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# CUBIC SPLINE INTERPOLATION WITH QUASIMINIMAL B-SPLINE COEFFICIENTS

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**Abstract.** The end conditions for cubic spline interpolation with equidistant knots will be defined so as to make the (slightly modified) B-spline coefficients minimal. This produces good approximation results as compared e.g. with the not-a-knot spline.

## 1. Introduction

For a natural $n$ let $\Omega_n = \{a + ih,\ i = 0, \dots, n\}$ be an equidistant (uniform) partition of the real interval $[a, b]$ with $h = (b - a)/n$. Let $S_3(\Omega_n)$ be the linear space of cubic splines with regard to this partition. Any such spline $s$ can be written uniquely as

$$s = \sum_{i=-3}^{n-1} c_i B_{3,i},$$

where $B_{3,i}$ are the cubic B-splines for the extended knot sequence $\Omega_\infty = \{x_i = a + ih,\ i \in \mathbf{Z}\}$. For convenience, we give the derivatives of the B-spline $B_{3,i}$ supported in $[x_i, x_{i+4}]$ at the relevant knots in the following table:

| | $x_i$ | $x_{i+1}$ | $x_{i+2}$ | $x_{i+3}$ | $x_{i+4}$ |
|---|---|---|---|---|---|
| $B_{3,i}(x)$ | 0 | $1/6$ | $2/3$ | $1/6$ | 0 |
| $B'_{3,i}(x)$ | 0 | $1/2h$ | 0 | $-1/2h$ | 0 |
| $B''_{3,i}(x)$ | 0 | $1/h^2$ | $-2/h^2$ | $1/h^2$ | 0 |

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Given a real function $f$ defined in $[a,b]$, the interpolatory conditions $s(x_i) = f(x_i)$, $0 \leqq i \leqq n$ assume

$$Mc = \hat{f}, \tag{1}$$

where $M = \frac{1}{6}\,\mathrm{tridiag}\,(1,4,1)$ is an $(n+1)\times(n+3)$ matrix, $\hat{f} \equiv f|_{\Omega_n}$ is the column of the function values $\big(f(x_i)\big)_{i=0}^{n}$, and $c$ is the column of the unknown coefficients $(c_i)_{i=-3}^{n-1}$. We use the notations of [5], Ch. 6.

Our aim is to fix the two end conditions such that the resulting spline

- *minimizes* the quadratic sum $\|c\|^2 = \sum_{i=-3}^{n-1} c_i^2$ of the coefficients, and
- *reproduces* the set of cubic polynomials.

Unfortunately, these requirements are conflicting. Hence we will introduce (in the form of a diagonal matrix) further parameters to 'scale down' the B-splines, especially the near-end ones.

The method derived has the optimal order of convergence. To prove this, we make use of the properties of the not-a-knot spline [3], cf. [4]: "(ii) It may be possible to carry out the argument by perturbation, ... showing that the change in the side conditions ... is gentle enough (at least for large $n$) to change $\|P''\|$ by a bounded amount..."

The new, (quasi)minimal spline will not bear comparison, of course, with splines using derivative information at the ends; however, it proves to be superior to the not-a-knot spline, as numerical tests suggest.

## 2. Determining the end conditions

Let the additional unknown rows be $r_a$ and $r_b$, where the subscripts indicate that they are related to $a$ and $b$. Then we get the enlarged system

$$\begin{pmatrix} r_a \\ M \\ r_b \end{pmatrix} c = \begin{pmatrix} 0 \\ \hat{f} \\ 0 \end{pmatrix} \tag{2}$$

of linear equations with a square matrix.

Our first statement concerns the problem of minimality.

LEMMA 1. *The solution $c$ of* (2) *is the minimal solution of* (1) *if and only if $r_a$ and $r_b$ are linearly independent and are orthogonal to the rows of $M$, i.e.*

$$r_a M^T = 0, \qquad r_b M^T = 0.$$

*A possible solution for this is*

$$r_a = (1, \lambda_1, \lambda_1^2, \dots, \lambda_1^{n+2}), \quad r_b = (\lambda_1^{n+2}, \dots, \lambda_1^2, \lambda_1, 1) \tag{3}$$

*with* $\lambda_1 = -2 + \sqrt{3}$.

