in the amount of

$$\ll U^{1/2} N^{-1/3} L^c \sum_{r_8 \leqq P} [r_1, \dots, r_8]^{-7/2+\varepsilon}$$

$$\times \sideset{}{^*}\sum_{\chi_8 \bmod r_8} \left(\int_{-1/(r_8Q)}^{1/(r_8Q)} \big|W(\chi_8, \lambda)\big|^2 \, d\lambda \right)^{1/2}$$

$$= U^{1/2} N^{-1/3} L^c K\big([r_1, \dots, r_7]\big) \ll [r_1, \dots, r_7]^{-7/2+\varepsilon} U N^{-2/3} L^c,$$

where we have used Lemma 3.4 again.

Inserting this last bound into (4.3), we can bound the sum over r_7 as

$$\ll U N^{-2/3} L^c \sum_{r_7 \leqq P} [r_1, \dots, r_7]^{-7/2+\varepsilon} \sideset{}{^*}\sum_{\chi_7 \bmod r_7} \max_{|\lambda| \leqq 1/(r_7Q)} \big|W(\chi_7, \lambda)\big|$$

$$\ll U N^{-2/3} L^c J\big([r_1, \dots, r_6]\big) \ll U^2 N^{-2/3} L^c [r_1, \dots, r_6]^{-7/2+\varepsilon}.$$

Similarly we can use Lemma 3.3 to bound the sums over $r_6, \dots, r_2$ and Lemma 3.2 to treat the sum over r_1 consecutively and find that

$$|I_9| \ll U^6 N^{-2/3} L^c \sum_{r_1 \leqq P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda| \leqq 1/(r_1Q)} \big|W(\chi_1, \lambda)\big| J(r_1) \tag{4.4}$$

$$\ll U^7 N^{-2/3} L^c J(1) \ll U^8 N^{-2/3} L^{-A}.$$

The other terms of $I_8, \dots, I_1$ can be estimated similarly in terms of K and J in Lemmas 3.2, 3.3, and 3.4. The only difference in bounding these terms is that we need two elementary estimates

$$\max_{|\lambda| \leqq 1/Q} \big|V(\lambda)\big| \ll U \quad \text{and} \quad \left(\int_{-1/Q}^{1/Q} \big|V(\lambda)\big|^2 \, d\lambda \right)^{1/2} \ll U^{1/2} N^{-1/3}.$$

In fact the first estimate is trivial and the second estimate can be easily obtained by partial summation and an elementary estimate for exponential sums. □

5. Estimation of $K(g)$

Let $Y \leqq X$ and $M_1, \ldots, M_{10}$ be positive integers such that

$$2^{-10}Y \leqq M_1 \cdots M_{10} < X \qquad \text{and} \qquad 2M_6, \ldots, 2M_{10} \leqq X^{1/5}. \tag{5.1}$$

For $j = 1, \ldots, 10$ define

$$a_j(m) = \begin{cases} \log m, & \text{if } j = 1, \\ 1, & \text{if } j = 2, \ldots, 5, \\ \mu(m), & \text{if } j = 6, \ldots, 10, \end{cases} \tag{5.2}$$

where $\mu(n)$ is the Möbius function. Then define the functions

$$f_j(s, \chi) = \sum_{m \sim M_j} \frac{a_j(m)\chi(m)}{m^s}$$

and

$$F(s, \chi) = f_1(s, \chi) \cdots f_{10}(s, \chi), \tag{5.3}$$

where χ is a Dirichlet character, and s is a complex variable.

The following hybrid estimate for $|F|$ is one of the key ingredients in carrying out the iterative procedure.

LEMMA 5.1. *Let $F(s, \chi)$ be as in* (5.3). *Then for any $1 \leqq R \leqq X^2$ and $T > 0$,*

$$\sum_{\substack{r \sim R \\ d \mid r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_T^{2T} \left| F\left(\frac{1}{2} + it, \chi\right) \right| dt \tag{5.4}$$

$$\ll \left\{ \frac{R^2}{d} T + \frac{R}{d^{1/2}} T^{1/2} X^{3/10} + X^{1/2} \right\} \log^c X.$$

PROOF. See Liu [9] for details.

