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NORMAL GENERALIZED TOPOLOGIES

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Abstract. The concept of normality is defined for generalized topologies in the sense of [1], a few properties of normal spaces are proved, and their characterization with the help of a suitable form of Urysohn's lemma is discussed.

1. Introduction

According to [1], a *generalized topology* (briefly GT) on a set X is a subset μ of the power set $\exp X$ such that $\emptyset \in \mu$ and the union of the elements of an arbitrary subset of μ belongs to μ. The elements of μ are said to be μ-open, their complements μ-closed. Clearly each topology is a GT.

As a generalization of the concept of normal topology, it is natural to say that a GT μ is *normal* iff, whenever F and F' are μ-closed sets such that $F \cap F' = \emptyset$, there exist μ-open sets G and G' satisfying $F \subset G$, $F' \subset G'$ and $G \cap G' = \emptyset$. In the literature, there are many papers discussing properties of normal GT's for some particular GT μ (cf. e.g. [2], [3], [5], [6], [7], [8]).

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In the present paper, our purpose is to indicate some general properties of a normal GT and, in particular, to give a characterization of them with the help of a form of the well-known Urysohn's lemma (see e.g. [4]).

2. Fundamental properties of normal GT's

A first obvious remark rises from the fact that it is possible for a GT μ that X is not μ-open:

PROPOSITION 2.1. *If μ is a GT such that $X \notin \mu$ then μ is (insipidly) normal.*

PROOF. There are now no disjoint μ-closed sets since $X - F, X - F' \in \mu$, $F \cap F' = \emptyset$ would imply $X = (X - F) \cup (X - F') \in \mu$. □

An almost obvious equivalent characterization is:

PROPOSITION 2.2. *A GT μ on X is normal iff $F \subset G$, $X - F, G \in \mu$ imply the existence of $G', X - F' \in \mu$ such that $F \subset G' \subset F' \subset G$.*

PROOF. If μ is normal then, for the disjoint μ-closed sets F and $X - G$, we choose the disjoint μ-open sets $G' \supset F$ and $X - F' \supset X - G$. Conversely, if $F \subset G$, $X - F, G \in \mu$ implies the existence of $G', X - F' \in \mu$ satisfying $F \subset G' \subset F' \subset G$, then, given disjoint μ-closed sets K, K', we can choose $G', X - F' \in \mu$ such that $K \subset G' \subset F' \subset X - K'$ and then G', $X - F'$ are disjoint μ-open sets satisfying $K \subset G'$, $K' \subset X - F'$ so that μ is normal. □

Consider now a GT μ on X and a map $g : X_0 \to X$. Similarly to the particular case of topologies, we say that the sets $g^{-1}(M) : M \in \mu$ constitute the *inverse image* $\mu_0 = g^{-1}(\mu)$ of μ.

PROPOSITION 2.3. *If μ is a GT on X then $g^{-1}(\mu)$ is a GT on X_0.*

PROOF. $\bigcup_{i \in I} g^{-1}(M_i) = g^{-1}\big(\bigcup_{i \in I} M_i\big)$. □

Of course, the μ_0-closed sets are those of the form $g^{-1}(F)$ where F is μ-closed.

In the particular case when $X_0 \subset X$ and $g(x) = x$ for $x \in X_0$ we say that μ_0 is the *restriction* of μ to X_0; we write now $\mu_0 = \mu | X_0$. In this case $g^{-1}(A) = A \cap X_0$ for $A \subset X$.