PROOF. As for the first part, it is enough to note that the minimal solution for (1) is given by

$$M^+ \hat{f} = M^T (MM^T)^{-1} \hat{f},$$

where $M^+$ stands for the Moore–Penrose pseudoinverse of $M$.

To prove (3), consider the homogeneous linear system $rM^T = 0$ as a recursion for $r$ with characteristic polynomial $\lambda^2 + 4\lambda + 1$. Its zeros are $\lambda_1$ and $\lambda_2 = 1/\lambda_1$, hence $r_a$ is appropriate, and so is $r_b$, for it is a scalar multiple of $(1, \lambda_2, \lambda_2^2, \dots, \lambda_2^{n+2})$.

COROLLARY. *Let us insert a diagonal positive definite matrix* $D_0$ *in the linear systems* (1-2) *to get the pair*

$$(MD_0^{-1})(D_0)c = \hat{f} \quad \text{and} \quad \begin{pmatrix} r_a D_0 \\ MD_0^{-1} \\ r_b D_0 \end{pmatrix} (D_0 c) = \begin{pmatrix} 0 \\ \hat{f} \\ 0 \end{pmatrix}.$$

*If* $r_a$ *and* $r_b$ *are chosen according to* (3)*, then the solution of the second is the minimal solution of the first equation – irrespective of* $D_0$*! This follows from* $(rD_0)(MD_0^{-1})^T = rM^T$*, with* $r = r_a$ *and* $r = r_b$.

REMARKS. 1. The spline obtained in this way is called *quasiminimal* because of the presence of $D_0$: note that in fact $\|D_0 c\|$ will be minimal.

2. The notation can be simplified by introducing $D = D_0^2$. With this, our system takes the form

$$\begin{pmatrix} r_a D \\ M \\ r_b D \end{pmatrix} c = \begin{pmatrix} 0 \\ \hat{f} \\ 0 \end{pmatrix}. \tag{4}$$

Thus, assuming $r_a$ and $r_b$ are the rows in (3), quasiminimality is assured, and we only have to care for the reproducing property.

3. Observe that $r_b$ is the reverse of $r_a$, or, by help of the so-called backward identity $J$ (where the ones lie on the secondary diagonal), $r_b = r_a J$ holds. We want to maintain this kind of symmetry for $D$ as well, by requiring $D = JDJ$, i.e. $DJ = JD$. Such matrices are called persymmetric; in our case we simply have

$$D = \operatorname{diag}(d_i), \quad d_i = d_{n+4-i}, \quad 1 \leqq i \leqq n+3.$$

Lemma 2. *Let $D$ be a positive definite persymmetric diagonal matrix, and $r_a$ and $r_b$ be such that the matrix on the left of* (4) *is invertible. Then the spline calculated on the basis of* (4) *is* reproducing *if and only if*

$$r_a D e^{(j)} = 0, \qquad 0 \leqq j \leqq 3, \tag{5}$$

*where* $e^{(j)} = \left(0^j, 1^j, \ldots, (n+2)^j\right)^T$.

Proof. The necessity follows from the unique representation of $\mathrm{id}^j$ in the form

$$\mathrm{id}^j = \sum_{i=-3}^{n-1} p_j(i)\ B_{3,i},$$

with $p_j$ a $j$-th degree polynomial, $0 \leqq j \leqq 3$. To prove sufficiency, first we show that (5) holds with $r_a$ replaced by $r_b$, too.

The case $j = 0$ is evident. If $j = 1$, then $r_b D e^{(1)} = r_a J D e^{(1)} = r_a D J e^{(1)}$, where $J e^{(1)}$ is a linear combination of $e^{(0)}$ and $e^{(1)}$, hence $r_b D e^{(1)} = 0$. Continuing this way, we get $r_b D e^{(j)} = 0$, $0 \leqq j \leqq 3$. The rest follows by the fact that the solution of (4) is unique.