PROOF OF LEMMA 3.4. Let

$$\hat{W}(\chi, \lambda) = \sum_{N_1 < m \leqq N_2} \left(\Lambda(m)\chi(m) - \delta_\chi \right) e(m^3 \lambda).$$

Then

$$W(\chi, \lambda) - \hat{W}(\chi, \lambda) \ll N^{1/6}. \tag{5.5}$$

Note that

$$\Big(\int_{-1/(rQ)}^{1/(rQ)}\big|W(\chi,\lambda)\big|^2\,d\lambda\Big)^{\frac12}\ll\Big(\int_{-1/(rQ)}^{1/(rQ)}\big(\big|\hat W(\chi,\lambda)\big|^2+|W-\hat W|^2\big)\,d\lambda\Big)^{\frac12}$$

$$\ll\Big(\int_{-1/(rQ)}^{1/(rQ)}\big|\hat W(\chi,\lambda)\big|^2\,d\lambda\Big)^{\frac12}+\Big(\int_{-1/(rQ)}^{1/(rQ)}\big|W-\hat W\big|^2\,d\lambda\Big)^{\frac12}.$$

Thus (5.5) contributes to (3.6) in the amount of

$$\ll N^{1/6}\sum_{r\leqq P}[g,r]^{-7/2+\varepsilon}\frac{r^{1/2}}{Q^{1/2}}\ll g^{-7/2+\varepsilon}N^{1/6}Q^{-1/2}\sum_{r\leqq P}\Big(\frac{r}{(g,r)}\Big)^{-3/4+\varepsilon}r^{1/2}$$

$$\ll g^{-7/2+\varepsilon}N^{1/6}Q^{-1/2}\sum_{\substack{d|g\\ d\leqq P}}d^{3/4-\varepsilon}\sum_{\substack{r\leqq P\\ d|r}}r^{-1/4+\varepsilon}\ll g^{-7/2+\varepsilon}N^{1/6}P^{3/4+\varepsilon}Q^{-1/2}$$

$$\ll g^{-7/2+\varepsilon}U^{1/2}N^{-1/3}L^c,$$

where we have used $[g,r](g,r)=gr$, (2.1), and (1.3).

Hence to establish Lemma 3.4, it suffices to show that

(5.6)
$$\sum_{r\sim R}[g,r]^{-7/2+\varepsilon}\sum_{\chi \bmod r}{}^{*}\Big(\int_{-1/(rQ)}^{1/(rQ)}\big|\hat W(\chi,\lambda)\big|^2\,d\lambda\Big)^{1/2}\ll g^{-7/2+\varepsilon}U^{1/2}N^{-1/3}L^c$$

holds for $R\leqq P$. By Gallagher's lemma (see [5], Lemma 1), we have

(5.7)
$$\int_{-1/(rQ)}^{1/(rQ)}\big|\hat W(\chi,\lambda)\big|^2\,d\lambda\ll\Big(\frac{1}{RQ}\Big)^2\int_{-\infty}^{\infty}\Big|\sum_{\substack{v<m^3\leqq v+rQ\\ N_1^3<m^3\leqq N_2^3}}\big(\Lambda(m)\chi(m)-\delta_\chi\big)\Big|^2\,dv$$

$$\ll\Big(\frac{1}{RQ}\Big)^2\int_{N_1^3-rQ}^{N_2^3}\Big|\sum_{\substack{v<m^3\leqq v+rQ\\ N_1^3<m^3\leqq N_2^3}}\big(\Lambda(m)\chi(m)-\delta_\chi\big)\Big|^2\,dv$$

$$\ll\Big(\frac{1}{RQ}\Big)^2\int_{N_1^3-rQ}^{N_2^3}\Big|\sum_{Y<m\leqq X}\big(\Lambda(m)\chi(m)-\delta_\chi\big)\Big|^2\,dv,$$

where

$$Y = \max(v^{1/3}, N_1), \qquad X = \min\big((v+rQ)^{1/3}, N_2\big).$$

We argue exactly as in the proof of Lemma 5.1 in [9] and see that the inner sum in (5.7) is a linear combination of $O(L^{10})$ terms, each of which has the form

$$\Sigma(u;\mathbf{M}) := \frac{1}{2\pi}\int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right)\frac{X^{1/2+it}-Y^{1/2+it}}{1/2+it}\,dt + O\left(\frac{N^{1/3}L^2}{T}\right),$$

where T is a parameter satisfying $2 \leqq T \leqq N^{1/3}$. One easily sees that

$$\frac{X^{1/2+it}-Y^{1/2+it}}{1/2+it} = \frac{1}{2}\int_{Y^3}^{X^3} u^{-5/6+it/3}\,du = \frac{1}{3}\int_{Y^3}^{X^3} u^{-5/6} e\left(\frac{t}{6\pi}\log u\right)du.$$

