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ON THE CURVATURE OF A GENERALIZATION OF CONTACT METRIC MANIFOLDS

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Abstract. We consider a genaralization of contact metric manifolds given by assignment of 1-forms $\eta^1, \dots, \eta^s$ and a compatible metric g on a manifold. With some integrability conditions they are called almost $\mathcal{S}$-manifolds. We give a sufficient condition regarding the curvature of an almost $\mathcal{S}$-manifold to be locally isometric to a product of a Euclidean space and a sphere.

Introduction

In recent years there has been a very extensive research done in contact geometry. Within the subject of contact geometry there is also the class of contact metric geometry and its generalizations [3]. The study of the curvature of such manifolds is of our interest in the present paper.

Let (M^{2n+s}, g) be a Riemannian manifold equipped with a metric f-structure, i.e. an endomorphism φ of the tangent bundle such that $\varphi^3 + \varphi = 0$ and which is compatible with g; the compatibility means that for each $X, Y \in \Gamma(TM)$ we have $g\big(\varphi(X), Y\big) = -g\big(X, \varphi(Y)\big)$. Such manifolds are a natural generalization of almost Hermitian manifolds (the case when φ is an isomorphism of TM). Moreover we assume that the kernel of φ is parallelizable, i.e. there exist global vector fields $\xi_1, \dots, \xi_s$ spanning $\ker \varphi$. The study of such manifolds was started by D. E. Blair, S. I. Goldberg, K. Yano, cf. [1, 7, 8]. Let $\eta^1, \dots, \eta^s$ be the dual 1-forms of $\xi_1, \dots, \xi_s$. According to the definitions of [6], the set consisting of M with the geometric structures $(\varphi, \xi_i, \eta^j, g)$ $(i, j = 1, \dots, s)$, g a compatible metric, is called an *almost $\mathcal{S}$-structure* if $d\eta^k = F$ for all $k = 1, \dots, s$ where F is the Sasaki 2-form defined by g and φ. The almost $\mathcal{S}$-structures were also studied by J. L. Cabrerizo, L. M. Fernández, M. Fernández [4].

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In the present paper we study almost $\mathcal{S}$-structures. In Section 2 we prove two theorems. In Theorem 2.1 we prove that given an almost $\mathcal{S}$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^j, g)$ $(n > 1)$ satisfying $R_{XY}\xi_i = 0$, for $X, Y \in \Gamma(TM)$ then M is locally isometric to the product of the flat $(n+s)$-dimensional Euclidean space and the n-dimensional sphere of curvature $4s$. If in the above theorem we assume that $n = 1$ then we get that M is flat; this is proved in Theorem 2.2.

In Section 3 we give a method of constructing a metric almost $\mathcal{S}$-structure starting from global 1-forms on M satisfying some non-degeneracy conditions. Under these conditions we prove the existence of a compatible Riemannian metric and s global orthonormal vector fields; these vector fields correspond to the Reeb vector field in the contact case. Then we apply this method in a construction of an example of an $\mathcal{S}$-structure on $\mathbf{R}^{2n+s}$; we calculate also the associated Riemannian and Ricci curvature tensors.

1. Preliminaries

Let M be a $(2n+s)$-dimensional manifold equipped with an *f-structure with a parallelizable kernel*, for brevity we call it an *f.pk-structure*; this means that there are given on M an f-structure φ, s global vector fields $\xi_1, \dots, \xi_s$ and 1-forms $\eta^1, \dots, \eta^s$ on M satisfying the conditions

$$\varphi(\xi_i) = 0, \quad \eta^i \circ \varphi = 0, \quad \varphi^2 = -I + \sum_{j=1}^{s} \eta^j \otimes \xi_j, \quad \eta^i(\xi_j) = \delta^i_j$$

for all $i, j = 1, \dots, s$. We denote by $\mathcal{D}$ the bundle $\mathrm{Im}\varphi$. On such a manifold a $(2,1)$-tensor

$$N_\varphi := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i$$

is defined where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ, i.e.

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y].$$

for all $X, Y \in \Gamma(TM)$; we denote here by $\Gamma(TM)$ the module of differentiable sections of the bundle TM. The structure (φ, ξ_i, η^j) on M $(i, j = 1, \dots, s)$ is said to be *normal* if and only if $N_\varphi = 0$.

On a manifold equipped with an f.pk-structure there always exists a *compatible* Riemannian metric g in the sense that for each $X, Y \in \Gamma(TM)$

$$g(X, Y) = g\big(\varphi(X), \varphi(Y)\big) + \sum_{j=1}^{s} \eta^j(X)\eta^j(Y). \tag{1.1}$$

However such a metric on M is not unique: we fix one of them; then the structure obtained is called a *metric f.pk-structure*. Let F be the Sasaki form of φ defined by $F(X,Y) := g(X,\varphi Y)$ for $X,Y \in \Gamma(TM)$. It may be observed that $\mathcal{D}$ is the orthogonal complement of the bundle $\ker\varphi = \langle \xi_1,\dots,\xi_s\rangle$. Then the manifold M is equipped with an f-structure φ, the complemented frame $\xi_1,\dots,\xi_s$, the 1-forms $\eta^1,\dots,\eta^s$, a compatible metric g and the Sasaki 2-form.

We recall the following definitions fundamental for our paper.

DEFINITION 1.1. The metric f.pk-manifold $(M,\varphi,\xi_i,\eta^j,g)$ is said to be an *almost $\mathcal{S}$-manifold* if and only if $d\eta^1 = \dots = d\eta^s = F$.

DEFINITION 1.2. The metric f.pk-manifold $(M,\varphi,\xi_i,\eta^j,g)$ is said to be an *$\mathcal{S}$-manifold* if and only if it is an almost $\mathcal{S}$-manifold and it is normal.