Remark. Since (5) represents only four equations, the majority of the $d_i$-s can be fixed:

$$d_i = 1, \qquad 5 \leqq i \leqq n-1. \tag{6}$$

Assuming now (3), (6), and persymmetry for $D$, (5) can be solved for any $n \geqq 6$. We obtain e.g.

$$d_1 = \frac{1}{454}, \quad d_2 = \frac{8}{227}, \quad d_3 = \frac{50}{227}, \quad d_4 = \frac{152}{227}$$

for $n = 6$, and

$$d_1 = \frac{1}{758}, \quad d_2 = \frac{17}{758}, \quad d_3 = \frac{58}{379}, \quad d_4 = \frac{202}{379}$$

for $n = 7$.

It is worth calculating the limits of these parameters for the sake of the convergence proof, for stability reasons, and also since their first 15 digits are, starting from $n = 36$, unchanged. Note that, in fact, $d_i = d_i^{(n)}$.

Lemma 3. *Denote* $d_i^* = \lim_{n\to\infty} d_i$, $1 \leqq i \leqq 4$. *We have*

$$d_1^* = \frac{7-4\sqrt{3}}{36}, \quad d_2^* = \frac{15-8\sqrt{3}}{36}, \quad d_3^* = \frac{21-8\sqrt{3}}{36}, \quad d_4^* = \frac{29-4\sqrt{3}}{36} \tag{7}$$

*with* $d_i - d_i^* = O(\lambda_1^n)$, $\lambda_1 = -2 + \sqrt{3}$.

PROOF. We only display the system (5) to be solved, using matrix formalism. Let $M_a$ be the Vandermonde matrix with second generating row $(0,1,2,3)$, i.e. let

$$M_a = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{pmatrix},$$

let $M_b$ be the Vandermonde matrix with second row $(n+2, n+1, n, n-1)$, define matrices $\Lambda_a = \operatorname{diag}(1, \lambda_1, \lambda_1^2, \lambda_1^3)$, $\Lambda_b = \operatorname{diag}(\lambda_1^{n+2}, \lambda_1^{n+1}, \lambda_1^n, \lambda_1^{n-1})$, introduce the column vector

$$\text{rhs} = -\left( \sum_{i=4}^{n-2} t^i, \sum_{i=4}^{n-2} i t^i, \sum_{i=4}^{n-2} i^2 t^i, \sum_{i=4}^{n-2} i^3 t^i \right)^T,$$

and the unknown column $d = (d_i)_{i=1}^4$. Then (5) is equivalent with

$$(M_a \Lambda_a + M_b \Lambda_b)\, d = \text{rhs}\,. \tag{8}$$

Focusing now on determining the limit values $(d_i^*)$, the second matrix $M_b \Lambda_b$ can be omitted, and the right hand vector rhs also simplifies to

$$\text{rhs}^* = -\lambda_1^4 \mu_1 \begin{pmatrix} 1 \\ \mu_1(4 - 3\lambda_1) \\ \mu_1^2(16 - 23\lambda_1 + 9\lambda_1^2) \\ \mu_1^3(64 - 131\lambda_1 + 100\lambda_1^2 - 27\lambda_1^3) \end{pmatrix}$$

with $\mu_1 = (1 - \lambda_1)^{-1}$. Now we have the simpler system $M_a \Lambda_a\ d^* = \text{rhs}^*$, with the solution stated.

The order of convergence follows from the standard estimation for linear systems ([8], [5]):

$$\frac{\|\Delta x\|}{\|x\|} \leqq \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\Delta A\|}{\|A\|}} \left( \frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta b\|}{\|b\|} \right), \quad \kappa(A) = \|A\|\, \|A^{-1}\|$$

applied to $A = M_a \Lambda_a$, $x = d^*$, $b = \text{rhs}^*$, $\Delta A = M_b \Lambda_b$, $\Delta b = \text{rhs} - \text{rhs}^*$. Since both $\|\Delta A\| = O(\lambda_1^n)$ and $\|\Delta b\| = O(\lambda_1^n)$, the denominator on the right of the inequality, $1 - \|A^{-1}\| \|\Delta A\|$ is positive for $n$ large enough, showing that the perturbed matrix is also invertible and the estimate holds. Note finally that, using 2-norm (i.e. operator norm), $\kappa(M_a \Lambda_a) = 54.4587$.