The integral can be easily estimated as

$$\ll X^{1/2}-Y^{1/2} \ll (v+rQ)^{1/6}-v^{1/6}$$

$$\ll v^{1/6}\big\{(1+rQ/v)^{1/6}-1\big\} \ll N^{-5/6}RQ.$$

On the other hand, one has trivially

$$\frac{X^{1/2+it}-Y^{1/2+it}}{1/2+it} \ll \frac{X^{1/2}}{|t|} \ll \frac{N_2^{1/2}}{|t|} \ll \frac{N^{1/6}}{|t|}.$$

Collecting the two upper bounds, we get

$$\frac{X^{1/2+it}-Y^{1/2+it}}{1/2+it} \ll \min\left(\frac{RQ}{N^{5/6}}, \frac{N^{1/6}}{|t|}\right).$$

Taking $T = N^{1/3}$, $T_0 = 12\pi N/(QR)$, we see that

$$\Sigma(u;\mathbf{M}) \ll \frac{RQ}{N^{5/6}}\int_{|t|\leqq T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt$$

$$+ N^{1/6}\int_{T_0<|t|\leqq T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|} + O(L^2).$$

Consequently (5.7) becomes

$$\int_{-1/(rQ)}^{1/(rQ)} \big|\hat{W}(\chi,\lambda)\big|^2\,d\lambda \ll UN^{-1}L^{20}\max_{\mathbf{M}}\left(\int_{|t|\leqq T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^2$$

$$+\frac{NUL^{20}}{(QR)^2}\max_{\mathbf{M}}\left(\int_{T_0<|t|\leqq T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|}\right)^2+\frac{N^{2/3}UL^{24}}{(QR)^2},$$

where we have used $N_2^3-N_1^3\ll N^{2/3}U$.

The last term above contributes to the left-hand side of (5.6) in the amount of

$$\ll\sum_{r\sim R}r^{-7/2+\varepsilon}\sum_{\chi\bmod r}\frac{\left(N^{2/3}U\right)^{1/2}L^{12}}{RQ}$$
$$\ll g^{-7/2+\varepsilon}\frac{N^{1/3}U^{1/2}L^{12}}{Q}\sum_{r\sim R}\left(\frac{r}{(g,r)}\right)^{-7/2+\varepsilon}$$
$$\ll g^{-7/2+\varepsilon}PU^{1/2}N^{1/3}Q^{-1}L^{12}\ll g^{-7/2+\varepsilon}U^{1/2}N^{-1/3}L^c,$$

and therefore the left-hand side of (5.6) is

$$\ll U^{1/2}N^{-1/2}L^{10}\max_{\mathbf{M}}\sum_{r\sim R}[g,r]^{-7/2+\varepsilon}\sideset{}{^*}\sum_{\chi\bmod r}\int_{|t|\leqq T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt$$
$$+\frac{N^{1/2}U^{1/2}L^{10}}{RQ}\max_{\mathbf{M}}\sum_{r\sim R}[g,r]^{-7/2+\varepsilon}\sideset{}{^*}\sum_{\chi\bmod r}\int_{T_0<|t|\leqq T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|}$$
$$+g^{-7/2+\varepsilon}U^{1/2}N^{-1/3}L^c.$$

Thus, to prove (5.6) it suffices to show that the estimate

$$\sum_{r\sim R}[g,r]^{-7/2+\varepsilon}\sideset{}{^*}\sum_{\chi\bmod r}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\ll g^{-7/2+\varepsilon}N^{1/6}L^c \tag{5.8}$$

holds for $R\leqq P$ and $0<T_1\leqq T_0$, and

$$\sum_{r\sim R}[g,r]^{-7/2+\varepsilon}\sideset{}{^*}\sum_{\chi\bmod r}\int_{T_2}^{2T_2}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\ll g^{-7/2+\varepsilon}(RQ)N^{-5/6}T_2L^c \tag{5.9}$$

holds for $R\leqq P$ and $T_0<T_2\leqq T$.

