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CONJUGATE DUALITY FOR MULTIOBJECTIVE COMPOSED OPTIMIZATION PROBLEMS

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Abstract. Given a multiobjective optimization problem with the components of the objective function as well as the constraint functions being composed convex functions, we introduce, by using the Fenchel–Moreau conjugate of the functions involved, a suitable dual problem. Under a standard constraint qualification and some convexity as well as monotonicity conditions we prove the existence of strong duality. Finally, some particular cases of this problem are presented.

1. Introduction

In the last decades convex composed programming (CCP) has received considerable attention since it offers a unified framework for treating different types of optimization problems. By (CCP) we mean a class of optimization problems in which the objective function as well as the constraint functions are composed convex functions. Among the large number of papers dealing with composed optimization problems in both finite and infinite dimensional spaces, we mention [1], [4], [8], [9], [10], [11], [12], [13], [14], [15], [21], [22] and [23].

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In this paper we consider a multiobjective composed problem of the form

$$(P) \qquad \underset{x\in\mathcal{A}}{\text{v-min}}\, f\big(F(x)\big), \qquad \mathcal{A} = \left\{x \in X : g\big(G(x)\big) \leqq_{\mathbf{R}^k_+} 0\right\},$$

where X is a nonempty subset of $\mathbf{R}^n$, $F = (F_1, \dots, F_m)^T : X \to \mathbf{R}^m$, $G = (G_1, \dots, G_l)^T : X \to \mathbf{R}^l$, $f = (f_1, \dots, f_s)^T : \mathbf{R}^m \to \mathbf{R}^s$ and $g = (g_1, \dots, g_k)^T : \mathbf{R}^l \to \mathbf{R}^k$ are vector-valued functions. The problem (P) has a quite general formulation and provides a unified framework for studying different multiobjective optimization problems which can be obtained as a special case.

Our purpose is to construct a multiobjective dual for the problem above. First, we associate to it a scalar problem for which we completely study the duality. We formulate the weak and strong duality theorems and give some optimality conditions regarding to this scalarized problem. The approach we adopt here is based on the conjugate duality approach, described in detail for instance in [6]. The optimality conditions which we derive in the scalar case allow to construct a multiobjective dual problem to the primal one. We prove weak and strong duality also for the multiobjective primal-dual pair. Once the general problem has been treated, some special cases are considered.

The main tool we use here to deal with the composed functions is the formula of the Fenchel–Moreau conjugate function of the composition of an increasing convex function with a convex function (see [5] and [24]).

This paper is organized as follows. In Section 2 we recall some notations and definitions and give some preliminary results. Section 3 is devoted to the study of the scalarized problem associated to problem (P). We introduce a dual problem in terms of the Fenchel–Moreau conjugate of the objective function and the constraint functions, respectively, and prove weak and strong duality statements. Necessary and sufficient optimality conditions linked to this scalarized problem are given. In Section 4 we deal with the multiobjective optimization problem. We introduce a multiobjective dual and prove weak and strong duality theorems. The last section contains some special cases of the original problem such as the classical multiobjective optimization problem with geometric and inequality constraints, as well as the multiobjective composed optimization problem only with geometric constraints.

2. Notations and preliminary results

Denote by $x^T y = \sum_{i=1}^{p} x_i y_i$ the inner product of the vectors

$$x = (x_1, \dots, x_p)^T, y = (y_1, \dots, y_p)^T \in \mathbf{R}^p$$

and by $\mathbf{R}^p_+$ the non-negative orthant of $\mathbf{R}^p$. For $x,\ y \in \mathbf{R}^p$, the inequality $x \leqq_{\mathbf{R}^p_+} y$ means that $y - x \in \mathbf{R}^p_+$, which is equivalent to $x_i \leqq y_i$ for all $i =$

$1,\dots,p$. Let X be a nonempty subset of $\mathbf{R}^p$. Denote by $\operatorname{ri}(X)$ the relative interior of the set X. Considering a function $f:\mathbf{R}^p\to\overline{\mathbf{R}}$, denote by $\operatorname{dom}(f) = \{x\in\mathbf{R}^p : f(x)<+\infty\}$ its effective domain. We say that f is proper if $\operatorname{dom}(f)\neq\emptyset$ and $f(x)>-\infty$ for all $x\in\mathbf{R}^p$.

DEFINITION 2.1. Let X be a nonempty subset of $\mathbf{R}^n$. The function $\delta_X:\mathbf{R}^n\to\overline{\mathbf{R}}$ defined by

$$\delta_X(x)=\begin{cases}0, & \text{if } x\in X,\\ +\infty, & \text{otherwise,}\end{cases}$$

is called the indicator function of the set X.

DEFINITION 2.2. When X is a nonempty subset of $\mathbf{R}^n$ and $f:X\to\mathbf{R}$, denote by

$$f_X^*:\mathbf{R}^n\to\overline{\mathbf{R}},\ f_X^*(x^*)=\sup_{x\in X}\{x^{*T}x-f(x)\}$$

the so-called conjugate relative to the set X. By taking $X=\mathbf{R}^n$ one obtains the classical Fenchel–Moreau conjugate of f.

DEFINITION 2.3. The function $f:\mathbf{R}^m\to\mathbf{R}$ is called componentwise increasing, if for $x=(x_1,\dots,x_m)^T$, $y=(y_1,\dots,y_m)^T\in\mathbf{R}^m$ where $x_i\leqq y_i$, $i=1,\dots,m$, we have $f(x)\leqq f(y)$.

PROPOSITION 2.1. *If $f:\mathbf{R}^m\to\mathbf{R}$ is a componentwise increasing function, then $f^*(q)=+\infty$ for all $q\in\mathbf{R}^m\setminus\mathbf{R}_+^m$.*

PROOF. Let $q\in\mathbf{R}^m\setminus\mathbf{R}_+^m$. Then there exists at least one $i\in\{1,\dots,m\}$ such that $q_i<0$. But

$$f^*(q)=\sup_{d\in\mathbf{R}^m}\{q^Td-f(d)\}\geqq\sup_{\substack{d=(0,\dots,d_i,\dots,0)^T,\\ d_i\in\mathbf{R}}}\{q^Td-f(d)\}$$

$$=\sup_{d_i\in\mathbf{R}}\{q_id_i-f(0,\dots,d_i,\dots,0)\}\geqq\sup_{d_i<0}\{q_id_i-f(0,\dots,d_i,\dots0)\}$$

$$\geqq\sup_{d_i<0}\{q_id_i\}-f(0,\dots,0)=+\infty.$$

Thus $f^*(q)=+\infty\ \forall q\in\mathbf{R}^m\setminus\mathbf{R}_+^m$. □

The following classical result plays an important role in this paper.

