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SOME RESULTS ON GENERALIZED TOPOLOGICAL SPACES AND GENERALIZED SYSTEMS

W. K. MIN (Chuncheon)*

Abstract. We characterize some properties of generalized topological spaces and (g, g')-continuity by using an interior operator defined on a generalized topological space. Also, we introduce the notions of (ψ, ψ')-open map, gn-continuity, gn-open map and investigate their properties by using new interior (or closure) operators defined on generalized neighborhood systems of a nonempty set.

1. Introduction

In [2], Á. Császár introduced the notions of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function ($= (\psi, \psi')$-continuous function) by using a closure operator defined on generalized neighborhood systems.

The generalized neighborhood systems and generalized topologies are generalizations of neighborhood structures [4] and supratopologies [7], respectively. Thus we investigate some properties of a generalized topological space defined in [2] by using interior operators as we did in [4] and introduce new concepts of interior and closure on generalized neighborhood systems. In order to introduce and characterize the new notions of continuity and open map on generalized neighborhood systems, we use the concepts of interior and closure defined on generalized neighborhood systems.

In Section 3, some properties of generalized topological spaces and (g, g')-continuity are characterized by strong generalized interior operators. We in-

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troduce and study the concept of (g, g')-open maps on generalized topological spaces.

In Section 4, we introduce the new concepts of interior and closure induced by a GNS and investigate some general properties by using convergence of m-families. We also introduce the concept of gn-continuity.

In Section 5, we introduce the notions of (ψ, ψ')-open maps and gn-open maps and characterize their properties by using interior operators defined on GNS's.

2. Preliminaries

We now recall some concepts and notations defined in [2]. Let X be a nonempty set and g be a collection of subsets of X. Then g is called a *generalized topology* on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \bigcup_{i \in I} G_i \in g$. The elements of g are called *g-open* sets and the complements are called *g-closed* sets. Let $\psi : X \to \exp\big(\exp(X)\big)$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighborhood* of $x \in X$ and ψ is called a *generalized neighborhood system* (briefly GNS) on X. If g is a generalized topology on X and $A \subset X$, the *interior* of A (denoted by $i_g(A)$) is the union of all $G \subset A$, $G \in g$, and the *closure* of A (denoted by $c_g(A)$) is the intersection of all g-closed sets containing A. If ψ is a generalized neighborhood system on X and $A \subset X$, the interior and closure of A on ψ (denoted by $\iota_\psi(A)$, $\gamma_\psi(A)$, respectively) are defined as follows:

$$\iota_\psi(A) = \big\{ x \in A : \text{there exists } V \in \psi(x) \text{ such that } V \subset A \big\};$$

$$\gamma_\psi(A) = \big\{ x \in X : V \cap A \neq \emptyset \text{ for all } V \in \psi(x) \big\}.$$

Let ψ be a GNS on X and $G \in g_\psi$ iff $G \subset X$ satisfies: if $x \in G$ then there is $V \in \psi(x)$ such that $V \subset G$. Let g and g' be generalized topologies on X and Y, respectively. Then a function $f : X \to Y$ is said to be (g, g')-*continuous* if $G' \in g'$ implies that $f^{-1}(G') \in g$. And let ψ and ψ' be generalized neighborhood systems on X and Y, respectively. Then a function $f : X \to Y$ is said to be (ψ, ψ')-*continuous* if for $x \in X$ and $U \in \psi'\big(f(x)\big)$, there is $V \in \psi(x)$ such that $f(V) \subset U$.

Recall the following concepts defined in [4]: Given a set X, a collection $\mathbf{C}$ of subsets of X is called a *stack* if $A \in \mathbf{C}$ whenever $B \in \mathbf{C}$ and $B \subseteqq A$.

A stack $\mathbf{H}$ on a set X is called a *p-stack* if it satisfies the following condition: $A, B \in \mathbf{H}$ implies $A \cap B \neq \emptyset$.

Let X be a set, $\dot{x} = \big\{ A \subset X : \{x\} \subset A \big\}$, and let $\nu = \big\{ \nu(x) : x \in X \big\}$, where for all $x \in X$, $\nu(x)$ is a p-stack and $\nu(x) \subseteqq \dot{x}$. Then ν is called a *neighborhood structure* on X, $\nu(x)$ is called the *ν-neighborhood stack at x*, and

(X,ν) is called a *neighborhood space*. A p-stack $\mathbf{H}$ on X ν-converges to x if $\nu(x) \subseteqq \mathbf{H}$.