To get the estimate (5.8), we note that $[g,r](g,r)=gr$. Then the left-hand side of (5.8) is

$$\ll g^{-7/2+\varepsilon}\sum_{\substack{d|g\\ d\leqq R}}\left(\frac{R}{d}\right)^{-7/2+\varepsilon}\sum_{\substack{r\sim R\\ d|r}}\sideset{}{^*}\sum_{\chi\bmod r}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \tag{5.10}$$

$$\ll g^{-7/2+\varepsilon}\sum_{\substack{d|g\\ d\leqq R}}\left(\frac{R}{d}\right)^{-1+\varepsilon}\sum_{\substack{r\sim R\\ d|r}}\ \sideset{}{^*}\sum_{\chi \bmod r}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt.$$

By Lemma 5.1, the above quantity can be estimated as

$$\ll g^{7/2+\varepsilon}\sum_{\substack{d|g\\ d\leqq R}}\left(\frac{R}{d}\right)^{-1+\varepsilon}\left(\frac{R^2}{d}T_1+\frac{R}{d^{1/2}}T_1^{1/2}N^{1/10}+N^{1/6}\right)L^c$$

$$\ll g^{-7/2+\varepsilon}\tau(g)\left\{R^{1+\varepsilon}T_1+R^{1/2+\varepsilon}T_1^{1/2}N^{1/10}+N^{1/6}\right\}L^c\ll g^{-7/2+\varepsilon}N^{1/6}L^c,$$

provided that $R\leqq P=N^{\frac{64}{555}}$. This establishes (5.8). Similarly we can prove (5.9) by taking $T=T_2$ in Lemma 5.1. Lemma 3.4 now follows. □

6. Estimation of $J(g)$ and $J(1)$

Proof of Lemma 3.2. Recall that $W(\chi,\lambda)-\hat{W}(\chi,\lambda)\ll N^{1/6}$. This contributes to (3.5) in the amount of

$$\ll N^{1/6}\sum_{r\leqq P}[g,r]^{-7/2+\varepsilon}r\ll g^{-7/2+\varepsilon}N^{1/6}\sum_{r\leqq P}\left(\frac{r}{(g,r)}\right)^{-7/2+\varepsilon}r$$

$$\ll g^{-7/2+\varepsilon}N^{1/6}\sum_{r\leqq P}\left(\frac{r}{(g,r)}\right)^{-1+\varepsilon}r\ll g^{-7/2+\varepsilon}N^{1/6}\sum_{\substack{d|g\\ d\leqq P}}d^{1-\varepsilon}\sum_{\substack{r\leqq P\\ d|r}}r^{\varepsilon}$$

$$\ll g^{-7/2+\varepsilon}N^{1/6}P^{1+\varepsilon}\ll g^{-7/2+\varepsilon}UL^c,$$

where we have used $[g,r](g,r)=gr$ and (2.1). Thus Lemma 3.2 is a consequence of the estimate

$$\sum_{r\sim R}[g,r]^{-7/2+\varepsilon}\ \sideset{}{^*}\sum_{\chi \bmod r}\ \max_{|\lambda|\leqq 1/(rQ)}\left|\hat{W}(\chi,\lambda)\right|\ll g^{-7/2+\varepsilon}UL^c, \tag{6.1}$$

where $R\leqq P$ and $c>0$ is some constant.

It is easy to establish (6.1) for $r=1$. In fact for $r=1$ the left-hand side of (6.1) is

$$\ll g^{-7/2+\varepsilon}\sum_{N_1<m\leqq N_2}\log m\ll g^{-7/2+\varepsilon}UL,$$

which is obviously acceptable. It therefore remains to show (6.1) in the case $r > 1$.

In this case we have $\delta_\chi = 0$ for all χ mod r. Thus arguing similarly as in the previous section, we find that

$$\left|\hat{W}(\chi,\lambda)\right| \ll L^{10} \max_{\mathbf{M}} \left| \int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right) \right.$$
$$\left. \times \int_{N_1^3}^{N_2^3} v^{-5/6} e\left(\frac{t}{6\pi}\log v + \lambda v\right) dv\, dt \right| + UN^{-\varepsilon}P^{-2},$$

where the maximum is taken over all $\mathbf{M} = (M_1, M_2, \ldots, M_{10})$ and

$$T = N^{1/3+2\varepsilon}U^{-1}P^2\bigl(1+|\lambda|N\bigr). \tag{6.2}$$

Since

$$\frac{d}{dv}\left(\frac{t}{6\pi}\log v + \lambda v\right) = \frac{t}{6\pi v} + \lambda, \qquad \frac{d^2}{dv^2}\left(\frac{t}{6\pi}\log v + \lambda v\right) = -\frac{t}{6\pi v^2},$$

by Lemmas 4.4 and 4.3 in [23], the inner integral above can be estimated as

$$\ll N^{-5/6} \min\left\{ UN^{2/3}, \frac{N}{\bigl(|t|+1\bigr)^{1/2}}, \frac{N}{\min_{N_1^3 < v \leqq N_2^3} |t+6\pi\lambda v|} \right\} \tag{6.3}$$

