Acta Math. Hungar., **111 (1–2)** (2006), 165–179.
DOI: 10.1556/AMath.111.2006.1-2.12

COMPARISON THEOREMS FOR OSCILLATION OF SECOND-ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS

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(Received October 29, 2004; revised March 7, 2005; accepted March 24, 2005)

Abstract. We establish new comparison theorems on the oscillation of solutions of a class of perturbed half-linear differential equations. These improve the work of Elbert and Schneider [6] in which connections are found between half-linear differential equations and linear differential equations. Our comparison theorems are not of Sturm type or Hille–Wintner type which are very famous. We can apply the main results in combination with Sturm's or Hille–Wintner's comparison theorem to a half-linear differential equation of the general form $\left(|x'|^{\alpha-1}x'\right)' + a(t)|x|^{\alpha-1}x = 0$.

1. Introduction

Over the past four decades a great deal of articles have been devoted to the study of oscillation of solutions of half-linear differential equations. For example, those results can be found in [1, 2, 3, 4, 5, 6, 9, 10, 11, 12]. Especially, it is well-known that all nontrivial solutions of a half-linear differential

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*Supported in part by Grant-in-Aid for Scientific Research 16540152.

Key words and phrases: oscillation, comparison theorems, half-linear differential equations, perturbation, differential inequalities.

2000 Mathematics Subject Classification: primary 34C10, 34C11; secondary 34C15.

equation of the form

$$(1.1) \qquad \left(|x'|^{\alpha-1}x'\right)' + \frac{\lambda}{t^{\alpha+1}}|x|^{\alpha-1}x = 0, \qquad t > t_0$$

with $\alpha > 0$, $\lambda > 0$ and $t_0 \geqq 0$, are oscillatory if

$$\lambda > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1};$$

otherwise, they are nonoscillatory. This fact means that $\left(\alpha/(\alpha+1)\right)^{\alpha+1}$ is the lower bound for all nontrivial solutions of (1.1) to be oscillatory. Such a number is generally called the *oscillation constant* (for example, see [7, 8, 14, 15, 16]).

Let us add a perturbation to equation (1.1) when λ is the oscillation constant and consider the perturbed half-linear differential equation

$$(\mathrm{E}_\alpha) \qquad \left(|x'|^{\alpha-1}x'\right)' + \frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\delta(t)\right\}|x|^{\alpha-1}x = 0,$$

where $\delta(t)$ is positive and continuous on some half-line (t_0, ∞). Elbert and Schneider [6] have investigated the asymptotic behaviour of solutions of (E_α). Using their results, we can present the following statements.

THEOREM A. *Let $\alpha > 1$. If equation (E_α) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_1) are oscillatory.*

THEOREM B. *Let $0 < \alpha < 1$. If equation (E_1) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_α) are oscillatory.*

REMARK 1.1. Sturm's separation theorem holds for half-linear differential equations as well as for linear differential equations. Hence, if there exists an oscillatory (respectively, a nonoscillatory) solution of (E_α), then all nontrivial solutions of (E_α) are oscillatory (respectively, nonoscillatory). In other words, oscillatory solutions and nonoscillatory solutions cannot coexist in equation (E_α).

It follows from the fact mentioned in the first paragraph and Sturm's comparison theorem for half-linear differential equations that if

$$(1.2) \qquad \liminf_{t\to\infty} \delta(t) > 0,$$

then all nontrivial solutions of (E_α) are oscillatory. As to Sturm's separation and comparison theorems, see for example [5, 12, 11]. On the other hand, if condition (1.2) fails to hold, then there is some possibility that equation (E_α)

has a nonoscillatory solution. One of the most interesting cases is that $\delta(t) = \lambda/(\log t)^2$ with $\lambda > 0$. In this case, if $\lambda > 1/2$, then all nontrivial solutions of (E_α) are oscillatory; otherwise, they are nonoscillatory (for details, see [6]).

We may regard Theorems A and B as comparison theorems between the linear differential equation

$$x'' + \frac{1}{t^2}\left\{\frac{1}{4} + \frac{1}{2}\delta(t)\right\}x = 0 \tag{E_1}$$

and half-linear differential equations of the form (E_α). Let α and β be positive numbers satisfying $\alpha < 1 < \beta$. Then, combining Theorems A and B, we get the following conclusion: if equation (E_β) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_α) are oscillatory. A natural question now arises as to whether or not the converse proposition is also true.