THEOREM 2.1 (cf. Theorem 16.4 in [16]). *Let $f_1,\dots,f_n:\mathbf{R}^m\to\overline{\mathbf{R}}$ be proper, convex functions. If the sets $\operatorname{ri}(\operatorname{dom}(f_i))$, $i=1,\dots,n$, have a point in common, then*

$$\left(\sum_{i=1}^n f_i\right)^*(p)=\inf\left\{\sum_{i=1}^n f_i^*(p_i):\sum_{i=1}^n p_i=p\right\},$$

where for each $p \in \mathbf{R}^m$ *the infimum is attained.*

In what follows let X be a nonempty convex subset of $\mathbf{R}^n$, $g : X \to \mathbf{R}^k$ a vector function with convex components and (CQ_a) the constraint qualification ([7], [16])

$$(\mathrm{CQ}_a) \qquad \exists\, x' \in \operatorname{ri}(X) : \begin{cases} g_i(x') \leqq 0, & i \in L_a, \\ g_i(x') < 0, & i \in N_a, \end{cases}$$

where

$$L_a := \left\{ i \in \{1, \dots, k\} \;\middle|\; \begin{array}{l} g_i : X \to \mathbf{R} \text{ is the restriction to } X \\ \text{of an affine function } \widetilde{g}_i : \mathbf{R}^n \to \mathbf{R} \end{array} \right\}$$

and $N_a := \{1, \dots, k\} \setminus L_a$.

Consider the optimization problem

$$(\mathrm{P}_a) \qquad \inf_{x \in \mathcal{A}_a} f(x), \qquad \mathcal{A}_a = \left\{ x \in X : g(x) \leqq_{\mathbf{R}^k_+} 0 \right\},$$

and its well-known Lagrange dual

$$(\mathrm{D}_a) \qquad \sup_{t \in \mathbf{R}^k_+} \inf_{x \in X} \left\{ f(x) + t^T g(x) \right\},$$

where $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}^k$.

The next theorem states the Lagrange duality for the problems (P_a) and (D_a).

THEOREM 2.2 (cf. [16]). *Assume that* X *is a nonempty convex subset of* $\mathbf{R}^n$ *and* $f : X \to \mathbf{R}$ *and the components* g_i, $i = 1, \dots, k$, *of* $g : X \to \mathbf{R}^k$ *are convex functions. If* $v(\mathrm{P}_a) > -\infty$ *and the constraint qualification* (CQ_a) *is fulfilled, then* $v(\mathrm{P}_a) = v(\mathrm{D}_a)$ *and the dual problem* (D_a) *has an optimal solution.*

REMARK 2.1. Denote by $v(P)$ the optimal objective value of the optimization problem (P).

Considering the multiobjective problem (P) assume that $X \subseteqq \mathbf{R}^n$ is a convex set, F_i, $i = 1, \dots, m$, G_j, $j = 1, \dots, l$, are convex functions on X and f_i, $i = 1, \dots, s$, and g_j, $j = 1, \dots, k$, are convex and componentwise increasing functions on $\mathbf{R}^m$ and $\mathbf{R}^l$, respectively.

For the multiobjective optimization problem (P) different notions of solution are known. Let us recall the definition of the efficient and properly efficient solutions.

DEFINITION 2.4. An element $\bar{x} \in \mathcal{A}$ is said to be efficient (or Pareto efficient) with respect to (P) if from $f(F(x)) \leqq_{\mathbf{R}^s_+} f(F(\bar{x}))$ for $x \in \mathcal{A}$ follows that $f(F(x)) = f(F(\bar{x}))$.

DEFINITION 2.5. An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient with respect to (P) if there exists $\lambda = (\lambda_1, \ldots, \lambda_s)^T \in \operatorname{int}(\mathbf{R}^s_+)$ (i.e. $\lambda_i > 0$, $i = 1, \ldots, s$), such that $\lambda^T f(F(\bar{x})) \leqq \lambda^T f(F(x))$ for all $x \in \mathcal{A}$.

REMARK 2.2. It is straightforward to realize that a properly efficient solution turns out to be efficient, too.

3. Duality for the scalarized problem

Inspired by Definition 2.5 we consider the following scalarized problem (P_λ) to (P)

$$(\mathrm{P}_\lambda) \qquad \inf_{x \in \mathcal{A}} \lambda^T f(F(x)),$$

where $\lambda = (\lambda_1, \ldots, \lambda_s)^T$ is a fixed vector in $\operatorname{int}(\mathbf{R}^s_+)$.

By using the perturbation theory developed by Ekeland and Temam (cf. [6]), Wanka, Boţ and Vargyas had introduced in [21] the following dual problem to the scalar problem (P_λ)

$$(\mathrm{D}_\lambda) \qquad \sup_{\substack{p \in \mathbf{R}^n,\ q \in \mathbf{R}^m, \\ q' \in \mathbf{R}^l,\ t \in \mathbf{R}^k_+}} \left\{ -(\lambda^T f)^*(q) - (t^T g)^*(q') - (q^T F)^*_X(p) - (q'^T G)^*_X(-p) \right\}.$$

REMARK 3.1. When $\lambda \in \mathbf{R}^s_+$, $f = (f_1, \ldots, f_s)^T : \mathbf{R}^m \to \mathbf{R}^s$ and f_i, $i = 1, \ldots, s$, are componentwise increasing functions it follows that $\lambda^T f : \mathbf{R}^m \to \mathbf{R}$ is also a componentwise increasing function.

By Remark 3.1 and Proposition 2.1 one can take $q \in \mathbf{R}^m_+$, $q' \in \mathbf{R}^l_+$ in (D_λ) and so the dual problem becomes

$$(\mathrm{D}_\lambda) \qquad \sup_{\substack{p \in \mathbf{R}^n,\ q \in \mathbf{R}^m_+, \\ q' \in \mathbf{R}^l_+,\ t \in \mathbf{R}^k_+}} \left\{ -(\lambda^T f)^*(q) - (t^T g)^*(q') - (q^T F)^*_X(p) - (q'^T G)^*_X(-p) \right\}.$$

Because of Theorem 2.1 we have

$$(1) \qquad (\lambda^T f)^*(q) = \left(\sum_{i=1}^s \lambda_i f_i \right)^*(q) = \inf \left\{ \sum_{i=1}^s (\lambda_i f_i)^*(r^i) : \sum_{i=1}^s r^i = q \right\}$$

and the infimum is attained for all $q \in \mathbf{R}^n$. According to Proposition 2.1, r^i, $i = 1, \ldots, s$, must belong to $\mathbf{R}^m_+$ and the dual (D_λ) looks like

(D_λ)

$$\sup_{(p,q,q',r,t) \in Y_\lambda} \left\{ -\sum_{i=1}^{s} (\lambda_i f_i)^*(r^i) - (t^T g)^*(q') - (q^T F)^*_X(p) - (q'^T G)^*_X(-p) \right\},$$

with

$$Y_\lambda = \left\{ (p, q, q', r, t) : p \in \mathbf{R}^n,\ q \in \mathbf{R}^m_+,\ q' \in \mathbf{R}^l_+,\ r = (r^1, \ldots, r^s), \right.$$
$$\left. r^i \in \mathbf{R}^m_+,\ i = 1, \ldots, s,\ \sum_{i=1}^{s} r^i = q,\ t \in \mathbf{R}^k_+ \right\}.$$