REMARKS. 1. The limit values $(d_i^*)$ can be checked by the Inverse Symbolic Calculator [6] on the internet; calculating the $d_i$-s from (8) for $n$ large ($n \geqq 36$), the program recognizes the irrational values (7).

2. Using the limits $(d_i^*)$ instead of the values $(d_i)$, the resulting spline is, of course, only *asymptotically* reproducing with indicated error $O(\lambda_1^n)$.

3. The case of quadratic splines has been handled in [7], where the matrix $M$ was $(n+1)\times(n+2)$, and there were only two scaling parameters: $\alpha_n$ and $\beta_n$. In the present notation they are $d_1$ and $d_2$ with limits $\frac{1}{4}$ and $\frac{3}{4}$.

## 3. The convergence

To prove convergence, we will exploit the same property of the not-a-knot spline, the definition of which requires the third derivative to be continuous across $x_1$ and $x_{n-1}$. As the following table shows, we use overlined variables for the not-a-knot spline, to distinguish between the two kinds of splines:

| | first row | last row | coef | spline |
|---|---|---|---|---|
| Not-a-knot | $\overline{r}_a$ | $\overline{r}_b$ | $\overline{c}$ | $\overline{s}$ |
| Quasiminimal | $r_aD$ | $r_bD$ | $c$ | $s$ |

Thus, using equally spaced knots we have

$$\overline{r}_a = (1,-4,6,-4,1,0,\dots 0), \qquad \overline{r}_b = \overline{r}_a J,$$

the end-conditions assume $\overline{r}_a\,\overline{c} = 0$, $\overline{r}_b\,\overline{c} = 0$, and the not-a-knot spline has the representation $\overline{s} = \sum_{i=-3}^{n-1} \overline{c}_i B_{3,i}$ on $[a,b]$. Observe that $\overline{r}_a$ and $\overline{r}_b$ are special cases of the rows

$$\overline{r}_i = (0,\dots 0,1,-4,6,-4,1,0,\dots 0),$$

with the trailing '1' in the $i$-th position.

Our last perparatory lemma concerns the connection of both methods.

LEMMA 4. *The system*

$$\sum_{i=1}^{n-1} t_i \overline{r}_i = \frac{1}{d_i} r_a D$$

*of linear equations with* (3-6) *holding is consistent with solution*

$$t_i = \lambda_1^{i-1} + O(\lambda_1^n), \qquad \lambda_1 = -2+\sqrt{3}.$$

PROOF. The Kronecker–Kapelli theorem tells us that $r_a D$ is a linear combination of the $\overline{r}_i$-s if and only if $r_a D$ is orthogonal to the subspace $H$ of all rows, orthogonal to the $\overline{r}_i$-s. To find a basis in $H$, take the characteristic equation $(\mu - 1)^4 = 0$ of this homogeneous recursion. Its solution $\mu_1 = 1$ is of multiplicity 4, giving the basis

$$\{\mu_1^i,\ i\mu_1^i,\ i^2\mu_1^i,\ i^3\mu_1^i\} = \{1,\ i,\ i^2,\ i^3\}$$

for $H$. However, this is exactly the set $\left(e^{(j)}\right)_{j=0}^{3}$, therefore solvability is guaranteed, cf. (5). As regards the form of the solution $t$, we replace $d_i$ by $d_i^*$, $i = 1, \ldots, 4$ and distinguish three cases.

a) The majority – the 'middle' – of the equations has the form

$$t_{i-4} - 4t_{i-3} + 6t_{i-2} - 4t_{i-1} + t_i = \frac{1}{d_i^*}\lambda_1^{i-1}, \qquad 5 \leqq i \leqq n-5,$$

where the right hand side can be written as $36\lambda_1^{i-3}$, due to the immediately calculated equality $\lambda_1^2 = 36d_1^*$. The trial $t_i = \lambda_1^{i-1}$ gives now $(\lambda_1 - 1)^4 = 36\lambda_1^2$, which is true owing to the factorization

$$(\lambda - 1)^4 - 36\lambda^2 = (\lambda^2 + 4\lambda + 1)(\lambda^2 - 8\lambda + 1)$$

and the fact that $\lambda_1^2 + 4\lambda_1 + 1 = 0$.

b) The first four equations can immediately be verified to be true:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 6 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \\ \lambda_1^3 \end{pmatrix} = \frac{1}{d_1^*} \begin{pmatrix} d_1^* \\ \lambda_1 d_2^* \\ \lambda_1^2 d_3^* \\ \lambda_1^3 d_4^* \end{pmatrix}.$$

c) The last four equations are false either with $d_i$-s or with $d_i^*$-s, however, both sides are within $O(\lambda_1^n)$.