The definition of an $\mathcal{S}$-structure was given by D. E. Blair in his seminal paper, cf. [1]. In that paper $\mathcal{K}$, $\mathcal{S}$, $\mathcal{C}$-structures are also defined which are direct generalizations of the normal almost contact metric, Sasakian and cosymplectic manifolds. The almost $\mathcal{S}$-structures were studied, without being precisely named, by J. L. Cabrerizo, L. M. Fernández and M. Fernández [4]. Then K. Duggal, S. Ianus and A. M. Pastore [6], also studied such manifolds and gave them the name almost $\mathcal{S}$-manifold.

On an almost $\mathcal{S}$-manifold $(M,\varphi,\xi_i,\eta^j,g)$ $(i,j=1,\dots,s)$ the operators $h_i := (1/2)L_{\xi_i}\varphi$ for $i=1,\dots,s$ are defined, cf. [4, (2.5)]. We use extensively the properties of these operators in the present paper. In particular these operators are self adjoint, anticommute with φ and for each $i,j=1,\dots,s$

$$h_i\xi_j = 0, \tag{1.2}$$

cf. [4]. Moreover we have the following identities, cf. [6]:

$$\nabla_X\xi_i = -\varphi X - \varphi h_i X, \tag{1.3}$$

$$\nabla_{\xi_i}\varphi = 0, \tag{1.4}$$

$$\nabla_{\xi_i}\xi_j = 0, \tag{1.5}$$

(1.6)
$$(\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = 2g(\varphi X,\varphi Y)\overline{\xi} + \overline{\eta}(Y)\varphi^2 X - \sum_{k=1}^{s}\eta^k(Y)h_k(X)$$

where ∇ is the Levi-Civita connection of g, $X \in \Gamma(TM)$ and $i,j=1,\dots,s$. Furthermore, we shall frequently use the following curvature identities related to the Levi-Civita connection of g:

$$R_{\xi_i X}\xi_i - \varphi(R_{\xi_i\varphi X}\xi_i) = 2(h_i^2X + \varphi^2X), \tag{1.7}$$

$$R_{\xi_i X}\xi_j - \varphi(R_{\xi_i \varphi X}\xi_j) = 2\big((h_i \circ h_j)X + \varphi^2 X\big) \tag{1.8}$$

for each $X \in \Gamma(TM)$ and $i, j = 1, \dots, s$. These immediately follow by combining the first equation on p. 158 of [4] and (1.3).

2. Curvature properties of almost $\mathcal{S}$-structures

Throughout this section we suppose that an almost $\mathcal{S}$-manifold $(M, \varphi, \xi_i, \eta^j, g)$ $(i, j = 1, \dots, s)$, with $\dim M = 2n + s$ is given. We denote by $\overline{\eta} := \eta^1 + \dots + \eta^s$, $\overline{\xi} := \xi_1 + \dots + \xi_s$, $\mathcal{D} := \mathrm{Im}\varphi$ and by F the associated Sasaki 2-form. We use the Levi-Civita connection ∇ associated with g; by R we denote the induced Riemannian curvature tensor.

LEMMA 2.1. *Let $(M, g, \varphi, \xi_i, \eta^j)$ be an almost $\mathcal{S}$-manifold. Then the curvature tensor satisfies the identities*

$$g(R_{\xi_i X}Y, Z) = -(\nabla_X F)(Y, Z) - g\big(\big(\nabla_Y(\varphi \circ h_i)\big)Z, X\big) + g\big(\big(\nabla_Z(\varphi \circ h_i)\big)Y, X\big) \tag{2.1}$$

and

$$\begin{aligned} &g(R_{\xi_i X}Y, Z) - g(R_{\xi_i X}\varphi Y, \varphi Z) + g(R_{\xi_i \varphi X}Y, \varphi Z) + g(R_{\xi_i \varphi X}\varphi Y, Z) \\ &= 2\Big((\nabla_{h_i X}F)(Y, Z) + \overline{\eta}(Z)g(X + h_i X, Y) - \overline{\eta}(Y)g(X + h_i X, Z) \\ &\qquad - \sum_{k=1}^{s} \eta^k(X)\big(\eta^k(Y)\overline{\eta}(Z) - \eta^k(Z)\overline{\eta}(Y)\big)\Big) \end{aligned} \tag{2.2}$$

for each $i = 1, \dots, s$ and $X, Y, Z \in \Gamma(TM)$.

PROOF. From (1.3) we have

$$R_{YZ}\xi_i = -(\nabla_Y \varphi)Z + (\nabla_Z \varphi)Y - \big(\nabla_Y(\varphi \circ h_i)\big)Z + \big(\nabla_Z(\varphi \circ h_i)\big)Y.$$

Then, since $g\big((\nabla_Y \varphi)Z, X\big) = (\nabla_Y F)(X, Z)$, we get

$$\begin{aligned} g\big(R_{\xi_i X}Y, Z\big) = g(R_{YZ}\xi_i, X) &= -(\nabla_Y F)(X, Z) + (\nabla_Z F)(X, Y) \\ &- g\big(\nabla_Y(\varphi \circ h_i)\big)Z, X) + g\big(\nabla_Z(\varphi \circ h_i)\big)Y, X). \end{aligned}$$

Using the last equation and the identity $(\nabla_X F)(Y,Z)+(\nabla_Y F)(Z,X)+(\nabla_Z F)(X,Y)=0$, we obtain (2.1). We introduce the operators A and B_i, $i\in\{1,\dots,s\}$ defined by

$$A(X,Y,Z):=-(\nabla_X F)(Y,Z)+(\nabla_X F)(\varphi Y,\varphi Z)-(\nabla_{\varphi X}F)(Y,\varphi Z)-(\nabla_{\varphi X}F)(\varphi Y,Z) \tag{2.3}$$

and

$$B_i(X,Y,Z):=-g\big(\varphi X,\big(\nabla_Y(\varphi\circ h_i)\big)(\varphi Z)\big)-g\big(\varphi X,\big(\nabla_{\varphi Y}(\varphi\circ h_i)\big)Z\big)-g\big(X,\big(\nabla_Y(\varphi\circ h_i)\big)Z\big)+g\big(X,\big(\nabla_{\varphi Y}(\varphi\circ h_i)\big)(\varphi Z)\big) \tag{2.4}$$