Since $\lambda_i > 0$, it follows that

$$(\lambda_i f_i)^*(r^i) = \lambda_i f_i^* \left(\frac{1}{\lambda_i} r^i \right), \tag{2}$$

for all $i = 1, \ldots, s$. Redenoting $\frac{1}{\lambda_i} r^i$ by r^i, $i = 1, \ldots, s$, we obtain the following formulation for the dual:

(3) (D_λ)

$$\sup_{(p,q,q',r,t) \in Y_\lambda} \left\{ -\sum_{i=1}^{s} \lambda_i f_i^*(r^i) - (t^T g)^*(q') - (q^T F)^*_X(p) - (q'^T G)^*_X(-p) \right\},$$

$$Y_\lambda = \left\{ (p, q, q', r, t) : p \in \mathbf{R}^n, q \in \mathbf{R}^m_+, q' \in \mathbf{R}^l_+, r = (r^1, \ldots, r^s), \right.$$
$$\left. r^i \in \mathbf{R}^m_+, i = 1, \ldots, s, \sum_{i=1}^{s} \lambda_i r^i = q, t \in \mathbf{R}^k_+ \right\}.$$

The next theorem states the existence of weak duality between (P_λ) and (D_λ).

THEOREM 3.1 (weak duality for (P_λ)). *We have* $v(\mathrm{D}_\lambda) \leqq v(\mathrm{P}_\lambda)$.

PROOF. Let $(p, q, q', r, t) \in Y_\lambda$ be an arbitrary element. By the Young–Fenchel inequality

$$-f_i^*(r^i) \leqq -(r^i)^T F(x) + f_i\big(F(x)\big), \quad \forall x \in \mathbf{R}^n,\ \forall i = 1, \ldots, s,$$

$$-(t^T g)^*(q') \leqq -q'^T G(x) + t^T g\big(G(x)\big), \quad \forall x \in \mathbf{R}^n,$$

$$-(q^T F)^*_X(p) \leqq -p^T x + q^T F(x), \quad \forall x \in X,$$

$$-(q'^T G)^*_X(-p) \leqq p^T x + q'^T G(x), \quad \forall x \in X.$$

Multiplying each of the first inequalities by $\lambda_i > 0$, $i = 1, \dots s$, respectively, and adding their sum for $i = 1, \dots, s$, to the other ones, it follows that

$$-\sum_{i=1}^{s} \lambda_i f_i^*(r^i) - (t^T g)^*(q') - (q^T F)^*_X(p) - (q'^T G)^*_X(-p)$$

$$\leqq \sum_{i=1}^{s} \lambda_i f_i\big(F(x)\big) + t^T g\big(G(x)\big),$$

for all $x \in X$. Because $(p, q, q', r, t) \in Y_\lambda$, we have $t^T g\big(G(x)\big) \leqq 0$ for all $x \in \mathcal{A}$, which together with the inequality above implies that

$$-\sum_{i=1}^{s} \lambda_i f_i^*(r^i) - (t^T g)^*(q') - (q^T F)^*_X(p) - (q'^T G)^*_X(-p)$$

$$\leqq \sum_{i=1}^{s} \lambda_i f_i\big(F(x)\big) = \lambda^T f\big(F(x)\big),$$

for all $x \in \mathcal{A}$. Taking on the left side of this inequality the supremum over $(p, q, q', r, t) \in Y_\lambda$ and on the right one the infimum over $x \in \mathcal{A}$, it follows that $v(\mathrm{D}_\lambda) \leqq v(\mathrm{P}_\lambda)$. □

Further we study the existence of strong duality between (P_λ) and (D_λ), namely the situation when the optimal objective values are equal and the dual has an optimal solution. In order to do this, we introduce a constraint qualification that guarantees the validity of strong duality, but first, let us divide the index set $\{1, \dots, k\}$ into two subsets,

$$L := \left\{ i \in \{1, \dots, k\} \;\middle|\; \begin{array}{l} g_i \circ G : X \to \mathbf{R} \text{ is the restriction to } X \text{ of an} \\ \text{affine function } \widetilde{g_i \circ G} : \mathbf{R}^n \to \mathbf{R} \end{array} \right\}$$

and $N := \{1, \dots, k\} \setminus L$. The constraint qualification follows (cf. [7], [16])

$$(\mathrm{CQ}) \qquad \exists\, x' \in \mathrm{ri}\,(X) : \begin{cases} g_i\big(G(x')\big) \leqq 0, & i \in L, \\ g_i\big(G(x')\big) < 0, & i \in N. \end{cases}$$

THEOREM 3.2 (strong duality for (P_λ)). *Assume that the constraint qualification* (CQ) *is fulfilled. Then* $v(\mathrm{P}_\lambda) = v(\mathrm{D}_\lambda)$. *Moreover, provided* $v(\mathrm{P}_\lambda) > -\infty$, *the dual problem has an optimal solution.*

PROOF. If $v(\mathrm{P}_\lambda) = -\infty$, by Theorem 3.1 it follows that $v(\mathrm{D}_\lambda) = -\infty$.

Assume that $v(\mathrm{P}_\lambda) > -\infty$. Because the constraint qualification (CQ) is fulfilled, by Theorem 2.2 there exists an element $\bar{t} \in \mathbf{R}_k^+$ such that (cf. Lagrange duality)

$$v(\mathrm{P}_\lambda) = \inf_{x \in X} \left\{ \lambda^T f\big(F(x)\big) + \bar{t}^T g\big(G(x)\big) \right\},$$

where $v(\mathrm{P}_\lambda)$ denotes the optimal objective value of the problem (P_λ). Further we attach to $\mathbf{R}^m$ a greatest element with respect to "$\leqq_{\mathbf{R}_+^m}$" denoted $\infty_{\mathbf{R}^m}$ and let $(\mathbf{R}^m)^\bullet = \mathbf{R}^m \cup \{\infty\}$. Then for any $x \in (\mathbf{R}^m)^\bullet$ one has $x \leqq_{\mathbf{R}_+^m} \infty_{\mathbf{R}^m}$ and we consider the following operations on $(\mathbf{R}^m)^\bullet : x + \infty_{\mathbf{R}^m} = \infty_{\mathbf{R}^m} + x = \infty_{\mathbf{R}^m}$ and $t\infty_{\mathbf{R}^m} = \infty_{\mathbf{R}^m}$ $\forall t \geqq 0$. The same will be done for $\mathbf{R}^l$. We can define now the functions

$$\widetilde{F} : \mathbf{R}^n \to (\mathbf{R}^m)^\bullet,\ \widetilde{F}(x) = \begin{cases} F(x), & \text{if } x \in X, \\ \infty_{\mathbf{R}^m}, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{G} : \mathbf{R}^n \to (\mathbf{R}^l)^\bullet,\ \widetilde{G}(x) = \begin{cases} G(x), & \text{if } x \in X, \\ \infty_{\mathbf{R}^l}, & \text{otherwise.} \end{cases}$$

We also make the conventions that for all $i = 1, \ldots, s$, $f_i(\infty_{\mathbf{R}^m}) = +\infty$ and for all $j = 1, \ldots, k$, $g_j(\infty_{\mathbf{R}^l}) = +\infty$.

