



TOEPLITZ DETERMINANTS IN ONE AND HIGHER DIMENSIONS*

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Abstract In this study, we derive the sharp bounds of certain Toeplitz determinants whose entries are the coefficients of holomorphic functions belonging to a class defined on the unit disk \mathbb{U} . Furthermore, these results are extended to a class of holomorphic functions on the unit ball in a complex Banach space and on the unit polydisc in \mathbb{C}^n . The obtained results provide the bounds of Toeplitz determinants in higher dimensions for various subclasses of normalized univalent functions.

Key words holomorphic mappings; starlike mappings; Toeplitz determinants; coefficient inequalities

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1 Introduction

Coefficient problems play a significant role in the growth of Geometric function theory. In fact, Bieberbach conjecture, which deals with the bounds of the coefficients of normalized analytic univalent functions defined on the unit disk, took 68 years to prove. While solving the conjecture, various new concepts were developed. Koebe one-quarter theorem illustrates an application of the coefficient problem, as it is proved using the second coefficient bound. However, Cartan [4] stated that Bieberbach conjecture does not hold in case of several complex variables. There are counterexamples, which show that many results in the Geometric function theory of one complex variable are not applicable for several complex variables (see [6]). We use the following notations for functions of one complex variable:

Let \mathcal{S} be the class of analytic univalent functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, having the series expansion of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Let \mathcal{S}^* , $\mathcal{S}^*(\alpha)$ and $\mathcal{SS}^*(\gamma)$ respectively denote the subclasses of \mathcal{S} , which contain starlike functions, starlike functions of order α ($0 \leq \alpha < 1$) and strongly starlike functions of order γ ($0 < \gamma \leq 1$). For more details about these classes, we refer [9].

For the class \mathcal{S} and its subclasses, various coefficient problems are studied. In particular, Ali *et al.* [2] obtained the sharp bounds of Toeplitz determinants, when the entries of the Toeplitz matrix are the coefficients of function $g \in \mathcal{S}$ and certain of its subclasses.

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For the coefficients $\{b_k\}_{k \geq 2}$ of the function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Toeplitz matrix is defined by

$$T_{m,n}(g) = \begin{bmatrix} b_n & b_{n+1} & \cdots & b_{n+m-1} \\ b_{n+1} & b_n & \cdots & b_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n+m-1} & b_{n+m-2} & \cdots & b_n \end{bmatrix}.$$

Thus, the second order Toeplitz determinant is

$$\det T_{2,2}(g) = b_2^2 - b_3^2 \tag{1.1}$$

and the third order Toeplitz determinant is given by

$$\det T_{3,1}(g) = \begin{vmatrix} 1 & b_2 & b_3 \\ b_2 & 1 & b_2 \\ b_3 & b_2 & 1 \end{vmatrix} = 2b_2^2 b_3 - 2b_2^2 - b_3^2 + 1. \tag{1.2}$$

Ye and Lim [24] showed that any $n \times n$ matrix over \mathbb{C} generically can be written as the product of some Toeplitz matrices or Hankel matrices. Toeplitz matrices and Toeplitz determinants have numerous applications in the field of pure as well as applied mathematics. They arise in partial differential equations, algebra, signal processing and time series analysis. For more applications of Toeplitz matrices and Toeplitz determinants, we refer [24] and the references cited therein.

In 2017, Ali *et al.* [2] determined the bound of $|\det T_{2,2}(g)|$ and $|\det T_{3,1}(g)|$, when entries of $T_{m,n}(g)$ are the coefficients of starlike functions. Recently, Ahuja *et al.* [1] obtained the bounds for various subclasses of starlike functions.

Theorem A ([2]) If $g \in \mathcal{S}^*$, then the following sharp bounds hold:

$$|\det T_{2,2}(g)| \leq 13 \text{ and } |\det T_{3,1}(g)| \leq 24.$$

Theorem B ([1]) If $g \in \mathcal{S}^*(\alpha)$, then

$$|\det T_{2,2}(g)| \leq (1 - \alpha)^2(4\alpha^2 - 12\alpha + 13).$$

For $\alpha \in [0, 2/3]$, the following inequality holds:

$$|\det T_{3,1}(g)| \leq 12\alpha^4 - 52\alpha^3 + 91\alpha^2 - 74\alpha + 24.$$

All these estimations are sharp.

Theorem C ([1]) Let $g \in \mathcal{S}^*[D, E]$, then the following sharp estimations hold:

(i) If $|D - 2E| \geq 1$, then

$$|\det T_{2,2}(g)| \leq \frac{(D - E)^2(D^2 + 4E^2 - 4DE + 4)}{4}.$$

(ii) If $E \leq \min\{(D - 1)/2, (3D - 1)/2\}$, then

$$|\det T_{3,1}(g)| \leq 1 + 2(D - E)^2 + (3D^2 - 5DE + 2E^2)(D^2 - 3DE + 2E^2)/4.$$

The class $\mathcal{S}^*[D, E]$ denotes the class of Janowski starlike functions [14], where $-1 \leq E < D \leq 1$.