3. Some results on generalized topology

Consider a function $I: 2^X \to 2^X$ satisfying these axioms:

($\mathbf{C}1$) $I(A) \subset A$, for all $A \subset X$;
($\mathbf{C}2$) $(A \subset B) \Rightarrow I(A) \subset I(B)$, for all $A, B \in 2^X$;
($\mathbf{C}3$) $I\big(I(A)\big) = I(A)$, for all $A \subset X$;
($\mathbf{C}4$) $I(X) = X$.

Then the function I is called:
(a) a strong generalized interior operator (shortly sgio) if ($\mathbf{C}1$), ($\mathbf{C}2$) and ($\mathbf{C}3$) hold;
(b) a generalized interior operator (shortly gio) if ($\mathbf{C}1$) and ($\mathbf{C}2$) hold;
(c) a quasi-interior operator [6, 7] if ($\mathbf{C}1$),($\mathbf{C}2$), ($\mathbf{C}3$) and ($\mathbf{C}4$) hold;
(d) an interior operator [4, 5] if ($\mathbf{C}1$), ($\mathbf{C}2$) and ($\mathbf{C}4$) hold.
We call (X,I) an *sgi (resp. gi)-space* when I is an sgio (resp. gio).

LEMMA 3.1. *Let ψ be a GNS on X and let $I: 2^X \to 2^X$ be defined as $I(A) = \iota_\psi(A)$ for each $A \subset X$. Then $I = \iota_\psi$ is a gio.*

PROOF. By definition of ι_ψ, this is obvious.

EXAMPLE 3.2. Consider the example introduced in [2]: Let $X = R$ and $\psi(x) = \big\{ (x-1, x+1) \big\}$ for $x \in R$. For an open interval $A = (1,4) \subset R$, we have $\iota_\psi(A) = [2,3] \subset R$ but $\iota_\psi\big(\iota_\psi(A)\big) = \emptyset$. Thus $\iota_\psi\big(\iota_\psi(A)\big) \neq \iota_\psi(A)$.

LEMMA 3.3. *Let g be a GTS on X and let $I: 2^X \to 2^X$ be defined as $I(A) = i_g(A)$ for each $A \subset X$. Then $I = i_g$ is an sgio.*

PROOF. By definition of i_g, this is obvious.

REMARK. (a) Let ψ be a GNS on X such that for all $x \in X$, $\psi(x)$ is a p-stack and $\psi(x) \subseteqq \dot{x}$. Then i_ψ is a quasi-interior operator, and i_ψ induces a supratopological space (X, τ_{i_ψ}) since i_ψ satisfies ($\mathbf{C}4$) [4].
(b) Let g be a GT on X with $X \in g$. Then g is a supratopology, and so i_g is a quasi-interior operator [4].

Let $\lambda \subset 2^X$ for a set X and define $i_\lambda A = \bigcup\{L \in \lambda : L \subset A\}$ (in particular, $i_\lambda A = \emptyset$ if no $L \in \lambda$ satisfies $L \subset A$).

COROLLARY 3.4. *The operation $i_\lambda : 2^X \to 2^X$ is an sgio.*

Proof. Obvious by Lemma 1.1 in [3].

Lemma 3.5 [3, Lemma 1.2]. *If* $\iota : 2^X \to 2^X$ *satisfies* $(\mathbf{C}1)$, $(\mathbf{C}2)$ *and* $(\mathbf{C}3)$, *then* $\iota = i_\lambda$ *for some* $\lambda \subset 2^X$; λ *can be chosen in the manner that it is a GT.*

Lemma 3.6 [3, Corollary 1.3]. *If* λ *is a GT in* X *and* $\iota : 2^X \to 2^X$ *satisfies* $(\mathbf{C}1)$, $(\mathbf{C}2)$ *and* $(\mathbf{C}3)$, *then* $\iota = i_\lambda$ *implies* $\lambda = \{L \subset X : \iota L = L\}$.

Let $SGI(X)$ be the set of all sgio's on X and let $GT(X)$ be the set of all generalized topologies on X.