Thus the optimal objective value of the primal problem can be written as

$$v(\mathrm{P}_\lambda) = \inf_{x \in \mathbf{R}^n} \left\{ \lambda^T f\big(\widetilde{F}(x)\big) + \bar{t}^T g\big(\widetilde{G}(x)\big) \right\}, \tag{4}$$

where $\lambda^T f \circ \widetilde{F}$ and $\bar{t}^T g \circ \widetilde{G}$ are functions with values in the extended real-valued space with $\operatorname{dom}(\lambda^T f \circ \widetilde{F}) = \operatorname{dom}(\bar{t}^T g \circ \widetilde{G}) = X$. Since this is a nonempty convex set, by Theorem 2.1 there exists $\bar{p} \in \mathbf{R}^n$ such that the infimum in (4) is equal to

$$\begin{aligned} v(\mathrm{P}_\lambda) &= -(\lambda^T f \circ \widetilde{F} + \bar{t}^T g \circ \widetilde{G})^*(0) \\ &= \max_{p \in \mathbf{R}^n} \left\{ -(\lambda^T f \circ \widetilde{F})^*(p) - (\bar{t}^T g \circ \widetilde{G})^*(-p) \right\} \\ &= -(\lambda^T f \circ \widetilde{F})^*(\bar{p}) - (\bar{t}^T g \circ \widetilde{G})^*(-\bar{p}). \end{aligned} \tag{5}$$

Further, by Proposition 4.11 in [5] (see also [24]), there exist some $\bar{q} \in \mathbf{R}^m_+$ and $\bar{q}' \in \mathbf{R}^l_+$ such that

(6)
$$(\lambda^T f \circ \widetilde{F})^*(\bar{p}) = \min_{q \in \mathbf{R}^m_+} \left\{ (\lambda^T f)^*(q) + (q^T \widetilde{F})^*(\bar{p}) \right\} = (\lambda^T f)^*(\bar{q}) + (\bar{q}^T \widetilde{F})^*(\bar{p})$$

and

(7)
$$(\bar{t}^T g \circ G)^*(-\bar{p}) = \min_{q' \in \mathbf{R}^l_+} \left\{ (\bar{t}^T g)^*(q') + (q'^T \widetilde{G})^*(-\bar{p}) \right\}$$
$$= (\bar{t}^T g)^*(\bar{q}') + (\bar{q}'^T \widetilde{G})^*(-\bar{p}).$$

The relations (4), (5), (6) and (7) imply actually the existence of the vectors $\bar{t} \in \mathbf{R}^k_+$, $\bar{p} \in \mathbf{R}^n$, $\bar{q} \in \mathbf{R}^m_+$ and $\bar{q}' \in \mathbf{R}^l_+$ such that

$$v(\mathrm{P}_\lambda) = -(\lambda^T f)^*(\bar{q}) - (\bar{t}^T g)^*(\bar{q}') - (\bar{q}^T F)^*_X(\bar{p}) - (\bar{q}'^T G)^*_X(-\bar{p}).$$

Applying again Theorem 2.1 one can find some $\bar{r}^i \in \mathbf{R}^m_+$, $i = 1, \dots s$, such that

$$(\lambda^T f)^*(\bar{q}) = \sum_{i=1}^{s} (\lambda_i f_i)^*(\bar{r}^i) \quad \text{and} \quad \sum_{i=1}^{s} \bar{r}^i = \bar{q}.$$

Since $\lambda_i > 0$, we have

$$(\lambda_i f_i)^*(\bar{r}^i) = \lambda_i f_i^* \left(\frac{1}{\lambda_i} \bar{r}^i \right),$$

for all $i = 1, \dots, s$. Redenoting $\frac{1}{\lambda_i} \bar{r}^i$ by $\bar{r}^i$, $i = 1, \dots, s$, we obtain the tuple $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})$ which is an optimal solution to the dual problem (D_λ) fulfilling

$$v(\mathrm{P}_\lambda) = -\sum_{i=1}^{s} \lambda_i f_i^*(\bar{r}^i) - (\bar{t}^T g)^*(\bar{q}') - (\bar{q}^T F)^*_X(\bar{p}) - (\bar{q}'^T G)^*_X(-\bar{p}) = v(\mathrm{D}_\lambda),$$

which actually means that strong duality between (P_λ) and (D_λ) holds. □

To investigate later the multiobjective duality for (P) we need the optimality conditions regarding to the scalar problem (P_λ) and its dual (D_λ). These are formulated in the following theorem.

THEOREM 3.3. (a) *Let the assumptions of Theorem 3.2 be fulfilled and let $\bar{x}$ be an optimal solution to* (P_λ). *Then there exists a tuple* $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}) \in Y_\lambda$, *optimal solution to* (D_λ), *such that the following optimality conditions are satisfied:*

(i) $f_i\big(F(\bar{x})\big) + f_i^*(\bar{r}^i) = (\bar{r}^i)^T F(\bar{x})$, $i = 1, \ldots, s$,

(ii) $\bar{q}^T F(\bar{x}) + (\bar{q}^T F)_X^*(\bar{p}) = \bar{p}^T \bar{x}$,

(iii) $\bar{t}^T g\big(G(\bar{x})\big) + (\bar{t}^T g)^*(\bar{q}') = \bar{q}'^T G(\bar{x})$,

(iv) $\bar{q}'^T G(\bar{x}) + (\bar{q}'^T G)_X^*(-\bar{p}) = (-\bar{p})^T \bar{x}$,

(v) $\bar{t}^T g\big(G(\bar{x})\big) = 0$.

(b) *Let $\bar{x}$ be admissible to* (P_λ) *and* $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})$ *be admissible to* (D_λ) *satisfying* (i)–(v). *Then $\bar{x}$ is an optimal solution to* (P_λ), $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})$ *is an optimal solution to* (D_λ) *and strong duality holds.*

PROOF. By Theorem 3.2 there exists a tuple $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}) \in Y_\lambda$, optimal solution to (D_λ), such that

$$\lambda^T f\big(F(\bar{x})\big) = v(\mathrm{P}_\lambda) = v(\mathrm{D}_\lambda) \tag{8}$$

$$= -\sum_{i=1}^{s} \lambda_i f_i^*(\bar{r}^i) - (\bar{t}^T g)^*(\bar{q}') - (\bar{q}^T F)_X^*(\bar{p}) - (\bar{q}'^T G)_X^*(-\bar{p}).$$

Equality (8) is equivalent to

$$\Big\{\lambda^T f\big(F(\bar{x})\big) + \sum_{i=1}^{s} \lambda_i f_i^*(\bar{r}^i) - \bar{q}^T F(\bar{x})\Big\} \tag{9}$$

$$+ \Big\{\bar{t}^T g\big(G(\bar{x})\big) + (\bar{t}^T g)^*(\bar{q}') - \bar{q}'^T G(\bar{x})\Big\} + \big\{\bar{q}^T F(\bar{x}) + (\bar{q}^T F)_X^*(\bar{p}) - \bar{p}^T \bar{x}\big\}$$

$$+ \big\{\bar{q}'^T G(\bar{x}) + (\bar{q}'^T G)_X^*(-\bar{p}) - (-\bar{p})^T \bar{x}\big\} + \Big\{ -\bar{t}^T g\big(G(\bar{x})\big)\Big\} = 0.$$