Take

$$T_0 = N^{2/3}U^{-2} \qquad \hat{T}_0 = 12\pi N/(RQ). \tag{6.4}$$

Here the choice of $\hat{T}_0$ is to ensure that $|t + 6\pi\lambda v| > |t|/2$ whenever $|t| > \hat{T}_0$. Thus in order to prove Lemma 3.2 it is enough to show that for $R \leqq P$ and $0 < T_1 \leqq T_0$,

$$\sum_{r\sim R} [g,r]^{-7/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll g^{-7/2+\varepsilon}N^{1/6}L^c; \tag{6.5}$$

for $R \leqq P$ and $T_0 < T_2 \leqq \hat{T}_0$,

(6.6)
$$\sum_{r\sim R} [g,r]^{-7/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll g^{-7/2+\varepsilon}UN^{-1/6}T_2^{1/2}L^c,$$

while for $R \leqq P$ and $\hat{T}_0 < T_3 \leqq T$,

$$\sum_{r\sim R}[g,r]^{-7/2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_3}^{2T_3}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \ll g^{-7/2+\varepsilon}UN^{-1/6}T_3L^c. \tag{6.7}$$

Following the same procedure that used to prove (5.8) and (5.9), we can establish these estimates by taking $T = T_1$, T_2, T_3 in Lemma 5.1 respectively. Thus Lemma 3.2 follows.

Proof of Lemma 3.3. The proof of Lemma 3.3 is the same as that of Lemma 3.2 except for L^{-A} on the right hand side. In order to save this factor, we have to distinguish two cases $L^B < R \leqq P$ and $R \leqq L^B$ where B is a constant depending on A. The proof of the first case is the same as that of Lemma 3.2. Here for a certain sufficiently large B, $L^B < R \leqq P$ guarantees that the term $g^{-7/2+\varepsilon}UL^c$ can be replaced by $g^{-7/2+\varepsilon}UL^{-A}$. So we omit the details.

Now we prove the second case $R \leqq L^B$. We use the well-known explicit formula

$$\sum_{m\leqq u}\Lambda(m)\chi(m) = \delta_\chi u - \sum_{|\gamma|\leqq T}\frac{u^\rho}{\rho} + O\left\{\left(\frac{u}{T}+1\right)\log^2(ruT)\right\} \tag{6.8}$$

where $\rho = \beta + i\gamma$ is a non-trivial zero of the function $L(s,\chi)$, and $2 \leqq T \leqq u$ is a parameter. Taking $T = N^{69/500}$ in (6.8), and then inserting it into $\hat{W}(\chi,\lambda)$, we get

$$\hat{W}(\chi,\lambda) = \int_{N_1}^{N_2} e(u^3\lambda)d\Big\{\sum_{n\leqq u}\big(\Lambda(m)\chi(m)-\delta_\chi\big)\Big\}$$

$$= \int_{N_1}^{N_2} e(u^3\lambda)\sum_{|\gamma|\leqq N^{69/500}} u^{\rho-1}\,du + O\big(N^{293/1500}\big(1+|\lambda|N^{2/3}U\big)\,L^2\big)$$

$$\ll U\sum_{|\gamma|\leqq N^{69/500}} N^{(\beta-1)/3} + O\big(N^{431/500}UQ^{-1}L^2\big)$$

$$\ll U\sum_{|\gamma|\leqq N^{69/500}} N^{(\beta-1)/3} + O(UN^{-\varepsilon}),$$

where we have used (2.1).

Now let $\eta(T) = c_2 \log^{-4/5} T$. By Prachar [21], $\prod_{\chi \bmod r} L(s,\chi)$ is zerofree in the region $\sigma \geqq 1 - \eta(T)$, $|t| \leqq T$ except for the possible Siegel zero. But by Siegel's theorem (see [4], Section 21), the Siegel zero does not exist in the present situation, since $r \sim R \leqq L^B$. Thus by the large-sieve type zero-density estimates for Dirichlet L-functions (see [7]),

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma| \leqq N^{69/500}} N^{(\beta-1)/3}$$

$$\ll L^c \int_0^{1-\eta(N^{69/500})} \left(N^{69/500}\right)^{12(1-\alpha)/5} N^{(\alpha-1)/3}\, d\alpha$$

$$\ll L^c N^{-0.002\eta(N^{69/500})} \ll \exp\left(-c_3 L^{1/5}\right).$$

Consequently

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda| \leqq 1/(rQ)} \left|\hat{W}(\chi,\lambda)\right| \ll UL^{-A},$$

where $A > 0$ is arbitrary. This proves Lemma 3.3 in the second case.

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