The first purpose of this paper is to extend Theorems A and B to a comparison theorem between any two half-linear differential equations. The second purpose is to give an answer to the above question. Our main results are stated as follows:

THEOREM 1.1. *Let* $0 < \alpha < \beta$. *If equation* (E_β) *has a nontrivial oscillatory solution, then all nontrivial solutions of* (E_α) *are oscillatory.*

REMARK 1.2. Theorem 1.1 is a generalization of Theorems A and B. To put it precisely, Theorem 1.1 coincides with Theorem A (respectively, Theorem B) when $\alpha = 1$ (respectively, $\beta = 1$).

THEOREM 1.2. *Let* $0 < \alpha < \beta$. *If equation* (E_α) *has a nontrivial oscillatory solution, then all nontrivial solutions of*

$$\left(|x'|^{\beta-1}x'\right)' + \frac{1}{t^{\beta+1}}\left\{\left(\frac{\beta}{\beta+1}\right)^{\beta+1} + \nu\delta(t)\right\}|x|^{\beta-1}x = 0 \tag{1.3}$$

are oscillatory, where $\nu > \left(\beta/(\beta+1)\right)^{\beta}$.

REMARK 1.3. It is essential that ν is greater than $\left(\beta/(\beta+1)\right)^{\beta}$ in Theorem 1.2. Unfortunately, even if equation (E_α) has a nontrivial oscillatory solution, we cannot judge whether all nontrivial solutions of (E_β) are oscillatory or not.

REMARK 1.4. From Theorems 1.1 and 1.2, we see that the oscillation constant for equation (E_α) with $\delta(t) = \lambda/(\log t)^2$ is $1/2$ for any $\alpha > 0$ (for a detailed explanation, see Section 4).

2. Riccati technique

Consider the half-linear differential equation

$$(2.1)\qquad \left(|x'|^{p-1}x'\right)' + \frac{1}{t^{p+1}}\left\{\left(\frac{p}{p+1}\right)^{p+1} + h(t)\right\}|x|^{p-1}x = 0$$

with $p > 0$ a fixed real number, where $h(t)$ is positive and continuous on $(0, \infty)$. Using Riccati's transformation, we prepare some lemmas below. To this end, we denote

$$H_p(\xi) = p\left\{\xi^{(p+1)/p} - \xi + \frac{p^p}{(p+1)^{p+1}}\right\}$$

for $\xi > 0$ and

$$\gamma_p = \left(\frac{p}{p+1}\right)^p.$$

Remark 2.1. The number γ_p is decreasing with respect to p. In fact, if $f(x) = x\bigl(\log x - \log(x+1)\bigr)$ for $x > 0$, then we have

$$\frac{d}{dx}f(x) = \log\frac{x}{x+1} + 1 - \frac{x}{x+1} < 0$$

for $x > 0$. Hence, we obtain

$$\log\gamma_\beta = f(\beta) < f(\alpha) = \log\gamma_\alpha < 0,$$

namely, $\gamma_\beta < \gamma_\alpha < 1$ for any α and β satisfying $0 < \alpha < \beta$. It is also clear that $\gamma_p > 1/e$ for all $p > 0$.

Lemma 2.1. *Let $\xi(s)$ be a positive function on $[s_0, \infty)$ with $s_0 > 0$ satisfying*

$$(2.2)\qquad \dot{\xi}(s) + H_p\bigl(\xi(s)\bigr) \leqq 0.$$

Then it is nonincreasing and tends to γ_p as $s \to \infty$.

Proof. From

$$H_p(\gamma_p) = p\left\{\left(\frac{p}{p+1}\right)^{p+1} - \left(\frac{p}{p+1}\right)^p + \frac{p^p}{(p+1)^{p+1}}\right\} = 0$$

and

$$\frac{d}{d\xi}H_p(\xi) = (p+1)\xi^{1/p} - p,$$

we see that $H_p(\xi) \geqq 0$ for $\xi > 0$ and $H_p(\xi) = 0$ if and only if $\xi = \gamma_p$.