Coefficient problems are also studied in the case of several variables. For instance, Xu and Liu [18] solved the Fekete-Szegő problem for a subclass of normalized starlike mappings on the unit ball of a complex Banach space. Xu *et al.* [23] obtained the bound of the same for a subclass of normalized quasi-convex mappings of type B on the unit ball of complex Banach space. Generalizing this work, Hamada *et al.* [12, 23] also determined the bound of Fekete-Szegő type inequality. Contrary to Fekete-Szegő inequality for various subclasses of \mathcal{S} , very few results are known for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in several complex variables [8, 10, 11, 21]. Numerous best-possible results concerning the coefficient estimates for subclasses of holomorphic mappings in higher dimensions are obtained in [3, 5, 7, 15, 17, 19, 20].

In higher dimensions, the following notations are used throughout the paper. Let X be a complex Banach space with respect to a norm $\|\cdot\|$ and \mathbb{C}^n denote the space of n complex variables $z = (z_1, z_2, \dots, z_n)'$. Also, let $\mathbb{B} = \{z \in X : \|z\| < 1\}$ be the unit ball in X and \mathbb{B}^n be the Euclidean unit ball in \mathbb{C}^n . Further, the boundary and distinguished boundary of \mathbb{U}^n is denoted by $\partial\mathbb{U}^n$ and $\partial_0\mathbb{U}^n$, respectively.

For each $z \in X \setminus \{0\}$, consider the set

$$T_z = \{l_z \in L(X, \mathbb{C}) : l_z(z) = \|z\|, \|l_z\| = 1\},$$

where $L(X, Y)$ denotes the set of continuous linear operators from X into a complex Banach space Y . Let I denote the identity in $L(X, X)$. This set is non-empty according to the Hahn-Banach theorem.

Let $\mathcal{H}(\Omega)$ denote the set of all holomorphic mappings from Ω into X . If $g \in \mathcal{H}(\mathbb{B})$, then for each $k = 1, 2, \dots$, there is a bounded symmetric linear mapping

$$D^k g(z) : \prod_{j=1}^k X \rightarrow X,$$

called the k -th order Fréchet derivative of g at z such that

$$g(w) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k g(z)((w - z)^k)$$

for all w in some neighborhood of z . It is understood that

$$D^0 g(z)((w - z)^0) = g(z),$$

and for $k \geq 1$,

$$D^k g(z)((w - z)^k) = D^k g(z) \underbrace{(w - z, w - z, \dots, w - z)}_{k\text{-times}}.$$

We say that g is normalized if $g(0) = 0$ and $Dg(0) = I$. A holomorphic mapping $g : \Omega \rightarrow X$ is said to be biholomorphic on the domain Ω if $g(\Omega)$ is a domain in X and the inverse g^{-1} exists and is holomorphic on $g(\Omega)$. If for each $z \in \Omega$, $Dg(z)$ has a bounded inverse, the mapping g is said to be locally biholomorphic. As in the finite dimensional case, let $\mathcal{S}(\mathbb{B})$ denote the set of normalized biholomorphic mappings from \mathbb{B} into X . The class $\mathcal{S}(\mathbb{B})$ of normalized biholomorphic mappings is not a normal family on the unit ball in \mathbb{C}^n .

On a bounded circular domain $\Omega \subset \mathbb{C}^n$, the first and the m -th Fréchet derivative of a holomorphic mapping $g : \Omega \rightarrow X$ are written by $Dg(z)$ and $D^m g(z)(a^{m-1}, \cdot)$, respectively. The matrix representations are

$$Dg(z) = \left(\frac{\partial g_j}{\partial z_k} \right)_{1 \leq j, k \leq n},$$

$$D^m g(z)(a^{m-1}, \cdot) = \left(\sum_{p_1, p_2, \dots, p_{m-1}=1}^n \frac{\partial^m g_j(z)}{\partial z_k \partial z_{p_1} \dots \partial z_{p_{m-1}}} a_{p_1} \dots a_{p_{m-1}} \right)_{1 \leq j, k \leq n},$$

where $g(z) = (g_1(z), g_2(z), \dots, g_n(z))'$, $a = (a_1, a_2, \dots, a_n)' \in \mathbb{C}^n$.

The following class was defined by Hamada *et al.* [13].

Definition 1.1 Let $g : \mathbb{B} \rightarrow X$ be a normalized locally biholomorphic mapping and $\alpha \in (0, 1)$. We say that g is starlike of order α if

$$\left| \frac{1}{\|z\|} l_z([Dg(z)]^{-1}g(z)) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad \forall z \in \mathbb{B} \setminus \{0\}, l_z \in T(z).$$

In case of $X = \mathbb{C}^n$ and $\mathbb{B} = \mathbb{U}^n$, the above condition is equivalent to

$$\left| \frac{q_k(z)}{z_k} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad \forall z \in \mathbb{U}^n \setminus \{0\},$$

where

$$q(z) = (q_1(z), q_2(z), \dots, q_n(z))' = (D(g(z)))^{-1}g(z)$$

is a column vector in \mathbb{C}^n and k satisfies

$$|z_k| = \|z\| = \max_{1 \leq j \leq n} \{|z_j|\}.$$

For $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, the relation is equivalent to

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > \alpha, \quad z \in \mathbb{U}.$$

Let $\mathcal{S}_\alpha^*(\mathbb{B})$ denote the class of starlike mappings of order α on \mathbb{B} . When $X = \mathbb{C}$ and $\mathbb{B} = \mathbb{U}$, the class $\mathcal{S}_\alpha^*(\mathbb{U})$ is denoted by $\mathcal{S}^*(\alpha)$.