for each $X,Y,Z\in\Gamma(TM)$. By a direct computation and using (2.1) we get that the left hand side of (2.2) equals $A(X,Y,Z)+B_i(X,Y,Z)-B_i(X,Z,Y)$. Since $\eta^j\big((\nabla_{\varphi Y}h_i)Z\big)=\eta^j\big(\nabla_{\varphi Y}(h_iZ)\big)$ we can write

$$\begin{aligned}B_i(X,Y,Z)&=-g\big(X,\nabla_Y\big((\varphi\circ h_i)Z\big)\big)+g\big(X,(\varphi\circ h_i)(\nabla_Y Z)\big)\\&+g\big(X,\nabla_{\varphi Y}\big((\varphi\circ h_i\circ\varphi)Z\big)\big)+g\big(X,(\varphi\circ h_i)(\nabla_{\varphi Y}\varphi Z)\big)\\&+g\big(X,(\varphi\circ h_i)\big(\nabla_{\varphi Y}(\varphi Z)\big)\big)-g\big(\varphi X,\nabla_Y\big((\varphi\circ h_i\circ\varphi)Z\big)\big)\\&+g\big(\varphi X,(\varphi\circ h_i)\big(\nabla_Y(\varphi Z)\big)\big)-g\big(\varphi X,\nabla_{\varphi Y}\big((\varphi\circ h_i)Z\big)\big)\\&+g\big(\varphi X,(\varphi\circ h_i)\big(\nabla_{\varphi Y}(h_iZ)\big)\big)=-g\big(X,(\nabla_Y\varphi)(h_iZ)\big)\\&+g\big(X,h_i\big((\nabla_Y\varphi)Z\big)\big)+g\big(X,(h_i\circ\varphi)\big((\nabla_{\varphi Y}\varphi)Z\big)\big)\\&+g\big(X,\varphi\big((\nabla_{\varphi Y}\varphi)(h_iZ)\big)\big)+\sum_{j=1}^s\eta^j\big((\nabla_{\varphi Y}h_i)Z\big)\eta^j(X).\end{aligned} \tag{2.5}$$

Moreover, from (1.3)–(1.6) it follows that

$$\begin{aligned}\big(\varphi\circ(\nabla_{\varphi X}\varphi)\big)Y&=(\nabla_{\varphi X}\varphi^2)Y-(\nabla_{\varphi X}\varphi)(\varphi Y)=\sum_{j=1}^s\big((\nabla_{\varphi X}\eta^j)Y\xi_j\big)\\&+\sum_{j=1}^s\big(\eta^j(Y)\nabla_{\varphi X}\xi_j\big)-(\nabla_{\varphi X}\varphi)(\varphi Y)=\sum_{j=1}^s\big((\nabla_{\varphi X})\big(g(\xi_j,Y)\big)\xi_j\end{aligned}$$

$$- g(\nabla_{\varphi X}Y, \xi_j)\xi_j\big) - \sum_{j=1}^{s} \eta^j(Y)h_jX + \overline{\eta}(Y)\Big(X - \sum_{j=1}^{s} \eta^j(X)\xi_j\Big)$$

$$- 2g(\varphi X, \varphi Y)\overline{\xi} - \overline{\eta}(Y)\varphi^2 X + \sum_{j=1}^{s} \eta^j(Y)h_jX + (\nabla_X\varphi)Y.$$

Hence

(2.6)

$$\big(\varphi \circ (\nabla_{\varphi X}\varphi)\big)Y = -\Big(g(X,Y) - \sum_{j=1}^{s} \eta^j(X)\eta^j(Y)\Big)\overline{\xi} - \sum_{j=1}^{s} g(h_jX, Y)\xi_j$$

$$- 2\overline{\eta}(Y)\Big(X - \sum_{j=1}^{s} \eta^j(X)\xi_j\Big) + (\nabla_X\varphi)Y.$$

Furthermore, from (1.6), for each $j = 1, \dots, s$ we have

$$\eta^i\big((\nabla_{\varphi Y}h_j)Z\big) = \eta^i\big(\nabla_{\varphi Y}(h_jZ)\big) = (\nabla_{\varphi Y}\eta^i)(h_jZ)$$

$$= -g(h_jZ, \nabla_{\varphi Y}\xi_i) = g(h_jZ, h_iY - Y).$$

Then, using (2.5) and (2.6) we get

$$B_i(X,Y,Z) = -g\big(X, (\nabla_Y\varphi)(h_iZ)\big) + g\Big(X, \big(h_i(\nabla_Y\varphi)\big)Z\Big) + 2\overline{\eta}(Z)g(h_iX, Y)$$

$$+ g(h_iX, (\nabla_Y\varphi)Z) - g(Y, h_iZ)\overline{\eta}(X) - \sum_{j=1}^{s} g(h_iZ, h_jY)\eta^j(X)$$

$$+ \sum_{j=1}^{s} \eta^j(X)g(h_iZ, h_jY) + g\big(X, (\nabla_Y\varphi)(h_iZ)\big) - \overline{\eta}(X)g(h_iZ, Y)$$

$$= 2\Big(g\big(h_iX, (\nabla_Y\varphi)Z\big) + \overline{\eta}(Z)g(Y, h_iX) - \overline{\eta}(X)(Y, h_iZ)\Big).$$

Therefore we obtain

$$A(X,Y,Z) + B_i(X,Y,Z) - B_i(X,Z,Y) = 2(\nabla_YF)(h_iX, Z)$$

$$- 2(\nabla_ZF)(h_iX, Y) + 2\overline{\eta}(Z)g(X + h_iX, Y) - 2\overline{\eta}(Y)g(X + h_iX, Z)$$

$$- 2\sum_{k=1}^{s} \eta^k(X)\big(\eta^k(Y)\overline{\eta}(Z) - \eta^k(Z)\overline{\eta}(Y)\big)$$

and hence (2.2) follows. □

Lemma 2.1 is a generalization of Lemma 3.2 proved by Z. Olszak [11] in which he considers contact metric manifolds.