Since $\xi(s)$ is positive for $s \geqq s_0$, we have

$$\dot{\xi}(s) \leqq -H_p\big(\xi(s)\big) \leqq 0$$

by (2.2), namely, $\xi(s)$ is nonincreasing. Hence, there exists a $\mu \geqq 0$ such that $\xi(s) \searrow \mu$ as $s \to \infty$. Suppose that $\mu \neq \gamma_p$. If $\mu > \gamma_p$, then $\xi(s) > \mu > (\mu + \gamma_p)/2 > \gamma_p$ for $s \geqq s_0$. If $\mu < \gamma_p$, then $\mu < \xi(s) < (\mu + \gamma_p)/2 < \gamma_p$ for s sufficiently large. In either case,

$$\dot{\xi}(s) \leqq -H_p\big(\xi(s)\big) \leqq -H_p\big((\mu + \gamma_p)/2\big) < 0$$

for s sufficiently large, which yields that $\xi(s)$ tends to $-\infty$ as $s \to \infty$. This contradicts the assumption that $\xi(s)$ is positive for $s \geqq s_0$. Thus, $\xi(s)$ tends to γ_p as $s \to \infty$. $\square$

We next give a sufficient condition for all nontrivial solutions of (2.1) to be nonoscillatory.

LEMMA 2.2. *Let $\xi(s)$ be a positive function on $[s_0, \infty)$ with $s_0 > 0$ satisfying*

$$\dot{\xi}(s) + H_p\big(\xi(s)\big) + h(e^s) \leqq 0, \tag{2.3}$$

where h is the function defined in equation (2.1). *Then all nontrivial solutions of* (2.1) *are nonoscillatory.*

PROOF. Define $c(s) = -\dot{\xi}(s) - H_p\big(\xi(s)\big)$ for $s \geqq s_0$. Then we have

$$c(s) \geqq h(e^s) \quad \text{for} \quad s \geqq s_0. \tag{2.4}$$

Let $u(s)$ be the positive function defined by

$$u(s) = \exp\left(\int_{s_0}^{s} \xi(\sigma)^{1/p} d\sigma\right)$$

for $s \geqq s_0$. Then we get $\dot{u}(s) = u(s)\xi(s)^{1/p} > 0$ for $s \geqq s_0$, namely,

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)}\right)^p \quad \text{for} \quad s \geqq s_0.$$

Differentiate $\xi(s)$ to obtain

$$\dot{\xi}(s) = \frac{\big(\dot{u}(s)^p\big)^{\cdot}\, u(s)^p - pu(s)^{p-1}\dot{u}(s)^{p+1}}{u(s)^{2p}} = \frac{\big(\dot{u}(s)^p\big)^{\cdot}}{u(s)^p} - p\left(\frac{\dot{u}(s)}{u(s)}\right)^{p+1}$$

for $s \geqq s_0$. Hence, we have

$$c(s) = -\frac{\left(\dot{u}(s)^p\right)^{\cdot}}{u(s)^p} + p\left(\frac{\dot{u}(s)}{u(s)}\right)^{p+1} - p\left\{\left(\frac{\dot{u}(s)}{u(s)}\right)^{p+1} - \left(\frac{\dot{u}(s)}{u(s)}\right)^{p} + \frac{p^p}{(p+1)^{p+1}}\right\}$$

$$= -\frac{\left(\dot{u}(s)^p\right)^{\cdot}}{u(s)^p} + p\left(\frac{\dot{u}(s)}{u(s)}\right)^{p} - \left(\frac{p}{p+1}\right)^{p+1},$$

and therefore, we see that the positive function $u(s)$ is a nonoscillatory solution of the equation

$$\left(|\dot{u}|^{p-1}\dot{u}\right)^{\cdot} - p|\dot{u}|^{p-1}\dot{u} + \left\{\left(\frac{p}{p+1}\right)^{p+1} + c(s)\right\}|u|^{p-1}u = 0. \tag{2.5}$$

Changing variable $t = e^s$, we can transform equation (2.5) into the equation

$$\left(|x'|^{p-1}x'\right)' + \frac{1}{t^{p+1}}\left\{\left(\frac{p}{p+1}\right)^{p+1} + c(\log t)\right\}|x|^{p-1}x = 0. \tag{2.6}$$

Let $x(t)$ be the solution of (2.6) corresponding to $u(s)$. Then $x(t)$ is positive for $t \geqq e^{s_0}$. From (2.4) it follows that $c(\log t) \geqq h(t)$ for $t \geqq e^{s_0}$. Hence, by Sturm's comparison theorem for half-linear differential equations, all nontrivial solutions of (2.1) are nonoscillatory. $\square$

3. Proof of the main theorems

By means of Lemmas 2.1 and 2.2, we can prove our comparison theorems for half-linear differential equations of the form (E_α).