Definition 1.2 ([16]) Let $g : \mathbb{B} \rightarrow X$ be a normalized locally biholomorphic mapping and $\gamma \in (0, 1]$. We say that f is strongly starlike mapping of order γ if

$$|\arg l_z([Dg(z)]^{-1}g(z))| < \frac{\pi}{2}\gamma, \quad \forall z \in \mathbb{B} \setminus \{0\}, l_z \in T(z).$$

In case of $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, the above condition is equivalent to

$$\left| \arg \frac{q_j(z)}{z_j} \right| < \frac{\pi}{2}\gamma, \quad z \in \mathbb{U}^n \setminus \{0\}.$$

where

$$q(z) = (q_1(z), q_2(z), \dots, q_n(z))' = (D(g(z)))^{-1}g(z)$$

is a column vector in \mathbb{C}^n and j satisfies

$$|z_j| = \|z\| = \max_{1 \leq k \leq n} \{|z_k|\}.$$

In case of $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, the relation is equivalent to

$$\left| \arg \frac{zg'(z)}{g(z)} \right| < \frac{\pi}{2}\gamma, \quad z \in \mathbb{U}.$$

Let $\mathcal{SS}_\gamma^*(\mathbb{B})$ denote the class of starlike mappings of order γ on \mathbb{B} . When $X = \mathbb{C}$ and $\mathbb{B} = \mathbb{U}$, the class $\mathcal{SS}_\gamma^*(\mathbb{U})$ is denoted by $\mathcal{SS}^*(\gamma)$.

For a biholomorphic function $\Phi : \mathbb{U} \rightarrow \mathbb{C}$, which satisfies $\Phi(0) = 1$ and $\text{Re } \Phi(z) > 0$, Kohr [15] introduced the class \mathcal{M}_Φ containing the functions $p \in \mathcal{H}(\mathbb{B})$ such that $Dp(0) = I$ and $\|z\|/l_z(p(z)) \in \Phi(\mathbb{U})$. Here, we additionally take $\Phi'(0) > 0$, $\Phi''(0) \in \mathbb{R}$ and define the following:

Definition 1.3 Let $\Phi : \mathbb{U} \rightarrow \mathbb{C}$ be a biholomorphic function such that $\Phi(0) = 1$, $\text{Re } \Phi(z) > 0$, $\Phi'(0) > 0$ and $\Phi''(0) \in \mathbb{R}$. We define \mathcal{M}_Φ to be the class of mappings given by

$$\mathcal{M}_\Phi = \left\{ p \in \mathcal{H}(\mathbb{B}) : p(0) = 0, D(p(0)) = I, \frac{\|z\|}{l_z(p(z))} \in \Phi(\mathbb{U}), z \in \mathbb{B} \setminus \{0\}, l_z \in T(z) \right\}.$$

For $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, the above relation is equivalent to

$$\mathcal{M}_\Phi = \left\{ p \in \mathcal{H}(\mathbb{U}^n) : p(0) = 0, D(p(0)) = I, \frac{z_k}{p_k(z)} \in \Phi(\mathbb{U}), z \in \mathbb{U}^n \setminus \{0\} \right\},$$

where $p(z) = (p_1(z), p_2(z), \dots, p_n(z))'$ is a column vector in \mathbb{C}^n and k satisfies

$$|z_k| = \|z\| = \max_{1 \leq j \leq n} \{|z_j|\}.$$

For $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, the relation is equivalent to

$$\mathcal{M}_\Phi = \left\{ p \in \mathcal{H}(\mathbb{U}) : p(0) = 0, p'(0) = 1, \frac{z}{p(z)} \in \Phi(\mathbb{U}), z \in \mathbb{U} \right\}.$$

Also, note that, if $g \in \mathcal{H}(\mathbb{B})$ and $D(g(z))^{-1}g(z) \in \mathcal{M}_\Phi$, then suitable choices of Φ in Definition 1.3 provide different subclasses of holomorphic mappings. For instance, when $\Phi(z) = (1+z)/(1-z)$, $\Phi(z) = (1+(1-2\alpha)z)/(1-z)$ and $\Phi(z) = ((1+z)/(1-z))^\gamma$, we easily obtain that $g \in \mathcal{S}^*(\mathbb{B})$, $g \in \mathcal{S}_\alpha^*(\mathbb{B})$ and $g \in \mathcal{SS}_\gamma^*(\mathbb{B})$, respectively.

In this paper, we obtain the sharp bounds of $|\det T_{2,2}(g)|$ and $|\det T_{3,1}(g)|$ for a class of holomorphic functions in the unit disk, which contain the above results as special cases. Further, these results are generalized in higher dimensions.

2 Main Results

First, we find the bounds of $|\det T_{2,2}(g)|$ and $|\det T_{3,1}(g)|$ for a class of holomorphic mappings defined on \mathbb{U} .