Theorem 3.7. *Let X be a nonempty set. Then there exists a bijection between $SGI(X)$ and $GT(X)$.*

Proof. Define $\phi : GT(X) \to SGI(X)$ such as $\phi(g) = i_g$. Then by Lemma 3.3, ϕ is well defined. Let $g_1, g_2 \in GT(X)$ with $g_1 \neq g_2$; then there exists an element, say, $U \in g_1$ such that $U \notin g_2$ and $i_{g_1}(U) = U \neq i_{g_2}(U)$. Thus $i_{g_1} \neq i_{g_2}$, so that ϕ is injective.

From Lemma 3.5 and Lemma 3.6, it follows that ϕ is surjective.

Definition 3.8. Let (X, I) and (Y, I') be two sgi (resp. gi)-spaces. Then $f : (X, I) \to (Y, I')$ is said to be

(a) *sgi (resp. gi)-continuous* if for all $A \in Y$, $f^{-1}\big(I'(A)\big) \subset I\big(f^{-1}(A)\big)$.

(b) *sg-interior* if for all $A \in Y$, $I\big(f^{-1}(A)\big) \subset f^{-1}\big(I'(A)\big)$.

Theorem 3.9. *Let (X, g_1) and (Y, g_2) be generalized topological spaces. Then $f : (X, g_1) \to (Y, g_2)$ is (g_1, g_2)-continuous iff $f : (X, i_{g_1}) \to (Y, i_{g_2})$ is sgi-continuous.*

Proof. Suppose f is (g_1, g_2)-continuous; then $f^{-1}\big(i_{g_2}(A)\big)$ is g_1-open for $A \subset Y$. Thus we have $f^{-1}\big(i_{g_2}(A)\big) \subset i_{g_1}\big(f^{-1}(A)\big)$.

Conversely, let $U \in g_2$; then by hypothesis and $i_{g_2}(U) = U$, we get $f^{-1}(U) \subset i_{g_1}\big(f^{-1}(U)\big)$. Thus $f^{-1}(U)$ is g_2-open.

Definition 3.10. Let (X, g_1) and (Y, g_2) be generalized topological spaces. Then $f : (X, g_1) \to (Y, g_2)$ is said to be (g_1, g_2)-*open* if for any g_1-open set U in X, $f(U)$ is g_2-open in Y.

Theorem 3.11. *Let $f : (X, I_1) \to (Y, I_2)$ be a function between the sgi-spaces (X, I_1) and (Y, I_2). Then f is sg-interior iff $f\big(I_1(A)\big) \subset I_2\big(f(A)\big)$ for $A \subset X$.*

PROOF. Suppose that f is sg-interior and $y \in f\big(I_1(A)\big)$; then there exists $x \in I_1(A) \subset I_1\big(f^{-1}f(A)\big)$ such that $f(x) = y$. Since f is sg-interior, we get $f(x) \in I_2\big(f(A)\big)$.

For the converse, suppose $f\big(I_1(A)\big) \subset I_2\big(f(A)\big)$ and $x \in I_1\big(f^{-1}(A)\big)$. Then since I_2 is an sgio, we get $f(x) \in I_2\big(ff^{-1}(A)\big) \subset I_2(A)$, so that $x \in f^{-1}\big(I_2(A)\big)$.

THEOREM 3.12. *Let $f : (X, g_1) \to (Y, g_2)$ be a function between generalized topological spaces (X, g_1) and (Y, g_2). Then f is (g_1, g_2)-open iff $f : (X, i_{g_1}) \to (Y, i_{g_2})$ is sg-interior.*

PROOF. Suppose that f is sg-interior and A is a g_1-open set in X. Let $y \in f(A)$ such that $f(x) = y$ for some $x \in A$; then since f is sg-interior, $y \in i_{g_2}\big(f(A)\big)$ by Theorem 3.11. Hence $f(A)$ is a g_2-open set.

For the converse, let $x \in i_{g_1}\big(f^{-1}(B)\big)$ for a subset B in Y. Then there exists a g_1-open set V in X such that $x \in V \subset f^{-1}(B)$. Since f is a (g_1, g_2)-open map, we have $f(x) \in i_{g_2}(B)$, so that $x \in f^{-1}\big(i_{g_2}(B)\big)$.