THEOREM 2.1. *Let* $(M, \varphi, \xi_i, \eta^j, g)$ *be an almost* $\mathcal{S}$*-manifold of dimension* $2n + s$, $n \geqq 2$, *such that* $R_{XY}\xi_i = 0$, *for each* $X, Y \in \Gamma(TM)$, $i = 1, \ldots, s$. *Then* M *is locally isometric to* $\mathbf{E}^{n+s} \times \mathbf{S}^n(4s)$ *where* $\mathbf{E}^{n+s}$ *is the* $n + s$ *dimensional Euclidean space and* $\mathbf{S}^n(4s)$ *is the* n *dimensional sphere of radius* $\frac{1}{2\sqrt{s}}$.

PROOF. Let $X \in \mathcal{D}$. From the hypothesis and (1.7) it follows that for each $i = 1, \ldots, s$ we have $g(h_i^2 X + \varphi^2 X, X) = 0$. Since h_i is self-adjoint, cf. [4], and (1.1) holds then $\|h_i X\| = \|\varphi X\| = \|X\|$. It follows that if X is an eigenvector of h_i with respect to the eigenvalue λ then $|\lambda| \, \|X\| = \|X\|$, so that $\lambda = \pm 1$. Furthermore, since φX is an eigenvector with respect to the eigenvalue $-\lambda$ and, by virtue of (1.2), $\langle \xi_1, \ldots, \xi_s \rangle$ is the eigenspace associated to the eigenvalue 0, the multiplicity of the eigenvalues ± 1 is n. We denote by $\mathcal{D}_+^i$ the eigenspace of h_i with respect to the eigenvalue 1 and by $\mathcal{D}_-^i$ the eigenspace of h_i with respect to the eigenvalue -1. From (1.8) we get $h_i \circ h_j = -\varphi^2 = h_j \circ h_i$ for each $j = 1, \ldots, s$. Since $\mathcal{D} = \mathcal{D}_+^i \oplus \mathcal{D}_-^i$ then $X = X_+ + X_-$ where $X_+ \in \mathcal{D}_+^i$ and $X_- \in \mathcal{D}_-^i$. Hence

$$h_j X = h_j(X_+ + X_-) = h_j\big(h_i(X_+ - X_-)\big) = -\varphi^2(X_+ - X_-)$$
$$= X_+ - X_- = h_i(X_+ + X_-) = h_i X,$$

i.e. $h_i|_{\mathcal{D}} = h_j|_{\mathcal{D}}$. Again from (1.2) we get $h_i = h_j$. We put

$$h := h_1 = \cdots = h_s, \quad \mathcal{D}_+ := \mathcal{D}_+^1 = \cdots = \mathcal{D}_+^s, \quad \mathcal{D}_- := \mathcal{D}_-^1 = \cdots = \mathcal{D}_-^s.$$

Let $X, Y \in \mathcal{D}_-$. Then from (1.3) it follows that $\nabla_X \xi_i = \nabla_Y \xi_i = 0$ for each $i = 1, \ldots, s$. Hence

$$0 = R_{XY}\xi_i = -\nabla_{[X,Y]}\xi_i = -\varphi\big([X,Y]\big) - \varphi h\big([X,Y]\big). \tag{2.7}$$

On the other hand from (1.2) we get $\eta^k\big(h[X,Y]\big) = 0$ for each $k = 1, \ldots, s$; moreover, since $\varphi Y \in \mathcal{D}_+$ then

$$\eta^k\big([X,Y]\big) = -2d\eta^k(X,Y) = -2F(X,Y) = -2g(X, \varphi Y) = 0.$$

Then applying φ to (2.7) we get $h\big([X,Y]\big) = -[X,Y]$. It follows that the distribution $\mathcal{D}_-$ is integrable. Analogously, since $\nabla_{[\xi_k, X]}\xi_i = -R_{\xi_k X}\xi_i = 0$ for $X \in \mathcal{D}_-$ we have $h\big([\xi_k, X]\big) = -[\xi_k, X]$ which means that $[\xi_k, X] \in \mathcal{D}_-$. Hence, due to $[\xi_i, \xi_j] = 0$ for each $i, j = 1, \ldots, s$, also the distribution

$\mathcal{D}_- \oplus \langle \xi_1, \dots, \xi_s \rangle$ is integrable. We can choose local coordinates $x_1, \dots, x_{2n+s}$ such that

$$\left\{ \frac{\partial}{\partial x_{n+1}}, \dots, \frac{\partial}{\partial x_{2n}}, \frac{\partial}{\partial x_{2n+1}}, \dots, \frac{\partial}{\partial x_{2n+s}} \right\}$$

is a local basis of $\mathcal{D}_- \oplus \langle \xi_1, \dots, \xi_s \rangle$. Let ρ^j_α, $\alpha \in \{1, \dots, n\}$, $j \in \{n+1, \dots, 2n+s\}$ be local functions such that

$$X_\alpha = \frac{\partial}{\partial x_\alpha} + \sum_{j=n+1}^{2n+s} \rho^j_\alpha \frac{\partial}{\partial x_j} \in \mathcal{D}_+.$$

Then $X_1, \dots, X_n$ is a local basis of $\mathcal{D}_+$. Since $\left[\frac{\partial}{\partial x_j}, X_\alpha\right] \in \mathcal{D}_- \oplus \langle \xi_1, \dots, \xi_s \rangle$ for each $\alpha = 1, \dots, n$ and $j = n+1, \dots, 2n+s$ we can locally write $\left[\frac{\partial}{\partial x_j}, X_\alpha\right] = X + \sum_{j=1}^s \sigma^j \xi_j$ where $X \in \mathcal{D}_-$ and $\sigma^1, \dots, \sigma^s$ are differentiable functions. We get

$$\nabla_{\left[\frac{\partial}{\partial x_j}, X_\alpha\right]} \xi_i = \nabla_X \xi_i + \sum_{j=1}^s \sigma^j \nabla_{\xi_j} \xi_i = 0$$

from which we conclude that ξ_i is parallel along $\left[\frac{\partial}{\partial x_j}, X_\alpha\right]$. Then from (1.3)