Theorem 2.1 Let $g(z) = z + b_2z^2 + b_3z^3 + \dots \in \Phi(\mathbb{U})$, where Φ is same as given in Definition 1.3 and satisfy

$$|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0) > 0.$$

If $g(z)/g'(z) \in \mathcal{M}_\Phi$, then

$$|T_{2,2}(g)| \leq \frac{(\Phi'(0))^2}{4} \left(\frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right)^2 + (\Phi'(0))^2.$$

The bound is sharp.

Proof Since $g(z)/g'(z) \in \mathcal{M}_\Phi$, therefore we have

$$G(z) := \frac{zg'(z)}{g(z)} \in \Phi(\mathbb{U}),$$

which yields $G \prec \Phi$. For $g(z) = z + b_2z^2 + b_3z^3 + \dots$, the Taylor series expansion of $G(z)$ is given by

$$G(z) = 1 + b_2z + (2b_3 - b_2^2)z^2 + \dots .$$

Xu *et al.* [22] proved that

$$|b_3 - \lambda b_2^2| \leq \frac{|\Phi'(0)|}{2} \max \left\{ 1, \left| \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + (1 - 2\lambda)\Phi'(0) \right| \right\}, \quad \lambda \in \mathbb{C}. \tag{2.1}$$

Thus, whenever $|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)$, the equation (2.1) yields

$$|b_3| \leq \frac{\Phi'(0)}{2} \left| \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right|. \tag{2.2}$$

Further, using the bound $|G'(0)| \leq \Phi'(0)$, we obtain

$$|b_2| \leq \Phi'(0). \tag{2.3}$$

From (1.1), we have

$$|\det T_{2,2}(g)| = |b_3^2 - b_2^2| \leq |b_3|^2 + |b_2|^2.$$

Clearly, the required bound follows directly from the above relation together with the bounds of $|b_3|$ and $|b_2|$ given in (2.2) and (2.3), respectively.

To see the sharpness of the bound, consider the function $g_\Phi : \mathbb{U} \rightarrow \mathbb{C}$ given by

$$g_\Phi(z) = z \exp \int_0^z \frac{(\Phi(it) - 1)}{t} dt = 1 + i\Phi'(0)z - \frac{1}{2} \left((\Phi'(0))^2 + \frac{\Phi''(0)}{2} \right) z^2 + \dots . \tag{2.4}$$

It can be easily seen that $g_\Phi(z)/g'_\Phi(z) \in \mathcal{M}_\Phi$ and

$$|\det T_{2,2}(g_\Phi)| = \frac{1}{4} \left((\Phi'(0))^2 + \frac{\Phi''(0)}{2} \right)^2 + (\Phi'(0))^2,$$

which shows that the bound is sharp and completes the proof. □

Theorem 2.2 Let $g(z) = z + b_2z^2 + b_3z^3 + \dots \in \Phi(\mathbb{U})$, where Φ is same as given in Definition 1.3 and satisfy

$$2\Phi'(0) - 2(\Phi'(0))^2 \leq \Phi''(0) \leq 6(\Phi'(0))^2 - 2\Phi'(0).$$

If $g(z)/g'(z) \in \mathcal{M}_\Phi$, then

$$|\det T_{3,1}(g)| \leq 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4} \left(3\Phi'(0) - \frac{\Phi''(0)}{2\Phi'(0)} \right) \left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0) \right).$$

The bound is sharp.

Proof Since $\Phi''(0) + 2(\Phi'(0))^2 \geq 2\Phi'(0)$, by (2.2), we obtain

$$|b_3| \leq \frac{\Phi'(0)}{2} \left(\frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right). \tag{2.5}$$

Also, $6(\Phi'(0))^2 - \Phi''(0) \geq 2\Phi'(0)$ holds, hence (2.1) gives

$$|b_3 - 2b_2^2| \leq \frac{\Phi'(0)}{2} \left(3\Phi'(0) - \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} \right). \tag{2.6}$$

From (1.2), we have

$$|\det T_{3,1}(g)| = |2b_2^2b_3 - 2b_2^2 - b_3^2 + 1|$$

$$\leq 1 + 2|b_2|^2 + |b_3||b_3 - 2b_2^2|.$$

Using the estimates for the second and third coefficients given in (2.5) and (2.2) together with the bound of $|b_3 - 2b_2^2|$ given in (2.6), required bound follows.

The estimate is sharp for the function $g_\Phi(z) = z + \sum_{n=2}^\infty b_n z^n$ given by (2.4). For this function, we have

$$1 - 2b_2^2 - b_3(b_3 - 2b_2^2) = 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4} \left(3\Phi'(0) - \frac{\Phi''(0)}{2\Phi'(0)} \right) \left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0) \right),$$

which proves the sharpness of the bound. □

Remark 2.3 By taking $\Phi(z) = (1 + z)/(1 - z)$, $\Phi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$ and $\Phi(z) = (1 + Dz)/(1 + Ez)$, Theorems 2.1 and 2.2 can be deduced to Theorem A, Theorem B and Theorem C, respectively.

The bounds for other classes can also be obtained by changing the corresponding function Φ . For $\Phi(z) = ((1 + z)/(1 - z))^\gamma$, the following result follows for the class $\mathcal{SS}^*(\gamma)$.