From Definition 3.8, Theorem 3.11 and Theorem 3.12, we get the following theorem:

THEOREM 3.13. *Let $f : (X, g_1) \to (Y, g_2)$ be a function between generalized topological spaces (X, g_1) and (Y, g_2). Then the following are equivalent:*
(1) *f is (g_1, g_2)-open;*
(2) *$f\big(i_{g_1}(A)\big) \subset i_{g_2}\big(f(A)\big)$ for $A \subset X$;*
(3) *$i_{g_1}\big(f^{-1}(B)\big) \subset f^{-1}(i_{g_2}(B)$ for $B \subset X$.*

THEOREM 3.14. *Let X be a nonempty set and let $I : 2^X \to 2^X$ be a gio. Then $g_I = \big\{A : I(A) = A\big\}$ is a generalized topology.*

PROOF. It is obviously $\emptyset \in g_I$. In order to show that the collection g_I is closed with respect to arbitrary unions, let $U_i \in g_I$ for $i \in J$. Then since $U_i = I(U_i)$ for $i \in J$ and I is a gio, we get $\bigcup U_i \in g_I$.

THEOREM 3.15. *Let X be a nonempty set and let $I : 2^X \to 2^X$ be a gio. For $x \in X$, let $\psi_I(x) = \big\{A \subset X : I(A) = A$ and $x \in A\big\}$. Then $\psi = \big\{\psi_I(x) : x \in X\big\}$ is a GNS and g_I is clearly a generalized topology.*

PROOF. See Theorem 3.14.

DEFINITION 3.16. Let $f : (X, I) \to (Y, I')$ be a function between the gi-spaces (X, I) and (Y, I'). Then f is said to be *gi-continuous* if $f^{-1}\big(I'(A)\big) \subset I\big(f^{-1}(A)\big)$ for all $A \subset X$.

THEOREM 3.17. *Let ψ, ψ' be GNS's of X, Y, respectively. Then $f : (X, \psi) \to (Y, \psi')$ is (ψ, ψ')-continuous iff $f : (X, \iota_\psi) \to (Y, \iota_{\psi'})$ is gi-continuous.*

PROOF. Suppose f is (ψ, ψ')-continuous and $x \in f^{-1}\big(\iota_{\psi'}(A)\big)$ for $A \subset Y$. Then for $y = f(x)$, there exists a subset $V \in \psi'(y)$ such that $V \subset A$. Since f is (ψ, ψ')-continuous, there exists a subset $U \in \psi(x)$ such that $U \subset f^{-1}(V) \subset f^{-1}(A)$, so that $x \in \iota_\psi\big(f^{-1}(A)\big)$. Thus f is a gi-continuous.

For the converse, let $V \in \psi'\big(f(x)\big)$ for $x \in X$; then $f(x) \in \iota_{\psi'}(V)$. From definition of gi-continuity, $x \in f^{-1}\big(\iota_{\psi'}(V)\big) \subset \iota_\psi\big(f^{-1}(V)\big)$. Thus there exists a subset $U \in \psi(x)$ such that $U \subset f^{-1}(V)$.

4. A new interior operator on GNS's and gn-continuity

In this section, we introduce new concepts of interior and closure induced by a GNS and investigate some general properties. Also we introduce and investigate the concepts of generalized continuity by using concepts of new interior defined on a GNS.

DEFINITION 4.1. For a nonempty set X, a collection $\mathbf{H}$ of subsets of X is called an *m-family* on X if $\bigcap \mathbf{H} \neq \emptyset$. Let ψ be a GNS in X and let $\mathbf{H}$ be an m-family on X. Then we say that an m-family $\mathbf{H}$ converges to $x \in X$ if $\mathbf{H}$ is finer than $\psi(x)$ i.e. $\psi(x) \subset \mathbf{H}$. And we will call (X, ψ) *a generalized neighborhood space* (in short, *gn-space*).

DEFINITION 4.2. Let (X, ψ) be a gn-space, $A \subset X$.

(a) $I^*_\psi(A) = \big\{x \in A : A \in \psi(x)\big\}$.

(b) $\mathrm{cl}^*_\psi(A) = \big\{x \in X : X - A \notin \psi(x)\big\}$.