Because $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}) \in Y_\lambda$, we have $\sum\limits_{i=1}^{s} \lambda_i \bar{r}^i = \bar{q}$, and so

$$\lambda^T f\big(F(\bar{x})\big) + \sum_{i=1}^{s} \lambda_i f_i^*(\bar{r}^i) - \bar{q}^T F(\bar{x})$$

$$= \sum_{i=1}^{s} \lambda_i \Big\{ f_i\big(F(\bar{x})\big) + f_i^*(\bar{r}^i) - (\bar{r}^i)^T F(\bar{x})\Big\}.$$

According to the Young–Fenchel inequality the following inequalities hold:

$$f_i\big(F(\bar{x})\big) + f_i^*(\bar{r}^i) - (\bar{r}^i)^T F(\bar{x}) \geqq 0, \qquad i = 1, \ldots, s,$$

$$\bar{t}^T g\big(G(\bar{x})\big) + (\bar{t}^T g)^*(\bar{q}') - \bar{q}'^T G(\bar{x}) \geqq 0,$$

$$\bar{q}^T F(\bar{x}) + (\bar{q}^T F)_X^*(\bar{p}) - \bar{p}^T \bar{x} \geqq 0,$$

$$\bar{q}'^T G(\bar{x}) + (\bar{q}'^T G)_X^*(-\bar{p}) - (-\bar{p})^T \bar{x} \geqq 0.$$

Because $\bar{t} \in \mathbf{R}_+^k$ and $\bar{x} \in \mathcal{A}$ there is $-\bar{t}^T g\big(G(\bar{x})\big) \geqq 0$, and so, equation (9) together with the inequalities above imply the relations (i)–(v).

(b) By (i)–(v), making the above calculations in the opposite direction

$$v(\mathrm{D}_\lambda) \geqq -\sum_{i=1}^{s} \lambda_i f_i^*(\bar{r}^i) - (\bar{t}^T g)^*(\bar{q}') - (\bar{q}^T F)_X^*(\bar{p}) - (\bar{q}'^T G)_X^*(-\bar{p})$$

$$= \lambda^T f\big(F(\bar{x})\big) \geqq v(\mathrm{P}_\lambda),$$

which together with Theorem 3.1 ensures the strong duality for (P_λ) and (D_λ). □

4. The multiobjective dual problem

By using the duality developed above in the scalar case, we can formulate now a multiobjective dual (D) to the original problem (P) which will be actually a vector maximum problem. We define the Pareto optimal solutions to (D) in the sense of maximum and prove weak and strong duality theorems between (P) and its dual.

The dual multiobjective optimization problem (D) is introduced by

$$\text{(D)} \qquad \underset{(p,q,q',r,t,\lambda,u)\in\mathcal{B}}{\text{v-max}} h(p,q,q',r,t,\lambda,u),$$

with

$$h(p,q,q',r,t,\lambda,u) = \begin{pmatrix} h_1(p,q,q',r,t,\lambda,u) \\ \vdots \\ h_s(p,q,q',r,t,\lambda,u) \end{pmatrix},$$

$$h_i(p,q,q',r,t,\lambda,u)$$

$$= -f_i^*(r^i) - \frac{1}{s\lambda_i}\big((t^T g)^*(q') + (q^T F)_X^*(p) + (q'^T G)_X^*(-p)\big) + u_i,$$

for $i = 1, \dots s$, the dual variables

$$p = (p_1, \dots, p_n)^T \in \mathbf{R}^n, \quad q = (q_1, \dots, q_m)^T \in \mathbf{R}^m, \quad q' = (q_1', \dots, q_l')^T \in \mathbf{R}^l,$$

$$r = (r^1, \ldots, r^s) \in \mathbf{R}^m \times \ldots \times \mathbf{R}^m, \quad t = (t_1, \ldots, t_k)^T \in \mathbf{R}^k,$$

$$\lambda = (\lambda_1, \ldots, \lambda_s)^T \in \mathbf{R}^s, \quad u = (u_1, \ldots, u_s)^T \in \mathbf{R}^s,$$

and the set of constraints

$$\mathcal{B} = \Big\{ (p, q, q', r, t, \lambda, u) : q \in \mathbf{R}^m_+,\ q' \in \mathbf{R}^l_+,\ r^i \in \mathbf{R}^m_+,\ i = 1, \ldots, s,\ t \in \mathbf{R}^k_+,$$

$$\lambda \in \operatorname{int}(\mathbf{R}^s_+),\ \sum_{i=1}^{s} \lambda_i r^i = q,\ \sum_{i=1}^{s} \lambda_i u_i = 0 \Big\}.$$

DEFINITION 4.1. An element $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u}) \in \mathcal{B}$ is said to be efficient (or Pareto efficient) with respect to the problem (D) if from

$$h(p, q, q', r, t, \lambda, u) \geqq_{\mathbf{R}^s_+} h(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u}) \quad \text{for} \quad (p, q, q', r, t, \lambda, u) \in \mathcal{B}$$

it follows that $h(p, q, q', r, t, \lambda, u) = h(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})$.

The following theorem provides the weak duality between the vector problems (P) and (D).

THEOREM 4.1. *There is no $x \in \mathcal{A}$ and no $(p, q, q', r, t, \lambda, u) \in \mathcal{B}$ fulfilling $f\big(F(x)\big) \leqq_{\mathbf{R}^s_+} h(p, q, q', r, t, \lambda, u)$ and $f\big(F(x)\big) \neq h(p, q, q', r, t, \lambda, u)$.*

PROOF. Let us assume that there exist $x \in \mathcal{A}$ and $(p, q, q', r, t, \lambda, u) \in \mathcal{B}$ such that $f_i\big(F(x)\big) \leqq h_i(p, q, q', r, t, \lambda, u)$ for all $i = 1, \ldots, s$ and $f_j\big(F(x)\big) < h_j(p, q, q', r, t, \lambda, u)$ for at least one $j \in \{1, \ldots, s\}$. This implies

$$\lambda^T f\big(F(x)\big) = \sum_{i=1}^{s} \lambda_i f_i\big(F(x)\big) < \sum_{i=1}^{s} \lambda_i h_i(p, q, q', r, t, \lambda, u) = \lambda^T h(p, q, q', r, t, \lambda, u). \tag{10}$$