Corollary 2.4 If $g \in \mathcal{SS}^*(\gamma)$, then for $\gamma \in [1/3, 1]$, the followings sharp inequalities hold:

$$|\det T_{2,2}(g)| \leq 9\gamma^4 + 4\gamma^2 \quad \text{and} \quad |\det T_{3,1}(g)| \leq 15\gamma^4 + 8\gamma^2 + 1.$$

Next, we extend the above results on the unit ball \mathbb{B} and on the unit polydisc \mathbb{U}^n .

Theorem 2.5 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and suppose that $G(z) = zg(z)$. If $(DG(z))^{-1}G(z) \in \mathcal{M}_\Phi$ such that Φ satisfy

$$|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0),$$

then

$$\left| \left(\frac{l_z(D^2G(0)(z^2))}{2!||z||^2} \right)^2 - \left(\frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \right)^2 \right| \leq \frac{(\Phi'(0))^2}{4} \left(\frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right)^2 + (\Phi'(0))^2.$$

The bound is sharp.

Proof Xu *et al.* [22, Theorem 3.2] proved that

$$\begin{aligned} & \left| \frac{l_z(D^3G(0)(z^3))}{3!||z||^3} - \lambda \left(\frac{l_z(D^2G(0)(z^2))}{2!||z||^2} \right)^2 \right| \\ & \leq \frac{|\Phi'(0)|}{2} \max \left\{ 1, \left| \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + (1 - 2\lambda)\Phi'(0) \right| \right\}, \quad \lambda \in \mathbb{C}, z \in \mathbb{B} \setminus \{0\}. \end{aligned} \tag{2.7}$$

Since $|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)$, the above inequality gives

$$\left| \frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \right| \leq \frac{\Phi'(0)}{2} \left| \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right|. \tag{2.8}$$

On the other hand, applying a similar method as in [9, Theorem 7.1.14] (also see [22, Theorem 3.2]), we obtain

$$(DG(z))^{-1} = \frac{1}{g(z)} \left(I - \frac{\frac{zDg(z)}{g(z)}}{1 + \frac{Dg(z)z}{g(z)}} \right).$$

Therefore

$$(DG(z))^{-1}G(z) = z \left(\frac{zg(z)}{g(z) + Dg(z)z} \right), \quad z \in \mathbb{B},$$

which directly gives

$$\frac{\|z\|}{l_z((DG(z))^{-1}G(z))} = 1 + \frac{Dg(z)z}{g(z)}. \tag{2.9}$$

For fix $z \in X \setminus \{0\}$ and $z_0 = \frac{z}{\|z\|}$, define the function $h : \mathbb{U} \rightarrow \mathbb{C}$ such that

$$h(\zeta) = \begin{cases} \frac{\zeta}{l_z((DG(\zeta z_0))^{-1}G(\zeta z_0))}, & \zeta \neq 0, \\ 1, & \zeta = 0. \end{cases}$$

Then $h \in \mathcal{H}(\mathbb{U})$ and $h(0) = 1 = \Phi(0)$. Moreover, since $(DG(z))^{-1}G(z) \in \mathcal{M}_\Phi$, we find that

$$\begin{aligned} h(\zeta) &= \frac{\zeta}{l_z((DG(\zeta z_0))^{-1}G(\zeta z_0))} = \frac{\zeta}{l_{z_0}((DG(\zeta z_0))^{-1}G(\zeta z_0))} \\ &= \frac{\|\zeta z_0\|}{l_{\zeta z_0}((DG(\zeta z_0))^{-1}G(\zeta z_0))} \in \Phi(\mathbb{U}), \quad \zeta \in \mathbb{U}. \end{aligned}$$

Taking (2.9) into consideration, we obtain

$$h(\zeta) = \frac{\|\zeta z_0\|}{l_{\zeta z_0}((Dg(\zeta z_0))^{-1}g(\zeta z_0))} = 1 + \frac{Dg(\zeta z_0)\zeta z_0}{g(\zeta z_0)}. \tag{2.10}$$

In view of the Taylor series expansions of $h(\zeta)$ and $g(\zeta z_0)$, the above equation gives

$$\begin{aligned} &\left(1 + h'(0)\zeta + \frac{h''(0)}{2}\zeta^2 + \dots\right) \left(1 + Dg(0)(z_0)\zeta + \frac{D^2g(0)(z_0^2)}{2}\zeta^2 + \dots\right) \\ &= \left(1 + Dg(0)(z_0)\zeta + \frac{D^2g(0)(z_0^2)}{2}\zeta^2 + \dots\right) \left(Dg(0)(z_0)\zeta + D^2g(0)(z_0^2)\zeta^2 + \dots\right). \end{aligned}$$

Comparison of homogeneous expansions yield that $h'(0) = Dg(0)(z_0)$. That is

$$h'(0)\|z\| = Dg(0)(z). \tag{2.11}$$

Since $G(z) = zg(z)$, therefore, we have

$$\frac{D^2G(0)(z^2)}{2!} = Dg(0)(z)z. \tag{2.12}$$

Moreover, from (2.12), we conclude that

$$\frac{l_z(D^2G(0)(z^2))}{2!} = Dg(0)(z)\|z\|. \tag{2.13}$$

Thus, equation (2.13) together with (2.11) gives

$$\left|\frac{l_z(D^2G(0)(z^2))}{2!}\right| = |Dg(0)(z)\|z\|| = |h'(0)\|z\||^2.$$

Since $h \prec \Phi$, therefore $|h'(0)| \leq \Phi'(0)$. Consequently, we obtain

$$\left|\frac{l_z(D^2G(0)(z^2))}{2!\|z\|^2}\right| \leq \Phi'(0). \tag{2.14}$$

Using the bounds given in (2.14) and (2.8) together with

$$\left|\left(\frac{l_z(D^2G(0)(z^2))}{2!\|z\|^2}\right)^2 - \left(\frac{l_z(D^3G(0)(z^3))}{3!\|z\|^3}\right)^2\right| \leq \left|\frac{l_z(D^3G(0)(z^3))}{3!\|z\|^3}\right|^2 + \left|\frac{l_z(D^2G(0)(z^2))}{2!\|z\|^2}\right|^2$$

the required bound follows.