From Definition 4.2, we get the following theorem:

THEOREM 4.3. *Let (X, ψ) be a gn-space, $A \subset X$.*

(a) *$I^*_\psi(A) \subset A$ and $A \subset \mathrm{cl}^*_\psi(A)$.*

(b) *$\mathrm{cl}^*_\psi(A) = X - I^*_\psi(X - A)$ and $I^*_\psi(A) = X - \mathrm{cl}^*_\psi(X - A)$.*

(c) *$I^*_\psi(A) \subset \iota_\psi(A)$ and $\gamma_\psi(A) \subset \mathrm{cl}^*_\psi(A)$.*

REMARK. A neighborhood space defined in [4] is a gn-space with special properties: Let (X, ψ) be a gn-space, where ψ is a GNS on X such that for all $x \in X$, $\psi(x)$ is a p-stack and $\psi(x) \subseteqq \dot{x}$. Then ι_ψ is a generalized interior operator with the condition ($\mathbf{C}$4). Thus ι_ψ induces a neighborhood space (X, ψ).

Example 4.4. Let $X=\{a,b,c\}$ and $A=\{a,b\}$. Consider $\psi(a)=\big\{\{a\},\{a,b\}\big\}$, $\psi(b)=\big\{\{b\}\big\}$ and $\psi(c)=\big\{\{c\}\big\}$. Then $I^*_\psi(A)=\{a\}$ but $\iota_\psi(A)=\{a,b\}$, and so $I^*_\psi(A)\neq\iota_\psi(A)$. Similarly, we can show that $\gamma_\psi(A)\neq \mathrm{cl}^*_\psi(A)$. Furthermore, let $B=X\supsetneqq A$; but since $I^*_\psi(B)=\emptyset$, $I^*_\psi(A)\not\subseteq I^*_\psi(B)$.

Theorem 4.5. *Let (X,ψ) be a gn-space, $A\subset X$ and let $\mathbf{B}=\big\{A\subset X: I^*_\psi(A)=A\big\}$. Then $g_{I^*}=\big\{\bigcup\sigma:\sigma\subset\mathbf{B}\big\}$ is a generalized topology and it is coarser than the generalized topology g_ι defined as in Theorem* 3.14.

Proof. The first part follows from the proof of Theorem 3.14. If $I^*_\psi(A)=A$ for a subset A in X, then from Theorem 4.3(c) we get $\iota_\psi(A)=A$. Thus the second statement is proved.

Remark. Let (X,ψ) be a gn-space, where ψ is a GNS on X such that for all $x\in X$, $\psi(x)$ is a p-stack and $\psi(x)\subseteqq\dot{x}$. Then $\mathbf{B}=\big\{A\subset X: I^*_\psi(A)=A\big\}$ is a base for the supratopology $g_{I^*}=\big\{\bigcup\sigma:\sigma\subset\mathbf{B}\big\}$ generated by $\mathbf{B}$.

Theorem 4.6. *Let (X,ψ) be a gn-space, $A\subset X$.*

(a) $I^*_\psi(A)=\{x\in A: A\in\mathbf{H}$, *for every m-family* $\mathbf{H}$ *converging to* $x\}$.

(b) $\mathrm{cl}^*_\psi(A)=\{x\in X$*: there exists an m-family* $\mathbf{H}$ *such that* $\mathbf{H}$ *converges to* x *and* $X-A\notin\mathbf{H}\}$.

Proof. (a) For $x\in X$, let $x\in I^*_\psi(A)$ and any m-family $\mathbf{H}$ converge to x. Then from Definition 4.2 and Definition 4.1, it follows $A\in\psi(x)\subset\mathbf{H}$.

Suppose that for every m-family $\mathbf{H}$ converging to x, $A\in\mathbf{H}$. Then $A\in\psi(x)$, so that $x\in I^*_\psi(A)$.

(b) Let $x\in\mathrm{cl}^*_\psi(A)$; then $X-A\notin\psi(x)$. There are two cases to be considered:

In case $\psi(x)=\emptyset$, let $\mathbf{H}=\big\{\{x\}\cup A\big\}$; then $\mathbf{H}$ is an m-family converging to x and $X-A\notin\mathbf{H}$.