Proof of Theorem 1.1. By way of contradiction, we suppose that equation (E_β) has an oscillatory solution and equation (E_α) has a nonoscillatory solution $x(t)$. We may assume that $x(t)$ is eventually positive, because the proof of the case that $x(t)$ is eventually negative is carried out in the same way. Hence, there exists a $T > t_0$ such that $x(t) > 0$ for $t \geqq T$, and therefore,

(3.1)
$$\left(\left|x'(t)\right|^{\alpha-1}x'(t)\right)' = -\frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \gamma_\alpha\delta(t)\right\}\left|x(t)\right|^{\alpha-1}x(t) < 0$$

for $t \geqq T$. From this we see that $x'(t)$ is also positive for $t \geqq T$. In fact, if there exists a $t_1 \geqq T$ such that $x'(t_1) \leqq 0$, then by (3.1) we have

$$\left|x'(t)\right|^{\alpha-1}x'(t) < \left|x'(t_1)\right|^{\alpha-1}x'(t_1) \leqq 0$$

for $t > t_1$. Hence, we can find a $t_2 > t_1$ such that $x'(t_2) < 0$. By (3.1) again, we obtain

$$\left|x'(t)\right|^{\alpha-1}x'(t) \leqq \left|x'(t_2)\right|^{\alpha-1}x'(t_2) < 0$$

for $t \geqq t_2$. We therefore conclude that $x'(t) \leqq x'(t_2) < 0$ for $t \geqq t_2$, which implies that

$$x(t) \leqq x'(t_2)(t - t_2) + x(t_2) \to -\infty$$

as $t \to \infty$. This is a contradiction to the assumption that $x(t)$ is eventually positive.

Making the change of variable $s = \log t$, we can rewrite equation (E_α) in the form

$$\left(|\dot{u}|^{\alpha-1}\dot{u}\right)^{\boldsymbol{\cdot}} - \alpha|\dot{u}|^{\alpha-1}\dot{u} + \left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \gamma_\alpha\delta(e^s)\right\}|u|^{\alpha-1}u = 0. \tag{3.2}$$

Let $u(s)$ be the solution of (3.2) which corresponds to $x(t)$. Then $u(s) = x(t) > 0$ and $\dot{u}(s) = tx'(t) > 0$ for $s \geqq \log T$. Define

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)}\right)^\alpha$$

and differentiate $\xi(s)$ to obtain

$$\dot{\xi}(s) = \frac{\left(\dot{u}(s)^\alpha\right)^{\boldsymbol{\cdot}}}{u(s)^\alpha} - \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^{\alpha+1}.$$

Using (3.2), we have

$$\begin{aligned}\dot{\xi}(s) &= \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^\alpha - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} - \gamma_\alpha\delta(e^s) - \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^{\alpha+1} \\ &= -\alpha\left\{\xi(s)^{(\alpha+1)/\alpha} - \xi(s) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\right\} - \gamma_\alpha\delta(e^s) \\ &= -H_\alpha\big(\xi(s)\big) - \gamma_\alpha\delta(e^s)\end{aligned} \tag{3.3}$$

for $s \geqq \log T$.

Here we show that there exists an $\varepsilon_0 > 0$ such that

$$\frac{\gamma_\alpha}{\gamma_\beta} H_\beta \left(\frac{\gamma_\beta}{\gamma_\alpha} \xi \right) \leqq H_\alpha(\xi) \tag{3.4}$$

for $\gamma_\alpha \leqq \xi \leqq \gamma_\alpha + \varepsilon_0$. For this purpose, define

$$F_1(\xi) = H_\alpha(\xi) - \frac{\gamma_\alpha}{\gamma_\beta} H_\beta \left(\frac{\gamma_\beta}{\gamma_\alpha} \xi \right).$$

Then, differentiating $F_1(\xi)$ three times, we obtain

$$\frac{d}{d\xi} F_1(\xi) = (\alpha + 1)\xi^{1/\alpha} - \alpha - (\beta + 1) \left(\frac{\gamma_\beta}{\gamma_\alpha} \right)^{1/\beta} \xi^{1/\beta} + \beta,$$

$$\frac{d^2}{d\xi^2} F_1(\xi) = \frac{\alpha + 1}{\alpha} \xi^{(1-\alpha)/\alpha} - \frac{\beta + 1}{\beta} \left(\frac{\gamma_\beta}{\gamma_\alpha} \right)^{1/\beta} \xi^{(1-\beta)/\beta},$$