$$0 = \nabla_{\left[\frac{\partial}{\partial x_j}, X_\beta\right]} \xi_i = \nabla_{\frac{\partial}{\partial x_j}} (\nabla_{X_\beta} \xi_i) - \nabla_{X_\beta} \Big(\nabla_{\frac{\partial}{\partial x_j}} \xi_i \Big) = -2 \nabla_{\frac{\partial}{\partial x_j}} (\varphi X_\beta)$$

and, since $\varphi X_\alpha \in \mathcal{D}_-$, we have $\nabla_{\varphi X_\alpha} \varphi X_\beta = 0$. It follows that the integral manifolds of $\mathcal{D}_- \oplus \langle \xi_1, \dots, \xi_s \rangle$ are totally geodesic and flat. From the hypothesis and (2.2) we have $(\nabla_{hX} F)(Y, Z) = 0$ for each $X, Y, Z \in \mathcal{D}$ and then $g\big((\nabla_{hX} \varphi) Y, Z\big) = 0$. Since $h|_{\mathcal{D}}$ is an isomorphism we get

$$g\big((\nabla_X \varphi) Y, Z\big) = 0 \text{ for each } X, Y, Z \in \mathcal{D}. \tag{2.8}$$

Using (1.3), for each $X, Y \in \mathcal{D}_+$, $i = 1, \dots, s$ we have

$$0 = R_{XY} \xi_i = -2(\nabla_X \varphi) Y + 2(\nabla_Y \varphi) X - \varphi\big([X, Y]\big) + \varphi\Big(h\big([X, Y]\big)\Big).$$

Since $h \circ \varphi = -\varphi \circ h$, for each $Z \in \mathcal{D}_+$ we obtain

$$g\Big(-h\Big(\varphi\big([X, Y]\big)\Big) - \varphi\big([X, Y]\big), Z \Big) = 0$$

and then $g\big([X, Y], \varphi Z\big) = 0$. But φ is an isomorphism of $\mathcal{D}_+$ onto $\mathcal{D}_-$, so $[X, Y]$ is orthogonal to $\mathcal{D}_-$. In an analogous way, since

$$\eta^i\big([X, Y]\big) = -2 d\eta^i (X, Y) = -2F(X, Y) = -2g(X, \varphi Y) = 0$$

we have that $\mathcal{D}_+$ is integrable. We want to prove now that the integral submanifolds of $\mathcal{D}_+$ are totally geodesic. For this purpose we take $X \in \mathcal{D}_-$ and $Y \in \mathcal{D}_+$. We have

$$0 = R_{XY}\xi_i = -2\nabla_X(\varphi Y) + \varphi\big([X,Y]\big) + \varphi\Big(h\big([X,Y]\big)\Big)$$

$$= -2(\nabla_X\varphi)Y - \varphi(\nabla_X Y) - \varphi(\nabla_Y X) - h\big(\varphi(\nabla_X Y)\big) + h\big(\varphi(\nabla_Y X)\big).$$

We take the scalar product with $Z \in \mathcal{D}_-$ and using (2.8) we get

$$0 = -g\big(\varphi(\nabla_X Y), Z\big) - g\big(\varphi(\nabla_Y X), Z\big) - g\Big(h\big(\varphi(\nabla_X Y)\big), Z\Big)$$

$$+ g\Big(h\big(\varphi(\nabla_Y X)\big), Z\Big) = -2g\big(\varphi(\nabla_Y X), Z\big).$$

Since φ is an isomorphism of $\mathcal{D}_-$ onto $\mathcal{D}_+$ then $\nabla_Y X$ is orthogonal to $\mathcal{D}_+$. On the other hand, for each $i = 1, \dots, s$, $Y, Z \in \mathcal{D}_+$, $X \in \mathcal{D}_-$, $g(\nabla_Y Z, \xi_i) = -g(\nabla_Y \xi_i, Z) = 2g(\varphi Y, Z) = 0$ and $g(\nabla_Y Z, X) = -g(Z, \nabla_Y X) = 0$. It follows that $\nabla_Y Z$ is orthogonal to $\mathcal{D}_- \oplus \langle \xi_1, \dots, \xi_s\rangle$ and each integral submanifold of $\mathcal{D}_+$ is totally geodesic. At this point we can say that M is locally a Riemannian product and one of the factors is locally isometric to $\mathbf{E}^{n+s}$. We want to prove now that the second factor is isometric to $\mathbf{S}^n(4s)$. Since (2.2) holds and h is an isomorphism of $\mathcal{D}_-$ onto $\mathcal{D}_+$ then for each $X, Y \in \mathcal{D}_+$ and each $i = 1, \dots, s$

$$g\big((\nabla_X\varphi)Y, \xi_i\big) = -(\nabla_X F)(Y, \xi_i) = \overline{\eta}(\xi_i)g(X + hX, Y) = 2g(X, Y).$$

From (2.8) it follows that $(\nabla_X\varphi)Y = 2g(X,Y)\overline{\xi}$. Therefore, by using again (2.8) we get that for each $X, Y, Z, W \in \mathcal{D}_+$ we have

$$g(\nabla_X\nabla_Y\varphi Z, \varphi W) - g(\nabla_X\nabla_Y Z, W) = 2g(Y,Z)g(\nabla_X\overline{\xi}, \varphi W)$$

$$+ g\Big(\nabla_X\big(\varphi(\nabla_Y Z)\big), \varphi W\Big) - g(\nabla_X\nabla_Y Z, W)$$

$$= 2sg(Y,Z)g(-\varphi X - \varphi hX, \varphi W) = -4sg(X,W)g(Y,Z).$$

Finally, from (2.8), $g\big(\nabla_{[X,Y]}\varphi Z, \varphi W\big) - g\big(\nabla_{[X,Y]}Z, W\big) = 0$ and $R_{XY}\varphi Z = 0$ since $\varphi Z \in V_- \oplus \langle \xi_1, \dots, \xi_s\rangle$. Then we get that

$$g(R_{XY}\varphi Z, \varphi W) - g(R_{XY}Z, W)$$

$$= -4s\big(g(X,W)g(Y,Z) - g(Y,W)g(X,Z)\big). \quad \square$$

THEOREM 2.2. *Let $(M, \varphi, \xi_i, \eta^j, g)$ be an almost $\mathcal{S}$-manifold of dimension $2+s$. If $R_{XY}\xi_i = 0$ for each $X, Y \in \Gamma(TM)$ and each $i = 1, \dots, s$ then M is flat.*