In case $\psi(x)\neq\emptyset$, let $\mathbf{H}=\psi(x)$. Then $\mathbf{H}$ satisfies the condition.

For the reverse inclusion, let $\mathbf{H}$ be an m-family converging to x and $X-A\notin\mathbf{H}$; then since $\psi(x)$ is contained in $\mathbf{H}$, $X-A\notin\psi(x)$. Hence by Definition 4.2, we have $x\in\mathrm{cl}^*_\psi(A)$.

Definition 4.7. Let $f:(X,\psi)\to(Y,\psi')$ be a function between the gn-spaces (X,ψ) and (Y,ψ'). Then f is called *gn-continuous* if for every $A\in\psi'\big(f(x)\big)$, $f^{-1}(A)$ is in $\psi(x)$.

Remark. Let $\psi\in\mathrm{GNS}\,(X)$, $\psi'\in\mathrm{GNS}\,(Y)$ and let $f:(X,\psi)\to(Y,\psi')$ be a function. Then: gn-continuous $\Rightarrow$ (ψ,ψ')-continuous $\Rightarrow$ $(g_\psi,g_{\psi'})$-continuous.

But from Example 2.2 in [2] and the next example, we can show that the converse relations need not be true.

EXAMPLE 4.8. Let $X = \{a, b, c\}$. Consider two GNS's ψ and ψ' on X defined as follows: $\psi(a) = \{\{a\}, \{a, b\}\}$, $\psi(b) = \{\{b\}\}$, $\psi(c) = X$, $\psi'(a) = \{\{a\}, \{a, b\}\}$, $\psi'(b) = \{\{a, b\}\}$ and $\psi'(c) = X$.

Let $f : (X, \psi) \to (X, \psi')$ be a function defined by $f(x) = x$, for $x \in X$. Then f is (ψ, ψ')-continuous, but not gn-continuous.

THEOREM 4.9. *Let $f : (X, \psi) \to (Y, \psi')$ be a function between the gn-spaces (X, ψ) and (Y, ψ'). Then the following are equivalent:*

(a) *f is gn-continuous;*

(b) $f^{-1}\big(I^*_{\psi'}(A)\big) \subset I^*_{\psi}\big(f^{-1}(A)\big)$*;*

(c) $\mathrm{cl}^*_{\psi}(f^{-1}(B) \subset f^{-1}\big(\mathrm{cl}^*_{\psi'}(B)\big)$.

PROOF. (a) $\Rightarrow$ (b). Suppose f is gn-continuous and $x \in f^{-1}\big(I^*_{\psi'}(A)\big)$; then $A \in \psi'\big(f(x)\big)$. $f^{-1}(A) \in \psi(x)$ follows from the gn-continuity, so that $x \in I^*_{\psi}\big(f^{-1}(A)\big)$.

(b) $\Rightarrow$ (a). It is obtained from Definition 4.2 and (b).

(b) $\Leftrightarrow$ (c). It is obtained from Theorem 4.3.

THEOREM 4.10. *Let $f : (X, \psi) \to (Y, \psi')$ be a bijective function between the gn-spaces (X, ψ) and (Y, ψ'). Then f is gn-continuous iff $f\big(\mathrm{cl}^*_{\psi}(A)\big) \subset \mathrm{cl}^*_{\psi'}\big(f(A)\big)$ for $A \subset X$.*

PROOF. Suppose $f^{-1}\big(I^*_{\psi'}(A)\big) \subset I^*_{\psi}\big(f^{-1}(A)\big)$ for each $A \subset X$. From Theorem 4.9(b), Theorem 4.3 and f is injective,

$$f^{-1}\Big(\mathrm{cl}^*_{\psi'}\big(f(A)\big)\Big) = f^{-1}\Big(Y - I^*_{\psi'}\big(Y - f(A)\big)\Big) \supset \mathrm{cl}^*_{\psi}(A).$$

For the converse, suppose $f\big(\mathrm{cl}^*_{\psi}(A)\big) \subset \mathrm{cl}^*_{\psi'}\big(f(A)\big)$ for $A \subset X$. Since f is surjective and gn-continuous,

$$I^*_{\psi}\big(f^{-1}(A)\big) \supset X - f^{-1}\big(\mathrm{cl}^*_{\psi'}\big(ff^{-1}(Y - A)\big) = f^{-1}\big(I^*_{\psi'}(A)\big).$$

Thus the proof is complete.