$$\frac{d^3}{d\xi^3} F_1(\xi) = \frac{1 - \alpha^2}{\alpha^2} \xi^{(1-2\alpha)/\alpha} - \frac{1 - \beta^2}{\beta^2} \left(\frac{\gamma_\beta}{\gamma_\alpha} \right)^{1/\beta} \xi^{(1-2\beta)/\beta},$$

so that

$$F_1(\gamma_\alpha) = \frac{d}{d\xi} F_1(\xi) \bigg|_{\xi = \gamma_\alpha} = \frac{d^2}{d\xi^2} F_1(\xi) \bigg|_{\xi = \gamma_\alpha} = 0 \tag{3.5}$$

and

$$\frac{d^3}{d\xi^3} F_1(\xi) \bigg|_{\xi = \gamma_\alpha} = \frac{\beta - \alpha}{\alpha\beta} \left(\frac{\alpha + 1}{\alpha} \right)^{2\alpha} > 0. \tag{3.6}$$

From (3.6) we can choose an $\varepsilon_0 > 0$ such that

$$\frac{d^3}{d\xi^3} F_1(\xi) > 0 \quad \text{for} \quad \gamma_\alpha \leqq \xi \leqq \gamma_\alpha + \varepsilon_0.$$

Hence, taking account of this estimation and (3.5), we see that $F_1(\xi) \geqq 0$ for $\gamma_\alpha \leqq \xi \leqq \gamma_\alpha + \varepsilon_0$, as required.

Because of (3.3), Lemma 2.1 is available for $p = \alpha$ and $s_0 = \log T$, and therefore, there exists an $s_1 > s_0$ such that $\gamma_\alpha \leqq \xi(s) \leqq \gamma_\alpha + \varepsilon_0$ for $s \geqq s_1$. Hence, together with (3.3) and (3.4), we get

$$\dot{\xi}(s) + \frac{\gamma_\alpha}{\gamma_\beta} H_\beta \left(\frac{\gamma_\beta}{\gamma_\alpha} \xi(s) \right) + \gamma_\alpha \delta(e^s) \leqq 0$$

for $s \geqq s_1$. Let $\eta(s) = \gamma_\beta \xi(s)/\gamma_\alpha$. Then we see that $\eta(s)$ satisfies

$$\dot{\eta}(s) + H_\beta\big(\eta(s)\big) + \gamma_\beta \delta(e^s) \leqq 0$$

for $s \geqq s_1$. Hence, from Lemma 2.2 with $p = \beta$ and $h(e^s) = \gamma_\beta \delta(e^s)$ we conclude that all nontrivial solutions of (E_β) are nonoscillatory. This contradicts the assumption that equation (E_β) has an oscillatory solution. □

PROOF OF THEOREM 1.2. Suppose to the contrary that equation (E_α) has an oscillatory solution and equation (1.3) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. Let $T > t_0$ be a number satisfying $x(t) > 0$ for $t \geqq T$. The same manner as in the proof of Theorem 1.1, we see that $x'(t)$ is also positive for $t \geqq T$.

By putting $t = e^s$, equation (1.3) becomes

$$\big(|\dot{u}|^{\beta-1}\dot{u}\big)^{\boldsymbol{\cdot}} - \beta|\dot{u}|^{\beta-1}\dot{u} + \left\{ \left(\frac{\beta}{\beta+1}\right)^{\beta+1} + (\gamma_\beta + \varepsilon)\delta(e^s) \right\} |u|^{\beta-1}u = 0$$

for some $\varepsilon > 0$, where $u(s) = x(e^s) = x(t)$. Define

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)}\right)^\beta,$$

which is positive for $s \geqq \log T$. A simple calculation shows that

$$\dot{\xi}(s) = -H_\beta\big(\xi(s)\big) - (\gamma_\beta + \varepsilon)\delta(e^s) \tag{3.7}$$

for $s \geqq \log T$. Hence, it follows from Lemma 2.1 with $p = \beta$ and $s_0 = \log T$ that

$$\xi(s) \searrow \gamma_\beta \quad \text{as} \quad s \to \infty. \tag{3.8}$$

Let

$$c = \frac{\gamma_\beta + \varepsilon}{\gamma_\alpha} \quad \text{and} \quad \eta(s) = \frac{\xi(s) + \varepsilon}{c}.$$

Then, from (3.7) and (3.8) it turns out that

$$\dot{\eta}(s) + \frac{1}{c} H_\beta(c\eta(s) - \varepsilon) + \gamma_\alpha \delta(e^s) = 0 \tag{3.9}$$

for $s \geqq s_0$ and

$$\eta(s) \searrow \gamma_\alpha \quad \text{as} \quad s \to \infty, \tag{3.10}$$

respectively.