Proof. With the same argument as in the proof of Theorem 2.1 we get that $h_1 = \cdots = h_s = h$ has eigenvalues ± 1 and 0. In such a case the eigenspaces distibutions $\mathcal{D}_+$ and $\mathcal{D}_-$ are 1-dimensional and hence integrable. If we put $\mathcal{D}_- = \langle X \rangle$ then $\mathcal{D}_+ = \langle \varphi X \rangle$. From equation (3.2) of [5] we have $(\nabla_{\xi_i} h)X = \varphi X - \varphi h^2 X = 0$ and then $\nabla_{\xi_i} X \in \mathcal{D}_-$. Using (1.5) we get that the distribution $\mathcal{D}_- \oplus \langle \xi_1, \dots, \xi_s \rangle$ is integrable. With the same reasoning as in Theorem 2.1 we conclude that $\nabla_X X = 0$. We choose X such that $\|X\| = 1$. Then $\|\varphi X\| = 1$ and $g(\nabla_{\varphi X}(\varphi X), \varphi X) = 0$. Moreover, $g(\nabla_{\varphi X}(\varphi X), \xi_i) = -g(\nabla_{\varphi X}\xi_i, \varphi X) = -2g(X, \varphi X) = 0$. Since $g(\nabla_{\varphi X}(\varphi X), \varphi X) = 0$ it follows that $\nabla_{\varphi X}(\varphi X) = 0$. Therefore from (1.4) by easy calculations we get

$$\nabla_{\varphi X}\xi_i = 2X, \quad \nabla_X \xi_i = 0, \quad \nabla_{\varphi X} X = -2\bar{\xi},$$

$$\nabla_X \varphi X = 0, \quad \nabla_{\xi_i} X = 0, \quad \nabla_{\xi_i} \varphi X = 0.$$

Using the φ-basis $\{X, \varphi X, \xi_1, \dots, \xi_s\}$, and the formulas above we easily calculate the Riemannian curvature tensor and find that it vanishes. In such a way we obtain that the manifold M is flat. □

Theorems 2.1 and 2.2 are generalizations for almost $\mathcal{S}$-manifolds of the D. E. Blair's results proved for contact metric manifolds, cf. [2].

Remark 2.1. There exist examples of manifolds considered in Theorem 2.2. In fact, in our previous paper [5, Example 6.2], we have constructed a flat almost $\mathcal{S}$-manifold $(M^{2+s}, \varphi, \xi_i, \eta^j, g)$ on a toroidal bundle.

3. Almost $\mathcal{S}$-structures determined by 1-forms

The following two lemmas are generalizations of the existence theorem of the Reeb vector field on a contact manifold.

Lemma 3.1. *Let M be a manifold and let $\eta^1, \dots, \eta^s$ be 1-forms on M such that $\eta^1 \wedge \ldots \wedge \eta^s \neq 0$ at each point of M. Then there exist vector fields $\xi_1, \dots, \xi_s$ on M such that $\eta^i(\xi_j) = \delta^i_j$ for each $i, j = 1, \dots, s$; thus $\xi_1, \dots, \xi_s$ are usually not unique.*

Proof. $\mathcal{D} := \ker \eta^1 \cap \ldots \cap \ker \eta^s$ is a vector subbundle of TM of rank $\dim M - s$. Hence there exists a vector subbundle V of TM such that $V \oplus \mathcal{D} = TM$. Then consider $\Phi := (\eta^1, \dots, \eta^s) : V \to \mathbf{R}^s$ which is an isomorphism on each fibre of V. Hence there exist vector fields $\xi_1, \dots, \xi_s \in \Gamma(V)$ such that $\Phi(\xi_i)$ is the i-th element of the canonical basis of $\mathbf{R}^s$, that is $\eta^i(\xi_j) = \delta^i_j$. The vector fields $\xi_1, \dots, \xi_s$ depend on the choice of the complementary bundle V. □

LEMMA 3.2. *Let M^{2n+s} be a manifold, let $\eta^1,\dots,\eta^s$ be 1-forms on M and let F be a 2-form of constant rank $2n$ such that $\eta^1\wedge\dots\wedge\eta^s\wedge F^n\neq 0$ at each point of M. Then there exist unique vector fields $\xi_1,\dots,\xi_s$ on M such that $\eta^i(\xi_j)=\delta^i_j$ and $i_{\xi_j}F=0$ for each $i,j=1,\dots,s$.*

PROOF. Let $W=\{X\in TM \mid i_XF=0\}$ be the F-nullity subbundle of TM. Then W is a vector subbundle of TM of rank s. Moreover the map $\Psi:=(\eta^1,\dots,\eta^s):W\to\mathbf{R}^s$ is an isomorphism on each fibre of W. Then we proceed as in Lemma 3.1 and obtain vector fields $\xi_1,\dots,\xi_s$ which satisfy the requirements of our lemma. The uniqueness of the existence of $\xi_1,\dots,\xi_s$ follows from the unicity of W. $\square$

THEOREM 3.1. *Let M be a manifold of dimension $2n+s$. Suppose there exist 1-forms $\eta^1,\dots,\eta^s$ on M such that $d\eta^1=\dots=d\eta^s$ is a 2-form of constant rank $2n$ and $\eta^1\wedge\dots\wedge\eta^s\wedge(d\eta^1)^n\neq 0$. Then there exists an f.pk-structure (φ,ξ_i,η^j,g) $(i,j=1,\dots,s)$ on M where $\xi_1,\dots,\xi_s$ are the unique vector fields provided by Lemma 3.2 with respect to $\eta^1,\dots,\eta^s$ and $F=d\eta^1=\dots=d\eta^s$. Moreover, for each $X,Y\in\Gamma(TM)$, $g(X,\varphi Y)=d\eta^1(X,Y)$ i.e. $d\eta^1$ is the Sasaki 2-form of the f.pk-structure and hence $(M,\varphi,\xi_i,\eta^j,g)$ is an almost $\mathcal{S}$-manifold.*