But

$$\sum_{i=1}^{s} \lambda_i h_i(p, q, q', r, t, \lambda, u) = -\sum_{i=1}^{s} \lambda_i f_i^*(r^i) - \sum_{i=1}^{s} \lambda_i \frac{1}{s\lambda_i} \big((t^T g)^*(q')$$

$$+ (q^T F)^*_X(p) + (q'^T G)^*_X(-p) \big) + \sum_{i=1}^{s} \lambda_i u_i$$

$$= -\sum_{i=1}^{s} \lambda_i f_i^*(r^i) - \big((t^T g)^*(q') + (q^T F)^*_X(p) + (q'^T G)^*_X(-p) \big),$$

and applying then for f_i, $i=1,\dots,s$, t^Tg, q^TF and q'^TG the Young–Fenchel inequality we have

$$-f_i^*(r^i) \leqq f_i\big(F(x)\big) - (r^i)^T F(x), \qquad \forall\, i=1,\dots,s,$$
$$-\left(t^Tg\right)^*(q') \leqq t^Tg\big(G(x)\big) - q'^TG(x),$$
$$-\left(q^TF\right)_X^*(p) \leqq q^TF(x) - p^Tx, \quad -(q'^TG)_X^*(-p) \leqq q'^TG(x) + p^Tx.$$

Because of $\sum\limits_{i=1}^{s} \lambda_i r^i = q$, $t \in \mathbf{R}_+^k$ and $x \in \mathcal{A}$, we obtain

$$\sum_{i=1}^{s} \lambda_i h_i(p,q,q',r,t,\lambda,u) \leqq \sum_{i=1}^{s} \lambda_i f_i\big(F(x)\big) - \sum_{i=1}^{s} \lambda_i (r^i)^T F(x) + t^Tg\big(G(x)\big)$$
$$- q'^TG(x) + q^TF(x) - p^Tx + q'^TG(x) + p^Tx$$
$$= \sum_{i=1}^{s} \lambda_i f_i\big(F(x)\big) + t^Tg\big(G(x)\big) \leqq \sum_{i=1}^{s} \lambda_i f_i\big(F(x)\big).$$

But the inequality $\sum\limits_{i=1}^{s} \lambda_i h_i(p,q,q',r,t,\lambda,u) \leqq \sum\limits_{i=1}^{s} \lambda_i f_i\big(F(x)\big)$ contradicts relation (10). Thus the weak duality between (P) and (D) holds. □

Theorem 4.2 gives us the strong duality between the multiobjective problems (P) and (D).

THEOREM 4.2. *Assume that the constraint qualification* (CQ) *is fulfilled and let $\bar{x}$ be a properly efficient element to* (P). *Then there exists an efficient solution $(\bar{p},\bar{q},\bar{q}',\bar{r},\bar{t},\bar{\lambda},\bar{u}) \in \mathcal{B}$ to the dual* (D) *such that the strong duality $f\big(F(\bar{x})\big) = h(\bar{p},\bar{q},\bar{q}',\bar{r},\bar{t},\bar{\lambda},\bar{u})$ holds.*

PROOF. Let $\bar{x}$ be a properly efficient element to (P). By Definition 2.5, it follows that there exists a vector $\bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_s)^T \in \operatorname{int}(\mathbf{R}_+^s)$ such that $\bar{x}$ solves the scalar problem

$$(\mathrm{P}_{\bar{\lambda}}) \qquad \inf_{x\in\mathcal{A}} \bar{\lambda}^T f\big(F(x)\big).$$

Since the constraint qualification (CQ) is fulfilled, by Theorem 3.3 there exists an optimal solution $(\widetilde{p},\widetilde{q},\widetilde{q}',\widetilde{r},\widetilde{t})$ to the dual problem $(\mathrm{D}_{\bar{\lambda}})$ such that the optimality conditions (i)–(v) are satisfied.

By using the elements $\bar{x}$ and $(\widetilde{p},\widetilde{q},\widetilde{q}',\widetilde{r},\widetilde{t})$ we can construct now an efficient solution $(\bar{p},\bar{q},\bar{q}',\bar{r},\bar{t},\bar{\lambda},\bar{u})$ to (D). In order to do this let $\bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_s)^T$

be the vector given by the proper efficiency of $\bar{x}$, $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)^T := (\widetilde{p}_1, \ldots, \widetilde{p}_n)^T = \widetilde{p}$, $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_m)^T := (\widetilde{q}_1, \ldots, \widetilde{q}_m)^T = \widetilde{q}$, $\bar{q}' = (\bar{q}'_1, \ldots, \bar{q}'_l)^T := (\widetilde{q}'_1, \ldots, \widetilde{q}'_l)^T = \widetilde{q}'$, $\bar{r} = (\bar{r}^1, \ldots, \bar{r}^s) := (\widetilde{r}^1, \ldots, \widetilde{r}^s) = \widetilde{r}$ and $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_k)^T := (\widetilde{t}_1, \ldots, \widetilde{t}_k)^T = \widetilde{t}$. It remains to define the vector $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_s)^T$. For $i = 1, \ldots, s$, let

$$\bar{u}_i := \frac{1}{s\bar{\lambda}_i}\big((\bar{t}^T g)^*(\bar{q}') + (\bar{q}^T F)^*_X(\bar{p}) + (\bar{q}'^T G)^*_X(-\bar{p})\big) + (\bar{r}^i)^T F(\bar{x}).$$

For $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})$ one has $\bar{q} \in \mathbf{R}^m_+$, $\bar{q}' \in \mathbf{R}^l_+$, $\bar{r}^i \in \mathbf{R}^m_+$, $i = 1, \ldots, s$, $\bar{t} \in \mathbf{R}^k_+$, $\bar{\lambda} \in \operatorname{int}(\mathbf{R}^s_+)$ and

$$\sum_{i=1}^s \bar{\lambda}_i \bar{u}_i = \sum_{i=1}^s \bar{\lambda}_i \frac{1}{s\bar{\lambda}_i}\Big(\big(\bar{t}^T g\big)^*(\bar{q}') + \big(\bar{q}^T F\big)^*_X(\bar{p}) + \big(\bar{q}'^T G\big)^*_X(-\bar{p})\Big)$$

$$+ \sum_{i=1}^s \bar{\lambda}_i \big(\bar{r}^i\big)^T F(\bar{x})$$

$$= \big(\bar{t}^T g\big)^*(\bar{q}') + \big(\bar{q}^T F\big)^*_X(\bar{p}) + \big(\bar{q}'^T G\big)^*_X(-\bar{p}) + \sum_{i=1}^s \bar{\lambda}_i \big(\bar{r}^i\big)^T F(\bar{x}).$$

As $\sum\limits_{i=1}^s \bar{\lambda}_i \bar{r}^i = \bar{q}$, from the optimality conditions derived in Theorem 3.3 we obtain

$$\sum_{i=1}^s \bar{\lambda}_i \bar{u}_i = \bar{q}'^T G(\bar{x}) - \bar{t}^T g\big(G(\bar{x})\big) + \bar{p}^T \bar{x} - \bar{q}^T F(\bar{x})$$

$$+ (-\bar{p})^T \bar{x} - \bar{q}'^T G(\bar{x}) + \bar{q}^T F(\bar{x}) = 0,$$

which actually means that the element $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})$ is feasible to (D).