To show that there exists an $\varepsilon_0 > 0$ such that

$$H_\alpha(\eta) \leqq \frac{1}{c} H_\beta(c\eta - \varepsilon) \tag{3.11}$$

for $\gamma_\alpha \leqq \eta \leqq \gamma_\alpha + \varepsilon_0$, we define

$$F_2(\eta) = \frac{1}{c} H_\beta(c\eta - \varepsilon) - H_\alpha(\eta).$$

Differentiating $F_2(\eta)$ twice, we have

$$\frac{d}{d\eta} F_2(\eta) = (\beta+1)(c\eta - \varepsilon)^{1/\beta} - \beta - (\alpha+1)\eta^{1/\alpha} + \alpha,$$

$$\frac{d^2}{d\eta^2} F_2(\eta) = \frac{c(\beta+1)}{\beta}(c\eta - \varepsilon)^{(1-\beta)/\beta} - \frac{\alpha+1}{\alpha}\eta^{(1-\alpha)/\alpha},$$

so that

$$F_2(\gamma_\alpha) = \frac{d}{d\xi} F_2(\eta)\bigg|_{\eta=\gamma_\alpha} = 0$$

and

$$\frac{d^2}{d\xi^2} F_2(\eta)\bigg|_{\eta=\gamma_\alpha} = \frac{\varepsilon}{\gamma_\alpha \gamma_\beta} > 0.$$

Hence, we can select an $\varepsilon_0 > 0$ such that

$$\frac{d^2}{d\xi^2} F_2(\eta) > 0 \quad \text{for} \quad \gamma_\alpha \leqq \eta \leqq \gamma_\alpha + \varepsilon_0,$$

and therefore, $F_2(\eta) \geqq 0$ for $\gamma_\alpha \leqq \eta \leqq \gamma_\alpha + \varepsilon_0$. Thus, the inequality (3.11) is shown.

By (3.10), there exists an $s_1 > s_0$ such that $\gamma_\alpha \leqq \eta(s) \leqq \gamma_\alpha + \varepsilon_0$ for $s \geqq s_1$. Hence, together with (3.9) and (3.11), we have

$$\dot\eta(s) + H_\alpha\big(\eta(s)\big) + \gamma_\alpha \delta(e^s) \leqq 0$$

for $s \geqq s_1$. Using Lemma 2.2 with $p = \alpha$ and $h(e^s) = \gamma_\alpha \delta(e^s)$, we see that all nontrivial solutions of (E_α) are nonoscillatory. This is a contradiction to the assumption that equation (E_α) has an oscillatory solution. □

4. Discussion and another comparison theorem

From Theorem 1.1 it turns out that if equation

$$(\mathrm{E}_3) \qquad \big((x')^3\big)' + \frac{1}{t^4}\left\{\left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^3 \delta(t)\right\} x^3 = 0$$

has a nontrivial oscillatory solution, then all nontrivial solutions of

$$(\mathrm{E}_2) \qquad \big(|x'|x'\big)' + \frac{1}{t^3}\left\{\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 \delta(t)\right\} |x|x = 0$$

are oscillatory. Indeed, as mentioned in Section 1, all nontrivial solutions of (E_3) and those of (E_2) are oscillatory in the case that $\delta(t) = \lambda/(\log t)^2$ with $\lambda > 1/2$. However, the above relation between equations (E_3) and (E_2) is not made clear by Theorems A and B. To be precise, Theorems A and B are inapplicable to the case that both α and β are greater (respectively, less) than 1.

When $\alpha = 1$ and $\delta(t) = \lambda/(\log t)^2$, equation ($\mathrm{E}_\alpha$) becomes the linear equation

$$(4.1) \qquad x'' + \frac{1}{t^2}\left\{\frac{1}{4} + \frac{\lambda}{2(\log t)^2}\right\} x = 0,$$

which is called the Riemann–Weber version of Euler differential equations. It is well-known that if $\lambda > 1/2$, then all nontrivial solutions of (4.1) are oscillatory; otherwise they are nonoscillatory (see, for example [7, 13, 14, 16, 17]). To put it another way, the oscillation constant for equation (4.1) is $1/2$.