To see the sharpness, consider the mapping G given by

$$G(z) = z \exp \int_0^{l_u(z)} \frac{(\Phi(it) - 1)}{t} dt, \quad z \in \mathbb{B}, \quad \|u\| = 1. \tag{2.15}$$

It is a simple exercise to see that $(DG(z))^{-1}G(z) \in \mathcal{M}_\Phi$ and a quick calculation reveals that

$$\frac{D^2G(0)(z^2)}{2!} = i\Phi'(0)l_u(z)z \text{ and } \frac{D^3G(0)(z^3)}{3!} = -\frac{1}{2} \left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right) (l_u(z))^2z.$$

In view of the above equations, we have

$$\frac{l_z(D^2G(0)(z^2))}{2!} = i\Phi'(0)l_u(z)\|z\|$$

and

$$\frac{l_z(D^3G(0)(z^3))\|z\|}{3!} = -\frac{1}{2} \left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right) (l_u(z))^2\|z\|^2.$$

Setting $z = ru$ ($0 < r < 1$), we get

$$\frac{l_z(D^2G(0)(z^2))}{2!\|z\|^2} = i\Phi'(0) \text{ and } \frac{l_z(D^3G(0)(z^3))}{3!\|z\|^3} = -\frac{1}{2} \left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right). \tag{2.16}$$

Thus for the mapping G , we have

$$\left| \left(\frac{l_z(D^2G(0)(z^2))}{2!\|z\|^2} \right)^2 - \left(\frac{l_z(D^3G(0)(z^3))}{3!\|z\|^3} \right)^2 \right| = \frac{(\Phi'(0))^2}{4} \left(\frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right)^2 + |\Phi'(0)|^2,$$

which proves the sharpness of the bound. □

Theorem 2.6 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and suppose that $G(z) = zg(z)$. If $(DG(z))^{-1}G(z) \in \mathcal{M}_\Phi$ such that Φ satisfy

$$2\Phi'(0) - 2(\Phi'(0))^2 \leq \Phi''(0) \leq 6(\Phi'(0))^2 - 2\Phi'(0),$$

then

$$|2b_2^2b_3 - b_3^2 - 2b_2^2 + 1| \leq 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4} \left(3\Phi'(0) - \frac{\Phi''(0)}{2\Phi'(0)} \right) \left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0) \right),$$

where

$$b_3 = \frac{l_z(D^3G(0)(z^3))}{3!\|z\|^3} \text{ and } b_2 = \frac{l_z(D^2G(0)(z^2))}{2!\|z\|^2}.$$

The bound is sharp.

Proof Since $2\Phi'(0) < \Phi''(0) + 2(\Phi'(0))^2$, from (2.7), we have

$$\left| \frac{l_z(D^3G(0)(z^3))}{3!\|z\|^3} \right| \leq \frac{\Phi'(0)}{2} \left(\frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right). \tag{2.17}$$

And, since $2\Phi'(0) + \Phi''(0) \leq 6(\Phi'(0))^2$, the inequality (2.7) directly gives

$$\left| \frac{l_z(D^3G(0)(z^3))}{3!\|z\|^3} - 2 \left(\frac{l_z(D^2G(0)(z^2))}{2!\|z\|^2} \right)^2 \right| \leq \frac{\Phi'(0)}{2} \left(3\Phi'(0) - \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} \right). \tag{2.18}$$

Also, we have

$$|2b_2^2b_3 - b_3^2 - 2b_2^2 + 1| \leq 1 + 2|b_2|^2 + |b_3||b_3 - 2b_2^2|. \tag{2.19}$$

The required bound is derived by using the estimates given in (2.14) and (2.17), and the bound given by (2.18) in the above inequality.

The equality case holds for the mapping $G(z)$ defined by (2.15). It follows from (2.16) that for this mapping, we have $b_2 = i\Phi'(0)$, $b_3 = -(\Phi''(0) + 2(\Phi'(0))^2)/4$ and hence

$$1 - b_3(b_3 - 2b_2^2) - 2b_2^2 = 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4} \left(3\Phi'(0) - \frac{\Phi''(0)}{2\Phi'(0)} \right) \left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0) \right),$$

which shows the sharpness of the bound. □

Theorem 2.7 Let $g \in \mathcal{H}(\mathbb{U}^n, \mathbb{C})$ with $g(0) = 1$ and suppose that $G(z) = zg(z)$. If $(DG(z))^{-1}G(z) \in \mathcal{M}_\Phi$ such that Φ satisfies

$$|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0),$$

then

$$\begin{aligned} & \left\| \frac{1}{3!}D^3G(0)\left(z^2, \frac{D^3G(0)(z^3)}{3!}\right) - \frac{1}{2!}D^2G(0)\left(z, \frac{D^2G(0)(z^2)}{2!}\right) \right\| \\ & \leq \frac{(\Phi'(0))^2\|z\|^5}{4} \left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2 + (\Phi'(0))^2\|z\|^3, \quad z \in \mathbb{U}^n. \end{aligned} \tag{2.20}$$

The bound is sharp.