PROOF. We obtain the vector fields $\xi_1,\dots,\xi_s$ from Lemma 3.2. Let g_0 be any Riemannian metric on M. Put $\mathcal{D}:=\ker\eta^1\cap\dots\cap\ker\eta^s$. Define the 2-form on $\mathcal{D}$ by $\Omega(X,Y):=d\eta^1(X,Y)$; observe that Ω is non-degenerate on $\mathcal{D}$. There exists a bundle isomorphism $A:\mathcal{D}\to\mathcal{D}$ such that for all $X,Y\in\mathcal{D}$, $g_0(AX,Y)=\Omega(X,Y)$. Then A is anti-adjoint with respect to g_0, i.e. $A^t=-A$. We have the polar decomposition $A=JG$ where J is an isometry of $\mathcal{D}$ and G is self-adjoint and positive definite with respect to g_0. Furthermore, observe that J^tGJ is similar to G and then it is positive definite. Since J is an isometry, G is adjoint and A is anti-adjoint and we have

$$JG=A=-A^t=-GJ^t.$$

Hence $G=(-J^2)J^tGJ$. From the uniqueness of polar decomposition of G we have $J^2=-\mathrm{Id}$, $J^tGJ=G$ and $J=-J^{-1}=-J^t$. Then define a metric tensor on M by

$$g(X,Y):=\begin{cases} g_0(GX,Y) & \text{if } X,Y\in\Gamma(\mathcal{D})\\ 0 & \text{if } X\in\Gamma(\mathcal{D}),\ Y\in\Gamma\big(\langle\xi_1,\dots,\xi_s\rangle\big)\\ \delta_{ij} & \text{if } X=\xi_i,\ Y=\xi_j\end{cases}$$

and an f-structure

$$\varphi(X):=\begin{cases} -J(X) & \text{if } X\in\Gamma(\mathcal{D})\\ 0 & \text{if } X\in\Gamma\big(\langle\xi_1,\dots,\xi_s\rangle\big).\end{cases}$$

It is easy to observe that $(M, g, \varphi, \xi_i, \eta^j)$ is an almost $\mathcal{S}$-manifold. It may be proved, similarly as in the symplectic case, that the set of such metric f-structures is path connected [13]. $\square$

As an application of Theorem 3.1 we give the following example of an $\mathcal{S}$-structure on $\mathbf{R}^{2n+s}$ that generalizes the Sasakian structure on $\mathbf{R}^{2n+1}$ given by S. Sasaki [12]. It is well known that this Sasakian structure on $\mathbf{R}^{2n+1}$ is of constant φ-sectional curvature -3 and that it is η-Einstein [10]. Our example is neither of constant φ-sectional curvature nor η-Einstein, according to the definition given by M. Kobayashi and S. Tsuchiya in [9].

EXAMPLE 3.1. Let $(x^1, \ldots, x^n, y^1, \ldots, y^n, z^1, \ldots, z^s)$ be the natural coordinates of $M := \mathbf{R}^{2n+s}$. For each $i = 1, \ldots, s$, put

$$\eta^i := \frac{1}{2}\left(dz^i - \sum_{\alpha=1}^{n} y^\alpha dx^\alpha\right), \quad \xi_i := 2\frac{\partial}{\partial z^i}.$$

We have

$$\eta^1 \wedge \ldots \wedge \eta^s \wedge (d\eta^i)^n \neq 0, \quad d\eta^1 = \cdots = d\eta^s = \sum_{\alpha=1}^{n} dx^\alpha \wedge dy^\alpha$$

and $d\eta^i(\xi_j, X) = 0$, for each $i, j \in \{1, \ldots, s\}$, $X \in \Gamma(TM)$ so that $\xi_1, \ldots, \xi_s$ are the unique s vector fields provided by Theorem 3.1. Let

$$g := \sum_{i=1}^{s} (\eta^i)^2 + \frac{1}{4}\sum_{\alpha=1}^{n} (dx^\alpha)^2 + (dy^\alpha)^2.$$

The matrix of g with respect to the canonical basis of vector fields on TM is

$$\frac{1}{4}\begin{pmatrix} A & 0 & B \\ 0 & I_n & 0 \\ B^t & 0 & I_s \end{pmatrix} \tag{3.1}$$

where $A_{\alpha\beta} = \delta_{\alpha\beta} + sy^\alpha y^\beta$, $B_{\alpha i} = -y^\alpha$, $\alpha,\ \beta \in \{1, \ldots, n\}$, $i \in \{1, \ldots, s\}$ and I_n, I_s are the identity matrices of order n and s, respectively. The inverse matrix of (3.1) is

$$4\begin{pmatrix} I_n & 0 & -B \\ 0 & I_n & 0 \\ -B^t & 0 & C \end{pmatrix}$$

where $C_{ij} = \delta_{ij} + \sum_{\alpha=1}^{n} (y^\alpha)^2$, $i, j = 1, \ldots, s$. Define the metric f-structure φ by giving its matrix with respect to the canonical basis of vector fields of TM:

$$\begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & B^t & 0 \end{pmatrix}.$$