We will show that the situation for equation (E_α) with $\delta(t) = \lambda/(\log t)^2$ is the same as that for equation (4.1). Suppose that $\lambda > 1/2$. In the case that $0 < \alpha < 1$, it is clear from Theorem 1.1 ($\beta = 1$) that all nontrivial solutions of (E_α) with $\delta(t) = \lambda/(\log t)^2$ are oscillatory. Consider the case that $\alpha > 1$. Let $\bar\lambda$ be a number satisfying $\lambda > \bar\lambda > 1/2$. Then equation ($\mathrm{E}_1$) with $\delta(t) = \bar\lambda/(\log t)^2$ has a nontrivial oscillatory solution. Hence, by Theorem 1.2 ($\alpha = 1$ and β is rewritten as α), all nontrivial solutions of the equation

$$(4.2) \qquad \big(|x'|^{\alpha-1}x'\big)' + \frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \frac{\nu\bar\lambda}{(\log t)^2}\right\} |x|^{\alpha-1}x = 0$$

are oscillatory, where $\nu = \big(\alpha/(\alpha+1)\big)^{\alpha}\lambda/\bar\lambda > \big(\alpha/(\alpha+1)\big)^{\alpha}$. Since $\nu\bar\lambda = \big(\alpha/(\alpha+1)\big)^{\alpha}\lambda$, equation (4.2) is equivalent to equation (E_α) with $\delta(t)$

$= \lambda/(\log t)^2$. Thus, if $\delta(t) = \lambda/(\log t)^2$ with $\lambda > 1/2$, then all nontrivial solutions of (E_α) are oscillatory for any $\alpha > 0$. Next, we suppose that $0 < \lambda < 1/2$. In the case that $\alpha > 1$, it is obvious from the contraposition of Theorem 1.1 ($\alpha = 1$ and β is rewritten as α) that all nontrivial solutions of (E_α) with $\delta(t) = \lambda/(\log t)^2$ are nonoscillatory. Consider the case that $0 < \alpha < 1$. We can choose a $\hat{\lambda}$ such that $\lambda < \hat{\lambda} < 1/2$. Since $\hat{\lambda} < 1/2$, equation (E_1) with $\delta(t) = \hat{\lambda}/(\log t)^2$ has a nonoscillatory solution. Let $\nu = \hat{\lambda}/(2\lambda) > 1/2$. Then equation ($\mathrm{E}_1$) with $\delta(t) = \hat{\lambda}/(\log t)^2$ coincides with

$$x'' + \frac{1}{t^2}\left\{\frac{1}{4} + \frac{\nu\lambda}{(\log t)^2}\right\}x = 0.$$

Hence, by the contraposition of Theorem 1.2 ($\beta = 1$), all nontrivial solutions of (E_α) with $\delta(t) = \lambda/(\log t)^2$ are nonoscillatory. Thus, if $\delta(t) = \lambda/(\log t)^2$ with $0 < \lambda < 1/2$, then all nontrivial solutions of (E_α) are nonoscillatory for any $\alpha > 0$.

REMARK 4.1. Elbert and Schneider [6, Corollary 2] have already shown that if $\delta(t) = \lambda/(\log t)^2$, then the oscillation constant for equation (E_α) is 1/2 (their original statement is written in a slightly different form).

Let us now look at Theorem 1.2 from a different angle. To this end, consider the more general half-linear differential equation

$$\left(|x'|^{\alpha-1}x'\right)' + a(t)|x|^{\alpha-1}x = 0, \tag{4.3}$$

where $\alpha > 0$ and $a(t)$ is positive and continuous on (t_0, ∞) for some $t_0 \geqq 0$. Then, all solutions of (4.3) are continuable in the future (refer to [5] for details). Hence, it is worth while to discuss whether solutions of (4.3) are oscillatory or not.

The Hille–Wintner comparison theorem has been widely studied by many authors. For example, Kusano and Yoshida [9] presented the following comparison theorem of Hille–Wintner type for half-linear differential equations (see also [10]).

THEOREM C. *Consider*

$$\left(|x'|^{\alpha-1}x'\right)' + b(t)|x|^{\alpha-1}x = 0, \tag{4.4}$$

where $b(t)$ is positive and continuous on (t_0, ∞). Suppose that

$$\int_t^\infty a(s)\,ds \leqq \int_t^\infty b(s)\,ds$$

for all sufficiently large t. If all nontrivial solutions of (4.3) *are oscillatory, then those of* (4.4) *are also oscillatory.*

We can regard the number α in equations (4.3) and (4.4) as a positive parameter. In Theorem C, needless to say, the parameter α is fixed and the integral of the coefficient $a(t)$ is compared with that of the coefficient $b(t)$. Let us fix the coefficient $a(t)$ and move the parameter α to the contrary. Then we have another comparison theorem for half-linear differential equations.