Proof For $z \in \mathbb{U}^n \setminus \{0\}$, let $z_0 = \frac{z}{\|z\|}$. Define $h_k : \mathbb{U} \rightarrow \mathbb{C}$ such that

$$h_k(\zeta) = \begin{cases} \frac{\zeta z_k}{p_k(\zeta z_0)\|z_0\|}, & \zeta \neq 0, \\ 1, & \zeta = 0, \end{cases} \tag{2.21}$$

where $p(z) = (DG(z))^{-1}G(z)$ and k satisfies $|z_k| = \|z\| = \max_{1 \leq j \leq n} \{|z_j|\}$. Since $(D(G(z)))^{-1}G(z) \in \mathcal{M}_\Phi$, we have $h_k(\zeta) \in \Phi(\mathbb{U})$. Furthermore, using (2.10), we have

$$h_k(\zeta) = 1 + \frac{Dg(\zeta z_0)\zeta z_0}{g(\zeta z_0)}$$

or equivalently,

$$h_k(\zeta)g(\zeta z_0) = g(\zeta z_0) + Dg(\zeta z_0)\zeta z_0.$$

A comparison of homogeneous expansions obtained by the Taylor series expansions of g and h_k about ζ gives

$$h'_k(0) = Dg(0)(z_0), \quad \frac{h''_k(0)}{2} = D^2g(0)(z_0^2) - (Dg(0)(z_0))^2. \tag{2.22}$$

Also, using $G(z_0) = z_0g(z_0)$, we have

$$\frac{D^3G_k(0)(z_0^3)}{3!} = \frac{D^2g(0)(z_0^2)}{2!} \frac{z_k}{\|z\|} \quad \text{and} \quad \frac{D^2G_k(0)(z_0^2)}{2!} = Dg(0)(z_0) \frac{z_k}{\|z\|}. \tag{2.23}$$

In view of (2.22) and (2.23), we get

$$\begin{aligned} \left| \frac{1}{2!}D^2G_k(0)\left(z_0, \frac{D^2G(0)(z_0^2)}{2!}\right) \frac{\|z\|}{z_k} \right| &= \left| \frac{1}{2!}D^2G_k(0)(z_0, Dg(0)(z_0)z_0) \frac{\|z\|}{z_k} \right| \\ &= \left| Dg(0)(z_0) \frac{1}{2!}D^2G_k(0)(z_0, z_0) \frac{\|z\|}{z_k} \right| \\ &= \left| Dg(0)(z_0) \frac{1}{2!}D^2G_k(0)(z_0^2) \frac{\|z\|}{z_k} \right| \\ &= |(Dg(0)(z_0))^2|. \end{aligned} \tag{2.24}$$

Since $Dg(0)(z_0) = h'_k(0)$ and $|h'_k(0)| \leq \Phi'(0)$, therefore

$$\left| \frac{1}{2!}D^2G_k(0)\left(z_0, \frac{D^2G(0)(z_0^2)}{2!}\right) \frac{\|z\|}{z_k} \right| = |Dg(0)(z_0)|^2 \leq (\Phi'(0))^2. \tag{2.25}$$

If $z_0 \in \partial_0\mathbb{U}^n$, then

$$\left| \frac{1}{2!}D^2G_k(0)\left(z_0, \frac{D^2G(0)(z_0^2)}{2!}\right) \right| \leq (\Phi'(0))^2.$$

Moreover, since

$$\frac{1}{2!}D^2G_k(0)\left(z_0, \frac{D^2G(0)(z_0^2)}{2!}\right), \quad k = 1, 2, \dots, n$$

are holomorphic functions on \bar{U}^n , by virtue of the maximum modulus theorem of holomorphic functions on the unit polydisc, we obtain

$$\left|\frac{1}{2!}D^2G_k(0)\left(z, \frac{D^2G(0)(z^2)}{2!}\right)\right| \leq (\Phi'(0))^2\|z\|^3, \quad k = 1, 2, \dots \tag{2.26}$$

Using the same arguments as in (2.24), we get

$$\left|\frac{1}{3!}D^3G_k(0)\left(z_0^2, \frac{D^3G(0)(z_0^3)}{3!}\right)\frac{\|z\|}{z_k}\right| = \left|\left(\frac{D^2g(0)(z_0^2)}{2!}\right)\right|^2. \tag{2.27}$$

According to the result established by Xu *et al.* [22, Theorem 3.3], we have

$$\begin{aligned} & \left|\frac{D^3G_k(0)(z_0^3)}{3!}\frac{\|z\|}{z_k} - \lambda\frac{1}{2}D^2G_k(0)\left(z_0, \frac{D^2G(0)(z_0^2)}{2!}\frac{\|z\|}{z_k}\right)\right| \\ & \leq \frac{|\Phi'(0)|}{2} \max\left\{1, \left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + (1 - 2\lambda)\Phi'(0)\right|\right\}. \end{aligned} \tag{2.28}$$