THEOREM 4.11. *Let $f : (X, \psi) \to (Y, \psi')$ be an injective function between the gn-spaces (X, ψ) and (Y, ψ'). Then f is gn-continuous iff for an m-family $\mathbf{H}$ converging to $x \in X$, $f(\mathbf{H})$ converges to $f(x)$.*

PROOF. Suppose f is gn-continuous and $\mathbf{H}$ is an m-family converging to $x \in X$. It is obvious that $f(\mathbf{H}) = \{f(F) : F \in \mathbf{H}\}$ is an m-family on Y. By hypothesis, we get $\psi'(f(x)) \subset f(\psi(x)) \subset f(\mathbf{H})$, so that $f(\mathbf{H})$ converges to $f(x)$.

Conversely, let $G \in \psi'(f(x))$ for $G \subset Y$. By the condition we get $\psi'(f(x)) \subset f(\psi(x))$ for $x \in X$. Since f is injective, $f^{-1}(G) \in \psi(x)$, so that f is gn-continuous.

5. (ψ, ψ')-open maps and gn-open maps

In this section, we introduce the notions of (ψ, ψ')-open maps and gn-open maps and characterize their properties by using the interior operators defined on GNS's.

DEFINITION 5.1. Let ψ, ψ' be GNS's of X, Y, respectively. Then $f : X \to Y$ is said to be *(ψ, ψ')-open* if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \psi'(f(x))$ such that $V \subset f(U)$.

THEOREM 5.2. *Let ψ, ψ' be GNS's of X, Y, respectively. If $f : X \to Y$ is (ψ, ψ')-open, then it is $(g_\psi, g_{\psi'})$-open.*

PROOF. From the notions of generalized topologies g_ψ, $g_{\psi'}$ and (ψ, ψ')-open map, this is obvious.

The following example shows that a $(g_\psi, g_{\psi'})$-open map may not be (ψ, ψ')-open.

EXAMPLE 5.3. Let R be the set of all real numbers. Consider for $x \in R$,

$$\psi(x) = \begin{cases} \{[x, \infty)\}, & \text{if } x \text{ is a rational number,} \\ \{(-\infty, x]\}, & \text{if } x \text{ is an irrational number.} \end{cases}$$

and $\psi'(x) = R$. Then $g_\psi = g_{\psi'} = \{\emptyset, R\}$. If $f : R \to R$ is the identity function, then f is $(g_\psi, g_{\psi'})$-open but not (ψ, ψ')-open.

THEOREM 5.4. *Let ψ, ψ' be GNS's of X, Y, respectively and let $f : X \to Y$ be a function. Then the following are equivalent:*

(a) *f is (ψ, ψ')-open;*

(b) *$\iota_\psi(f^{-1}(A)) \subset f^{-1}(\iota_{\psi'}(A)$ for $A \subset Y$;*

(c) *$f^{-1}(\gamma_{\psi'}(A)) \subset \gamma_\psi(f^{-1}(A))$ for $A \subset Y$;*

(d) *$f(\iota_\psi(B)) \subset \iota_{\psi'}(f(B))$ for $B \subset X$.*

PROOF. (a) $\Rightarrow$ (b). Suppose f is (ψ, ψ')-open and $x \in \iota_\psi\big(f^{-1}(A)\big)$; then there is $U \in \psi(x)$ such that $U \subset f^{-1}(A)$. Since f is (ψ, ψ')-open, we can say there is $V \in \psi'\big(f(x)\big)$ such that $V \subset f(U) \subset A$, so that $f(x) \in \iota_{\psi'}(A)$.

(b) $\Rightarrow$ (a). Let $U \in \psi(x)$ for $x \in X$; then $x \in \iota_\psi(U) \subset \iota_\psi\big(f^{-1}f(U)\big)$. By hypothesis, $x \in f^{-1}(\iota_{\psi'}\big(f(U)\big)$, and so $f(x) \in \iota_{\psi'}\big(f(U)\big)$. From definition of the interior operator $\iota_{\psi'}$, there exists an element $V \in \psi'\big(f(x)\big)$ such that $V \subset f(U)$.