THEOREM. *For the quasiminimal spline $s$ defined by* (2) *to* (6) *it holds that*

$$f - s = O(h^4), \qquad f \in \mathbf{C}^4[a, b].$$

PROOF. We know that the not-a-knot spline $\overline{s}$ satisfies $f - \overline{s} = O(h^4)$. By the triangle inequality we have

$$\|f - s\| \leqq \|f - \overline{s}\| + \|\overline{s} - s\|$$

for the uniform norm, thus it suffices to investigate the difference $\overline{s} - s$ of the not-a-knot and quasiminimal splines. Since both of them satisfy the interpolatory conditions, their difference vanishes at the knots, and the homogeneous system $M(\overline{c} - c) = 0$ holds for the difference of the coefficients for the B-spline representations. Thus, $\overline{c} - c$ belongs to the two-dimensional subspace generated by $r_a^T$ and $r_b^T$, i.e.

$$\overline{c} - c = \alpha r_a^T + \beta r_b^T.$$

Multiplying by $r_a D$ and $r_b D$, resp., and using the definition

$$r_a D c = 0, \qquad r_b D c = 0$$

of quasiminimal splines, we obtain

$$\begin{pmatrix} r_a D r_a^T & r_a D r_b^T \\ r_b D r_a^T & r_b D r_b^T \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} r_a D \overline{c} \\ r_b D \overline{c} \end{pmatrix}.$$

The Gramian on the left can be calculated to tend to $\delta I_2$, a scalar multiple of the identity with $\delta = \frac{13}{\sqrt{3}} - \frac{15}{2} \approx 0.0055$. Hence it is enough to estimate the right hand vector. Since both coordinates are handled equally, we take $r_a D\overline{c}$, and apply Lemma 4 to get

$$r_a D \overline{c} = \sum_{i=1}^{n-1} t_i \overline{r}_i \overline{c} \quad \text{with} \quad t_i = \lambda_1^{i-1} + O(\lambda_1^n).$$

Here $\overline{r}_1 \overline{c} \equiv \overline{r}_a \overline{c} = 0$, $\overline{r}_{n-1} \overline{c} \equiv \overline{r}_b \overline{c} = 0$ by definition of the not-a-knot spline. For the remaining terms we have

$$\begin{aligned} \overline{r}_i \overline{c} &= \overline{c}_{i-4} - 4\overline{c}_{i-3} + 6\overline{c}_{i-2} - 4\overline{c}_{i-1} + \overline{c}_i \\ &= h^2 \big( \overline{s}''(x_{i+1}) - 2\overline{s}''(x_i) + \overline{s}''(x_{i-1}) \big) \\ &= 6h^2 \left( \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} - \overline{s}''(x_i) \right), \end{aligned}$$

due to the representations

$$s(x_j) = \frac{1}{6}\big(c_{j-3} + 4c_{j-2} + c_{j-1}\big), \quad s''(x_j) = \frac{1}{h^2}\big(c_{j-3} - 2c_{j-2} + c_{j-1}\big),$$

and the identity

$$\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} = \frac{\overline{s}''(x_{i-1}) + 4\overline{s}''(x_i) + \overline{s}''(x_{i+1})}{6}. \tag{9}$$

Since $\frac{1}{h^2}\big(f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\big) - f''(x_i) = O(h^2)$ by the assumption on $f$, and $f''(x_i) - \overline{s}''(x_i) = O(h^2)$ by the known property of the not-a-knot spline, we conclude that $\overline{r}_i\overline{c} = O(h^4)$. Consequently, $\overline{c} - c = O(h^4)$ and we have for all $x \in [a,b]$

$$\left|\overline{s}(x) - s(x)\right| \leqq \max|\overline{c}_i - c_i| \cdot \left|\sum_{i=-3}^{n-1} B_{3,i}(x)\right|,$$

i.e.

$$\|\overline{s} - s\|_{C[a,b]} \leqq \|\overline{c} - c\|_\infty = O(h^4)$$

by the partition of unity for B-splines. This completes the proof.