THEOREM 4.1. *Consider*

$$\left(|x'|^{\beta-1}x'\right)' + a(t)|x|^{\beta-1}x = 0, \tag{4.5}$$

where $a(t)$ is the same as in equation (4.3). *Suppose that $0 < \alpha < \beta$. If all nontrivial solutions of* (4.3) *are oscillatory, then those of* (4.5) *are also oscillatory.*

PROOF. The proof is by contradiction. We suppose that all nontrivial solutions of (4.3) are oscillatory and equation (4.5) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. As in the proof of Theorem 1.1, we see that $x'(t)$ is also eventually positive.

Define

$$\xi(t) = \left(\frac{x'(t)}{x(t)}\right)^{\beta}.$$

Then there exists a $T > t_0$ such that $\xi(t) > 0$ and

$$\xi'(t) = -a(t) - \beta\xi(t)^{(\beta+1)/\beta} < 0 \tag{4.6}$$

for $t \geqq T$, namely, $\xi(t)$ is decreasing and bounded from below. Hence, we can find a $\mu \geqq 0$ such that $\xi(t) \searrow \mu$ as $t \to \infty$, and therefore, we have

$$\xi'(t) = -a(t) - \beta\xi(t)^{(\beta+1)/\beta} \leqq -\beta\mu^{(\beta+1)/\beta}$$

for $t \geqq T$. If $\mu > 0$, then $\xi(t)$ has to tend to $-\infty$ as $t \to \infty$. This contradicts the fact that $\xi(t)$ is eventually positive. Thus, $\xi(t)$ tends to zero as $t \to \infty$. From this property of $\xi(t)$ and the assumption that $0 < \alpha < \beta$, we see that there exists a $t_1 > T$ such that

$$\alpha\xi(t)^{(\alpha+1)/\alpha} \leqq \beta\xi(t)^{(\beta+1)/\beta}$$

for $t \geqq t_1$. Hence, together with (4.6), we have

$$\xi'(t) \leqq -a(t) - \alpha\xi(t)^{(\alpha+1)/\alpha} \tag{4.7}$$

for $t \geqq t_1$.

It is easy to check that the function

$$y(t) = \exp\left(\int_{t_1}^{t} \xi(\tau)^{1/\alpha} d\tau\right)$$

is a nonoscillatory solution of

$$\left(|x'|^{\alpha-1}x'\right)' + b(t)|x|^{\alpha-1}x = 0,$$

where $b(t) = -\xi'(t) - \alpha\xi(t)^{(\alpha+1)/\alpha}$. From (4.7) it follows that $a(t) \leqq b(t)$ for $t \geqq t_1$. Hence, Sturm's comparison theorem implies that (4.3) also has a nonoscillatory solution. This is a contradiction. $\square$

In the case that

$$t^{\alpha+1}a(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{4.8}$$

for t sufficiently large, we can rewrite equation (4.3) in the form (E_α) with

$$\delta(t) = \left(\frac{\alpha+1}{\alpha}\right)^{\alpha} t^{\alpha+1}a(t) - \frac{\alpha}{\alpha+1} > 0.$$

Suppose that all nontrivial solutions of (4.3) are oscillatory. Then, from Theorem 1.2 we see that all nontrivial solutions of

$$\left(|x'|^{\beta-1}x'\right)' + c(t)|x|^{\beta-1}x = 0$$

with

$$c(t) = \frac{1}{t^{\beta+1}}\left\{\left(\frac{\beta}{\beta+1}\right)^{\beta+1} + \left(\left(\frac{\beta}{\beta+1}\right)^{\beta} + \varepsilon\right)\delta(t)\right\}$$

are oscillatory for some $\varepsilon > 0$. Since $0 < \alpha < \beta$, we have

$$\begin{aligned} c(t) &= \frac{1}{t^{\beta+1}}\left\{\left(\frac{\beta}{\beta+1}\right)^{\beta+1} + \left(\left(\frac{\beta}{\beta+1}\right)^{\beta} + \varepsilon\right)\delta(t)\right\} \\ &< \frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\delta(t)\right\} = a(t) \end{aligned}$$

for t sufficiently large. Hence, from Theorem C we conclude that all nontrivial solutions of (4.5) are also oscillatory. This means that Theorem 1.2 is sharper than Theorem 4.1 in the case (4.8).

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