Finally, we show that $f\big(F(\bar{x})\big) = h(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})$. By Theorem 3.3 we have for all $i = 1, \ldots, s$

$$h_i(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})$$

$$= -f_i^*(\bar{r}^i) - \frac{1}{s\bar{\lambda}_i}\big((\bar{t}^T g)^*(\bar{q}') + (\bar{q}^T F)^*_X(\bar{p}) + (\bar{q}'^T G)^*_X(-\bar{p})\big) + \bar{u}_i$$

$$= -f_i^*(\bar{r}^i) - \frac{1}{s\bar{\lambda}_i}\big((\bar{t}^T g)^*(\bar{q}') + (\bar{q}^T F)^*_X(\bar{p}) + (\bar{q}'^T G)^*_X(-\bar{p})\big)$$

$$+\frac{1}{s\bar{\lambda}_i}\Big(\big(\bar{t}^T g\big)^*(\bar{q}')+\big(\bar{q}^T F\big)_X^*(\bar{p})+\big(\bar{q}'^T G\big)_X^*(-\bar{p})\Big)+\big(\bar{r}^i\big)^T F(\bar{x})$$

$$=-f_i^*(\bar{r}^i)+\big(\bar{r}^i\big)^T F(\bar{x})=f_i\big(F(\bar{x})\big).$$

The maximality of $(\bar{p},\bar{q},\bar{q}',\bar{r},\bar{t},\bar{\lambda},\bar{u})$ follows by Theorem 4.1. □

5. Special cases

5.1. The classical multiobjective optimization problem with geometric and inequality constraints. The last section of this paper is devoted to some special cases of the primal problem (P). First, we consider the classical multiobjective optimization problem with inequality constraints

$$(\mathrm{P}') \qquad \inf_{x\in\mathcal{A}'} F(x),$$

where $\mathcal{A}'=\big\{x\in X : G(x)\leqq_{\mathbf{R}^k_+} 0\big\}$, $X\subseteqq\mathbf{R}^n$ is a convex subset,

$$F=(F_1,\ldots,F_s)^T : X\to\mathbf{R}^s,\quad G=(G_1,\ldots,G_k)^T : X\to\mathbf{R}^k$$

and F_i, $i=1,\ldots,s$, and G_j, $j=1,\ldots,k$, are convex functions.

One may observe that (P′) is a special case of the original problem (P). Taking the functions $f=(f_1,\ldots,f_s)^T:\mathbf{R}^s\to\mathbf{R}^s$ and $g=(g_1,\ldots,g_k)^T:\mathbf{R}^k\to\mathbf{R}^k$, such that $f_i(y)=y_i$ for all $y\in\mathbf{R}^s$ and $i=1,\ldots,s$, and $g_j(z)=z_j$ for all $z\in\mathbf{R}^k$ and $j=1,\ldots,k$, we actually obtain the multiobjective problem (P′). Defining f_i, $i=1,\ldots,s$, and g_j, $j=1,\ldots,k$, in this way, the functions $f=(f_1,\ldots,f_s)^T$ and $g=(g_1,\ldots,g_k)^T$ are obviously convex and componentwise increasing.

Applying the results in the previous sections, one can determine a multiobjective dual to (P′). Let us also mention that the scalarized problem becomes

$$(\mathrm{P}'_\lambda) \qquad \inf_{x\in\mathcal{A}'} \lambda^T F(x),$$

where $\lambda=(\lambda_1,\ldots,\lambda_s)^T$ is a fixed vector in $\operatorname{int}(\mathbf{R}^s_+)$, and its dual looks like (cf. (3))

$$(\mathrm{D}'_\lambda) \quad \sup_{(p,q,q',r,t)\in Y'_\lambda}\Big\{-\sum_{i=1}^s\lambda_i f_i^*(r^i)-\big(t^T g\big)^*(q')-\big(q^T F\big)_X^*(p)-\big(q'^T G\big)_X^*(-p)\Big\},$$

with

$$Y'_\lambda = \Big\{(p,q,q',r,t) : p \in \mathbf{R}^n,\ q \in \mathbf{R}^s_+,\ q' \in \mathbf{R}^k_+,\ r = (r^1,\dots,r^s),$$
$$r^i \in \mathbf{R}^s_+,\ i = 1,\dots,s,\ \sum_{i=1}^s \lambda_i r^i = q,\ t \in \mathbf{R}^k_+\Big\}.$$

Taking into consideration the definitions of the functions f_i, $i = 1,\dots,s$, and g_j, $j = 1,\dots,k$, respectively, we have for all $i = 1,\dots,s$,

$$f_i^*(r^i) = \begin{cases} 0, & \text{if } r_i^i = 1 \text{ and } r_j^i = 0, j = 1,\dots,s,\ j \neq i, \\ +\infty, & \text{otherwise}, \end{cases} \tag{11}$$

$$(t^T g)(q') = \begin{cases} 0, & \text{if } q' = t, \\ +\infty, & \text{otherwise}, \end{cases} \tag{12}$$

and

$$(q^T F)_X^*(p) = \left(\left(\sum_{i=1}^s \lambda_i r^i\right)^T F\right)_X^*(p) = (\lambda^T F)_X^*(p). \tag{13}$$

Thus (D'_λ) becomes

$$(14)\ (\mathrm{D}'_\lambda) \qquad \sup_{p \in \mathbf{R}^n,\ t \in \mathbf{R}^k_+} \left\{ -(\lambda^T F)_X^*(p) - (t^T G)_X^*(-p) \right\}.$$

Let us notice that (D'_λ) is nothing else but the so-called Fenchel—Lagrange dual problem which has proved to be useful in studying the duality in vector optimization (cf. [17], [18]).

The constraint qualification which will guarantee the existence of strong duality becomes

$$(\mathrm{CQ}') \qquad \exists\, x' \in \operatorname{ri}(X) : \begin{cases} G_i(x') \leqq 0, & i \in L, \\ G_i(x') < 0, & i \in N, \end{cases}$$

where

$$L := \left\{ i \in \{1,\dots,k\} \;\middle|\; \begin{array}{l} G_i : X \to \mathbf{R} \text{ is the restriction to } X \text{ of an} \\ \text{affine function } \widetilde{G}_i : \mathbf{R}^n \to \mathbf{R} \end{array} \right\}$$

and $N := \{1,\dots,k\} \setminus L$.

The vector dual problem of (P′) can be equivalently written as

$$(\mathrm{D}') \qquad \underset{(p,t,\lambda,u)\in\mathcal{B}'}{\text{v-max}}\, h'(p,t,\lambda,u), \qquad \text{with} \qquad h'(p,t,\lambda,u)=\begin{pmatrix} h'_1(p,t,\lambda,u)\\ \vdots \\ h'_s(p,t,\lambda,u)\end{pmatrix},$$

$$h'_i(p,t,\lambda,u) = -\frac{1}{s\lambda_i}\big((\lambda^T F)^*_X(p) + (t^T G)^*_X(-p)\big) + u_i, \quad i=1,\dots,s,$$

the dual variables

$$p=(p_1,\dots,p_n)^T\in\mathbf{R}^n, \quad t=(t_1,\dots,t_k)^T\in\mathbf{R}^k, \quad \lambda=(\lambda_1,\dots,\lambda_s)^T\in\mathbf{R}^s,$$
$$u=(u_1,\dots,u_s)^T\in\mathbf{R}^s,$$

and the set of constraints

$$\mathcal{B}'=\left\{(p,t,\lambda,u)\,:\, t\in\mathbf{R}^k_+,\ \lambda\in\operatorname{int}(\mathbf{R}^s_+),\ \sum_{i=1}^s \lambda_i u_i = 0\right\}.$$

For an overview of multiobjective dual problems for (P′) see [2] and [3].