Remarks. 1. Identity (9) is interesting in itself: it is true for any interpolation spline – irrespective of the two additional conditions. It can be found in [1] as well, in a more general context.

2. There is another natural (not minimal) choice for the supplementary conditions. Disregard to this the settings (6) and choose

$$\varrho_a = \left(\binom{n+1}{0}, \binom{n+1}{1}, \dots \binom{n+1}{n+1}, 0\right), \qquad \varrho_b = \varrho_a J$$

instead. One attains this by considering the last two columns of the inverse Vandermonde matrix, resulting in $\varrho_a e^{(j)} = 0$, $0 \leqq j \leqq 3$, a condition for reproducibleness, cf. (5). Applied to a $k$-th degree polynomial $f, k \leqq n+1$, the coefficients $c_i$ obtained here also are $k$-th degree *polynomials* of their subscript. This method is favourable for analytical functions, however fails to converge for functions of the class $C^4[a,b]$, by the analogy with Lagrange interpolation.

## 4. Numerical tests

The following notations will be used. The quasiminimal spline with scaling factors $(d_i)$ and their limits $(d_i^*)$-s will be denoted by $QM$ and $QM^*$, resp. The above mentioned method with 'polynomial coefficients' will be referred to as method $PC$, and '$n-a-kn$' will stand for the not-a-knot spline. Then we have the following list of $C[a,b]$ errors; notice that the common factor $\gamma$ is picked out for brevity. Further, Runge$\,(x) = 1/(1+x^2)$, and the MATLAB conventions are used.

| $f$ | $[a, b]$ | $n$ | $\gamma$ | $QM$ | $QM^*$ | $PC$ | $n-a-kn$ |
|---|---|---|---|---|---|---|---|
| $\mathrm{id}^4$ | $[0, 1]$ | 10 | $1e-5$ | 5.1 | 5.4 | 0.6 | 6.8 |
| $\mathrm{id}^4$ | $[0, 1]$ | 20 | $1e-6$ | 3 | 3 | 0.3 | 4.2 |
| atan | $[0, 5]$ | 10 | $1e-3$ | 5.1 | 5.1 | 1.4 | 5.6 |
| atan | $[0, 5]$ | 20 | $1e-3$ | 0.34 | 0.34 | 0.06 | 0.48 |
| atan | $[-5, 5]$ | 10 | $1e-2$ | 2.9 | 2.9 | 21 | 2.9 |
| atan | $[-5, 5]$ | 20 | $1e-3$ | 1.8 | 1.8 | 795 | 1.8 |
| Runge | $[-5, 5]$ | 10 | $1e-2$ | 2.2 | 2.2 | 174 | 2.2 |
| Runge | $[-5, 5]$ | 20 | $1e-3$ | 3.2 | 3.2 | 1e4 | 3.2 |
| abs | $[-1, 1]$ | 10 | $1e-2$ | 3.4 | 3.4 | 61 | 3.39 |
| abs | $[-1, 1]$ | 20 | $1e-2$ | 1.7 | 1.7 | 2e3 | 1.7 |
| exp | $[0, 1]$ | 10 | $1e-5$ | 0.5 | 5 | 0.07 | 0.7 |
| exp | $[0, 1]$ | 20 | $1e-7$ | 3.3 | 3.3 | 0.4 | 4.5 |
| tan | $[-1.5, 1.5]$ | 10 | 1 | 2.72 | 2.71 | 1.9 | 2.8 |
| tan | $[-1.5, 1.5]$ | 20 | $1e-5$ | 1.24 | 1.24 | 0.57 | 1.3 |
| sin | $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ | 10 | $1e-2$ | 2.52 | 2.85 | 39 | 2 |
| sin | $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ | 20 | $1e-3$ | 6.4 | 6.4 | 1.1 | 7.3 |

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