PROPOSITION 2.4. *If μ is a normal GT on X and $g : X_0 \to X$ is surjective then $g^{-1}(\mu)$ is normal.*

PROOF. Suppose F_0, F_0' are disjoint μ_0-closed sets. Then $F_0 = g^{-1}(F)$, $F_0' = g^{-1}(F')$, and $x \in F \cap F'$ would imply $x = g(y)$ for some $y \in X_0$ so that $y \in F_0 \cap F_0'$ would hold. Therefore F and F' are disjoint, there are disjoint μ-open sets G, G' such that $F \subset G$, $F' \subset G'$, and then $G_0 = g^{-1}(G)$, $G_0' = g^{-1}(G')$ are disjoint μ_0-open sets such that $F_0 \subset G_0$, $F_0' \subset G_0'$. □

For the case of a restriction, the above statement cannot be used because then g is not surjective in general. However, we have:

PROPOSITION 2.5. *If μ is a normal GT on X and $X_0 \subset X$ is μ-closed then the restriction $\mu_0 = \mu|X_0$ is normal.*

PROOF. If F_0, F_0' are disjoint μ_0-closed sets then $F_0 = F \cap X_0$ for some μ-closed set F, but so F_0 is μ-closed itself. Similarly F_0' is μ-closed. Therefore there exist disjoint μ-open sets G, G' satisfying $F_0 \subset G$, $F_0' \subset G'$ and then $G \cap X_0$, $G' \cap X_0$ are disjoint, μ_0-open and satisfy $F_0 \subset G \cap X_0$, $F_0' \subset G' \cap X_0$. $\square$

3. Urysohn's lemma for normal GT's

Let us recall (see [1]) that, if μ is a GT on X, ν is a GT on Y, then $f : X \to Y$ is said to be (μ,ν)-*continuous* iff $f^{-1}(N) \in \mu$ for each $N \in \nu$.

Let $\beta \subset \exp Y$ be arbitrary. Then:

LEMMA 3.1. *The family $\nu \subset \exp Y$ composed of $\emptyset$ and all sets $N \subset Y$ of the form $N = \bigcup_{i \in I} B_i$, where $B_i \in \beta$ and $I \neq \emptyset$ is arbitrary, is a GT on Y.* $\square$

We say that the *base* β *generates* the GT ν.

E.g. consider $Y = \mathbf{R}$ and $\beta = \{(-\infty,t) : t \in \mathbf{R}\} \cup \{(t,+\infty) : t \in \mathbf{R}\}$. Then the GT on $\mathbf{R}$ generated by the base β will be denoted by υ.

LEMMA 3.2. *Let μ be a GT on X and the GT ν on Y be generated by the base β. Then a map $f : X \to Y$ is (μ,ν)-continuous iff $f^{-1}(B) \in \mu$ for each $B \in \beta$.* $\square$

Now we are able the prove the following variant of Urysohn's lemma:

THEOREM 3.3. *Let μ be a normal GT on X and F, F' be disjoint μ-closed sets. Then there exists a (μ,υ)-continuous function $f : X \to \mathbf{R}$ such that $f(x) = 0$ for $x \in F$ and $f(x) = 1$ for $x \in F'$.*

PROOF. Let $\mathbf{D}$ denote the collection of all real numbers of the form $m/2^n$, $n = 0,1,2,\dots$, $m \in \mathbf{Z}$. We first define μ-open sets $G(r)$ and μ-closed sets $F(r)$ for $r \in \mathbf{D}$ satisfying

(3.3.1) $$G(r) \subset F(r) \quad \text{for} \quad r \in \mathbf{D}$$

and

(3.3.2) $$F(r) \subset G(s) \quad \text{for} \quad r,s \in \mathbf{D}, \quad (r < s).$$

First put for $r \in \mathbf{D}$

(3.3.3) $$G(r) = \emptyset \quad (r \leqq 0), \quad F(r) = \emptyset \quad (r < 0),$$

$$G(r) = X \quad (r > 1), \quad F(r) = X \quad (r \geqq 1), \tag{3.3.4}$$

$$F(0) = F, \quad G(1) = X - F'. \tag{3.3.5}$$

So the sets $G(r)$ and $F(r)$ are defined for $r \in \mathbf{D}$, $r \leqq 0$ and $r \geqq 1$ in a way that (3.3.1) and (3.3.2) are valid.