(b) $\Leftrightarrow$ (c). From Lemma 1.4 in [2], it is obvious.

(b) $\Rightarrow$ (d). It is easily obtained.

(d) $\Rightarrow$ (a). Let $U \in \psi(x)$ for $x \in X$. Then we have $f(x) \in f\big(\iota_\psi(U)\big)$, so that by hypothesis $f(x) \in \iota_{\psi'}\big(f(U)\big)$. Thus we can say that there exists an element $V \in \psi'\big(f(x)\big)$ such that $V \subset f(U)$.

DEFINITION 5.5. Let $f : (X, \psi) \to (Y, \psi')$ be a function between the gn-spaces (X, ψ) and (Y, ψ'). Then f is called a gn-open map if for $x \in X$ and for every $A \in \psi(x)$, $f(A) \in \psi'\big(f(x)\big)$.

REMARK. Let ψ, ψ' be GNS's of X, Y, respectively and let $f : (X, \psi) \to (Y, \psi')$ be a function. From Definition 5.1 and Definition 5.5, every gn-open map is also (ψ, ψ')-open.

$$\text{gn-open} \;\Rightarrow\; (\psi, \psi')\text{-open} \;\Rightarrow\; (g_\psi, g_{\psi'})\text{-open}.$$

But the converse is not always true as shown by the following example:

EXAMPLE 5.6. Let $X = \{a, b, c\}$. Consider two GNS's ψ and ψ' on X defined as in Example 4.8. Let $f : (X, \psi') \to (X, \psi)$ be a function defined by $f(x) = x$, for $x \in X$. Then f is (ψ', ψ)-open, but not gn-open.

THEOREM 5.7. *Let $f : (X, \psi) \to (Y, \psi')$ be a function between two gn-spaces (X, ψ) and (Y, ψ'). Then f is gn-open iff $f\big(I^*_\psi(A)\big) \subset I^*_{\psi'}\big(f(A)\big)$ for $A \subset X$.*

PROOF. Suppose f is gn-open and $y \in f\big(I^*_\psi(A)\big)$. Then there exists $x \in I^*_\psi(A)$ such that $f(x) = y$. It follows $A \in \psi(x)$. Since f is gn-open, $f(A) \in \psi'\big(f(x)\big)$, so that we have $y \in I^*_{\psi'}\big(f(A)\big)$.

For the converse, let $A \in \psi(x)$; then $f(x) \in f\big(I^*_\psi(A)\big)$ and by hypothesis $f(x) \in I^*_{\psi'}\big(f(A)\big)$. Hence we have $f(A) \in \psi'\big(f(x)\big)$.

THEOREM 5.8. *Let $f : (X, \psi) \to (Y, \psi')$ be a bijection between the gn-spaces (X, ψ) and (Y, ψ'). Then f is gn-open iff $I^*_\psi\big(f^{-1}(B)\big) \subset f^{-1}\big(I^*_{\psi'}(B)\big)$ for $B \subset Y$.*

PROOF. Suppose f is gn-open and $x \in I^*_\psi\bigl(f^{-1}(B)\bigr)$ for $B \subset Y$; then $f^{-1}(B) \in \psi(x)$. Since f is a surjective gn-open map, $B \in \psi'\bigl(f(x)\bigr)$ and $f(x) \in I^*_{\psi'}(B)$. Thus we get $x \in f^{-1}\bigl(I^*_{\psi'}(B)\bigr)$.

Conversely, suppose $I^*_\psi\bigl(f^{-1}(B)\bigr) \subset f^{-1}\bigl(I^*_{\psi'}(B)\bigr)$ for $B \subset Y$ and $A \in \psi(x)$; then $x \in I^*_\psi(A)$. Since f is injective, $x \in I^*_\psi f^{-1}\bigl(f(A)\bigr)$ and by hypothesis we get $x \in f^{-1}\bigl(I^*_{\psi'}\bigl(f(A)\bigr)\bigr)$, so that $f(A) \in I^*_{\psi'}\bigl(f(x)\bigr)$.

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DEPARTMENT OF MATHEMATICS
KANGWON NATIONAL UNIVERSITY
CHUNCHEON, 200-701
KOREA
E-MAIL: WKMIN@CC.KANGWON.AC.KR