The next two theorems yield the weak and strong duality for the multiobjective problems (P′) and (D′) and can be derived from Theorems 4.1 and 4.2.

THEOREM 5.1. *There is no $x\in\mathcal{A}'$ and no $(p,t,\lambda,u)\in\mathcal{B}'$ fulfilling $F(x) \leqq_{\mathbf{R}^s_+} h'(p,t,\lambda,u)$ and $F(x)\neq h'(p,t,\lambda,u)$.*

THEOREM 5.2. *Assume that the constraints qualification (CQ′) is fulfilled and let $\bar{x}$ be a properly efficient element to (P′). Then there exists an efficient solution $(\bar{p},\bar{t},\bar{\lambda},\bar{u})\in\mathcal{B}'$ to the dual (D′) and the strong duality $F(\bar{x}) = h(\bar{p},\bar{t},\bar{\lambda},\bar{u})$ holds.*

5.2. The multiobjective composed optimization problem with geometric constraints. In this subsection we consider the multiobjective optimization problem only with geometric constraints

$$(\mathrm{P}'') \qquad \underset{x\in X}{\text{v-min}}\, f\big(F(x)\big),$$

where $X\subseteqq\mathbf{R}^n$, $F=(F_1,\dots,F_m)^T : X\to\mathbf{R}^m$ and $f=(f_1,\dots,f_s)^T : \mathbf{R}^m\to\mathbf{R}^s$. Assume that F_i, $i=1,\dots,m$, are convex and f_j, $j=1,\dots,s$, are convex and componentwise increasing functions.

Problem (P″) was already treated by the authors in [19], the purpose hereby is to show how the results obtained in [19] can be obtained, as special case, from the general results formulated in Sections 3 and 4 of this paper. To this end, let us notice that problem (P$''$) can be obtained from (P) by taking the functions $g_i(y)=0$, for all $i=1,\dots,k$ and $y\in \mathbf{R}^l$. Analogously to the previous sections first we give the dual of the scalar primal problem

$$(\mathrm{P}''_\lambda) \qquad \inf_{x\in X} \lambda^T f\big(F(x)\big),$$

associated to (P$''$) where $\lambda=(\lambda_1,\dots,\lambda_s)^T\in \operatorname{int}(\mathbf{R}^s_+)$ is a fixed vector. By (3), the dual of (P$''_\lambda$) is

$$(\mathrm{D}''_\lambda) \qquad \sup_{(p,q,q',r,t)\in Y''_\lambda} \left\{-\sum_{i=1}^s \lambda_i f_i^*(r^i)-(t^Tg)^*(q')-(q^TF)^*_X(p)-(q'^TG)^*_X(-p)\right\},$$

with

$$Y''_\lambda=\bigg\{(p,q,q',r,t): p\in\mathbf{R}^n,\ q\in\mathbf{R}^m_+,\ q'\in\mathbf{R}^l_+,\ r=(r^1,\dots,r^s),$$
$$r^i\in\mathbf{R}^m_+,\ i=1,\dots,s,\ \sum_{i=1}^s\lambda_i r^i=q,\ t\in\mathbf{R}^k_+\bigg\}.$$

Since in this case

$$(t^Tg)^*(q')=(0)^*(q')=\sup_{y\in\mathbf{R}^l}\{y^Tq'\}=\begin{cases}0, & \text{if } q'=0,\\ +\infty, & \text{otherwise,}\end{cases}$$

and therefore

$$(q'^TG)^*_X(-p)=0^*_X(-p)=-\inf_{x\in X}p^Tx=\delta^*_X(-p),$$

the dual problem becomes

$$(\mathrm{D}''_\lambda) \qquad \sup_{\substack{p\in\mathbf{R}^n,\ q\in\mathbf{R}^m_+,\ r^i\in\mathbf{R}^m_+,\\ i=1,\dots,s,\ \sum_{i=1}^s\lambda_i r^i=q}} \left\{-\sum_{i=1}^s\lambda_i f_i^*(r^i)-(q^TF)^*_X(p)-\delta^*_X(-p)\right\}.$$

Let us mention that this dual has been introduced by Boţ and Wanka in [1].

The multiobjective dual to (P$''$) is then

$$(\mathrm{D}'') \qquad \operatorname*{v-max}_{(p,q,r,\lambda,u)\in\mathcal{B}''} h''(p,q,r,\lambda,u),$$

with

$$h''(p,q,r,\lambda,u) = \begin{pmatrix} h''_1(p,q,r,\lambda,u) \\ \vdots \\ h''_s(p,q,r,\lambda,u) \end{pmatrix},$$

$$h''_i(p,q,r,\lambda,u) = -f_i^*(r^i) - \frac{1}{s\lambda_i}\big((q^T F)_X^*(p) + \delta_X^*(-p)\big) + u_i, \quad i = 1,\dots,s,$$

the dual variables

$$p = (p_1,\dots,p_n)^T \in \mathbf{R}^n,\ q = (q_1,\dots,q_m)^T \in \mathbf{R}^m,\ r = (r^1,\dots,r^s),\ r^i \in \mathbf{R}^m,$$

$$i = 1,\dots,s,\ \lambda = (\lambda_1,\dots,\lambda_s)^T \in \mathbf{R}^s,\ u = (u_1,\dots,u_s)^T \in \mathbf{R}^s,$$

and the set of constraints

$$\mathcal{B}'' = \bigg\{(p,q,r,\lambda,u) : q \in \mathbf{R}^m_+,\ r^i \in \mathbf{R}^m_+,\ i = 1,\dots,s,\ \lambda \in \operatorname{int}(\mathbf{R}^s_+),$$

$$\sum_{i=1}^s \lambda_i r^i = q,\ \sum_{i=1}^s \lambda_i u_i = 0\bigg\}.$$

The next two theorems provide the weak and strong duality for the multiobjective problems (P$''$) and (D$''$) and can be derived from Theorem 4.1 and Theorem 4.2.

THEOREM 5.3. *There is no $x \in X$ and no $(p,q,r,\lambda,u) \in \mathcal{B}''$ fulfilling $f\big(F(x)\big) \leqq_{\mathbf{R}^s_+} h''(p,q,r,\lambda,u)$ and $f\big(F(x)\big) \neq h''(p,q,r,\lambda,u)$.*

THEOREM 5.4. *Let $\bar{x}$ be a properly efficient element to* (P$''$). *Then there exists an efficient solution $(\bar{p},\bar{q},\bar{r},\bar{\lambda},\bar{u}) \in \mathcal{B}''$ to the dual* (D$''$) *and the strong duality $f\big(F(\bar{x})\big) = h''(\bar{p},\bar{q},\bar{r},\bar{\lambda},\bar{u})$ holds.*

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