From Theorem 3.1 it follows that $(M, \varphi, \xi_i, \eta^j, g)$ $(i, j = 1, \ldots, s)$ is an almost $\mathcal{S}$-manifold. By a direct verification it may be proved that $N_\varphi = 0$ and hence $(M, \varphi, \xi_i, \eta^j, g)$ is an $\mathcal{S}$-manifold. We observe that in this case

$$\text{(3.2)} \quad \mathcal{D} = \operatorname{span}\left\{ 2\left(\frac{\partial}{\partial x^1} + y^1\overline{\xi}\right), \ldots, 2\left(\frac{\partial}{\partial x^n} + y^n\overline{\xi}\right), 2\frac{\partial}{\partial y^1}, \ldots, 2\frac{\partial}{\partial y^n} \right\},$$

where $\overline{\xi} = \sum_{j=1}^{s} \xi_j$. The f-structure φ may be also characterized by observing that the generators of $\mathcal{D}$ in (3.2) constitute a φ-basis, i.e. they are orthonormal and

$$\varphi\left(\frac{\partial}{\partial x^1} + y^1\overline{\xi}\right) = \frac{\partial}{\partial y^1}, \quad \ldots, \quad \varphi\left(\frac{\partial}{\partial x^n} + y^n\overline{\xi}\right) = \frac{\partial}{\partial y^n}.$$

We are going to write down the components of the Riemannian curvature tensor of g. For generic indices i, j, r put

$$G_{ij}^r = \frac{1}{2}\left(\frac{\partial g_{rj}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r}\right).$$

We use the Greek letters $\alpha, \beta, \ldots$ as the indices relative to $x^1, \ldots, x^n$, then we use $\alpha^*, \beta^*, \ldots$ as the indices relative to $y^1, \ldots, y^n$, and $i, j, \ldots$ as the indices relative to $z^1, \ldots, z^s$. We get

$$G_{\beta\gamma^*}^{\alpha} = \frac{1}{8}\big(\delta_{\beta\gamma}y^\alpha + \delta_{\alpha\gamma}y^\beta\big); \quad G_{\beta\gamma}^{\alpha^*} = -\frac{1}{8}\big(\delta_{\alpha\beta}y^\gamma + \delta_{\alpha\gamma}y^\beta\big);$$

$$G_{i\beta}^{\alpha^*} = \frac{1}{8}\delta_{\alpha\beta}; \quad G_{\alpha\beta^*}^{i} = -\frac{1}{8}\delta_{\alpha\beta}; \quad G_{i\beta^*}^{\alpha} = -\frac{1}{8}\delta_{\alpha\beta};$$

the other G_{ij}^r's are zero. It follows that the non zero Christoffel's symbols of the Riemannian structure are

$$\Gamma_{\beta\gamma^*}^{\alpha} = \frac{1}{2}\delta_{\alpha\gamma}y^\beta; \quad \Gamma_{\beta\gamma}^{\alpha^*} = -\frac{1}{2}\big(\delta_{\alpha\beta}y^\gamma + \delta_{\alpha\gamma}y^\beta\big); \quad \Gamma_{\beta^* i}^{\alpha} = -\frac{1}{2}\delta_{\alpha\beta};$$

$$\Gamma_{\alpha\beta^*}^{i} = \frac{1}{2}\big(y^\alpha y^\beta - \delta_{\alpha\beta}\big); \quad \Gamma_{j\alpha^*}^{i} = -\frac{1}{2}y^\alpha; \quad \Gamma_{\beta i}^{\alpha^*} = \frac{1}{2}\delta_{\alpha\beta}.$$

Finally the non zero components of the Riemannian curvature tensor are

$$R_{\alpha\beta\gamma\delta} = \frac{1}{16}\big(\delta_{\alpha\gamma}y^\beta y^\delta - \delta_{\alpha\delta}y^\beta y^\gamma - s\delta_{\gamma\beta}y^\alpha y^\delta + s\delta_{\beta\delta}y^\alpha y^\gamma\big);$$

$$R_{\alpha^*\beta^*\gamma\delta} = \frac{1}{16}\big((s-1)\big(\delta_{\alpha\gamma}y^\beta y^\delta - \delta_{\alpha\delta}y^\beta y^\gamma\big) + s\big(\delta_{\alpha\delta}\delta_{\gamma\beta} - \delta_{\beta\delta}\delta_{\alpha\gamma}\big)\big);$$

$$R_{\alpha\beta^*\gamma^*\delta} = \frac{1}{16}\big(2\delta_{\alpha\beta}\delta_{\gamma\delta} + s\delta_{\beta\delta}\delta_{\alpha\gamma} - s\delta_{\beta\gamma}y^\alpha y^\delta\big); \quad R_{i\beta^*\gamma^*\delta} = \frac{1}{16}\delta_{\beta\gamma}y^\delta;$$

$$R_{\alpha ij\delta} = -\frac{1}{16}\delta_{\alpha\delta}; \quad R_{i\beta\gamma\delta} = \frac{1}{16}\big(\delta_{\beta\gamma}y^\delta - \delta_{\beta\delta}y^\gamma\big); \quad R_{\alpha^* ij\delta^*} = -\frac{1}{16}\delta_{\alpha\delta}.$$

Observe that for each $\alpha \in \{1, \ldots, n\}$ the φ-sectional curvature of the planes generated by $\left\{\frac{\partial}{\partial y^\alpha}, \varphi\big(\frac{\partial}{\partial y^\alpha}\big)\right\}$ is $-2 - s + s(s-1)(y^\alpha)^2$. Hence the φ-sectional curvature of M is not constant. The components of the Ricci tensor are

$$R_{\alpha\beta} = \frac{1}{2}(sny^\alpha y^\beta - \delta_{\alpha\beta}) + \frac{1}{4}\left((s-1)y^\alpha y^\beta + (s-1)^2\delta_{\alpha\beta}\sum_{\rho=1}^{n}(y^\rho)^2\right),$$

$$R_{\alpha^*\beta^*} = \frac{1}{4}\delta_{\alpha\beta}\left(-2 + s(s-1)\sum_{\rho=1}^{n}(y^\rho)^2\right), \quad R_{\alpha\beta^*} = 0,$$

$$R_{\alpha i} = -\frac{1}{2}ny^\alpha + \frac{1}{4}(1-s)y^\alpha, \quad R_{ij} = \frac{1}{2}n, \quad R_{\alpha^* i} = 0.$$

Comparing with (1.12) of [9] we conclude that $\mathbf{R}^{2n+s}$ is not η-Einstein.

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