We have to define $G(r)$ and $F(r)$ for $r \in \mathbf{D}$, $0 < r < 1$. Let $\mathbf{D}_n$ denote the set composed of $m/2^n$ for $n = 0, 1, 2, \dots$ and $m = 0, 1, \dots, 2^n$. We shall define $G(r)$ and $F(r)$ for $r \in \mathbf{D}_n$, $n = 0, 1, 2, \dots$. The set $\mathbf{D}_0 = \{0, 1\}$ is settled by (3.3.3) to (3.3.5). This will be the starting point of a recursion.

Suppose that $G(r)$ and $F(r)$ are defined for $r \in \mathbf{D}_k$, $k = 0, \dots, n$ in such a way that (3.3.1) and (3.3.2) are fulfilled. Then, in particular, $F(m/2^n) \subset G\big((m+1)/2^n\big)$. Define the μ-open set $G\big((2m+1)/2^{n+1}\big)$ and the μ-closed set $F\big((2m+1)/2^{n+1}\big)$ using 2.2 so that

$$F(m/2^n) \subset G\big((2m+1)/2^{n+1}\big) \subset F\big((2m+1)/2^{n+1}\big) \subset G\big((m+1)/2^n\big).$$

This being done for $m = 0, \dots, 2^n - 1$, the set $\mathbf{D}_{n+1}$ is settled. At the end of this recursion, $G(r)$ and $F(r)$ is known for $r \in \mathbf{D}$ and (3.3.1), (3.3.2) are valid.

Define now

$$f(x) = \inf \big\{ r \in \mathbf{D} : x \in F(r) \big\}. \tag{3.3.6}$$

By (3.3.3) and (3.3.4), $0 \leqq f(x) \leqq 1$ and $f(x) = 0$ for $x \in F(0) = F$ by (3.3.5), $f(x) = 1$ for $x \in F'$ since $r \in \mathbf{D}$, $r < 1$ implies $F(r) \subset G(1) = X - F'$, thus $x \notin F(r)$, by (3.3.5) again.

We have to show that f is (μ, υ)-continuous. According to (3.2) it suffices to examine whether $f^{-1}\big((-\infty, t)\big)$ and $f^{-1}\big((t, +\infty)\big)$ belong to μ.

If x belongs to the first set, i.e. if $f(x) < t$ then there is $r \in \mathbf{D}$ such that $r < t$ and $x \in F(r)$, and then $x \in G(s)$ for $s \in \mathbf{D}$, $r < s < t$, according to (3.3.2). Now $y \in G(s)$ implies $y \in F(s)$ by (3.3.1), therefore $f(y) \leqq s < t$, so that $G(s)$ is a μ-open set satisfying $x \in G(s) \subset f^{-1}\big((-\infty, t)\big)$: the latter set is the union of μ-open sets and hence μ-open itself.

If $x \in f^{-1}\big((t, +\infty)\big)$ then $t < f(x)$ so that $x \notin F(r)$ whenever $r \in \mathbf{D}$, $r < f(x)$. Choose r such that $t < r < f(x)$. For the μ-open set $X - F(r)$, necessarily $x \in X - F(r) \subset f^{-1}\big((t, +\infty)\big)$; in fact, $y \in X - F(r)$ implies that $y \in F(s)$, $s \in \mathbf{D}$, $s < r$ is impossible since then $y \in F(s) \subset G(r) \subset F(r)$ would hold by (3.3.1) and (3.3.2). Therefore $y \notin F(s)$ for these s, $f(y) \geqq r > t$. Again, $f^{-1}\big((t, +\infty)\big)$ is the union of μ-open sets and so μ-open itself. □

The statement in 3.3 is sufficient for the normality of μ:

THEOREM 3.4. *If μ is a GT on X with the property that, if F, F' are disjoint μ-closed sets, there exists a (μ, υ)-continuous function $f : X \to \mathbf{R}$ satisfying $f(x) = 0$ for $x \in F$ and $f(x) = 1$ for $x \in F'$, then μ is normal.*

PROOF. The disjoint sets $f^{-1}\big((-\infty, 1/2)\big)$ and $f^{-1}\big((1/2, +\infty)\big)$ are μ-open and contain F and F', respectively. $\square$

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