Since $|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)$, therefore from (2.23) and (2.28), it follows that

$$\left|\frac{D^3G_k(0)(z_0^3)}{3!}\frac{\|z\|}{z_k}\right| = \left|\frac{D^2g(0)(z_0^2)}{2!}\right| \leq \frac{|\Phi'(0)|}{2} \left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right|. \tag{2.29}$$

Thus, from (2.27) and (2.29), we obtain

$$\left|\frac{1}{3!}D^3G_k(0)\left(z_0^2, \frac{D^3G(0)(z_0^3)}{3!}\right)\frac{\|z\|}{z_k}\right| \leq \frac{(\Phi'(0))^2}{4} \left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2.$$

For $z_0 \in \partial_0U^n$, we get

$$\left|\frac{1}{3!}D^3G_k(0)\left(z_0^2, \frac{D^3G(0)(z_0^3)}{3!}\right)\right| \leq \frac{(\Phi'(0))^2}{4} \left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2.$$

Again, since

$$\frac{1}{3!}D^3G_k(0)\left(z_0^2, \frac{D^3G(0)(z_0^3)}{3!}\right), \quad k = 1, 2, \dots, n$$

are holomorphic functions on \bar{U}^n , therefore the maximum modulus principle on the unit polydisc yields

$$\left|\frac{1}{3!}D^3G_k(0)\left(z^2, \frac{D^3G(0)(z^3)}{3!}\right)\right| \leq \frac{(\Phi'(0))^2\|z\|^5}{4} \left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2. \tag{2.30}$$

Now, using the bounds given in (2.26) and (2.30), we obtain

$$\begin{aligned} & \left|\frac{1}{3!}D^3G_k(0)\left(z^2, \frac{D^3G(0)(z^3)}{3!}\right) - \frac{1}{2!}D^2G_k(0)\left(z, \frac{D^2G(0)(z^2)}{2!}\right)\right| \\ & \leq (\Phi'(0))^2\|z\|^3 + \frac{(\Phi'(0))^2\|z\|^5}{4} \left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2 \end{aligned}$$

for $k = 1, 2, \dots, n$. Therefore,

$$\begin{aligned} & \left\|\frac{1}{3!}D^3G_k(0)\left(z^2, \frac{D^3G(0)(z^3)}{3!}\right) - \frac{1}{2!}D^2G_k(0)\left(z, \frac{D^2G(0)(z^2)}{2!}\right)\right\| \\ & \leq (\Phi'(0))^2\|z\|^3 + \frac{(\Phi'(0))^2\|z\|^5}{4} \left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2. \end{aligned}$$

which is the required bound.

To prove the sharpness, consider the mapping

$$G(z) = z \exp \int_0^{z_1} \frac{\Phi(it) - 1}{t} dt, \quad z \in \mathbb{U}^n. \quad (2.31)$$

It is a simple exercise to check that $D(G(z))^{-1}G(z) \in \mathcal{M}_\Phi$. From the above relation, we deduce that

$$\frac{D^2G(0)(z^2)}{2!} = i\Phi'(0)z_1z, \quad \frac{D^3G(0)(z^3)}{3!} = -\frac{1}{2} \left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right) (z_1)^2z.$$

By taking $z = (r, 0, \dots, 0)$, the equality in (2.20) holds. \square

Remark 2.8 (i) When $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, Theorem 2.5 and Theorem 2.7 are equivalent to Theorem 2.1.

(ii) In case of $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, Theorem 2.6 is equivalent to Theorem 2.2.

3 Applications for Various Subclasses

If we take $\Phi(z) = (1+z)/(1-z)$, $\Phi(z) = (1+(1-2\alpha)z)/(1-z)$ and $\Phi(z) = ((1+z)/(1-z))^\gamma$, Theorems 2.5–2.7 give the following bounds (the branch of the power function is taken such that $((1+z)/(1-z))^\gamma = 1$ at $z = 0$).

Corollary 3.1 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and $G(z) = zg(z) \in \mathcal{S}^*(\mathbb{B})$. Then the following holds:

$$\left| \left(\frac{l_z(D^2G(0)(z^2))}{2!||z||^2} \right)^2 - \left(\frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \right)^2 \right| \leq 13, \quad z \in \mathbb{B} \setminus \{0\}, l_z \in T_z.$$

If $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, then

$$\left\| \frac{1}{3!} D^3G(0) \left(z^2, \frac{D^3G(0)(z^3)}{3!} \right) - \frac{1}{2!} D^2G(0) \left(z, \frac{D^2G(0)(z^2)}{2!} \right) \right\| \leq 9||z||^5 + 4||z||^3, \quad z \in \mathbb{U}^n.$$

All the estimates are sharp.

Corollary 3.2 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and $G(z) = zg(z) \in \mathcal{S}_\alpha^*(\mathbb{B})$. Then the following holds:

$$\left| \left(\frac{l_z(D^2G(0)(z^2))}{2!||z||^2} \right)^2 - \left(\frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \right)^2 \right| \leq (1-\alpha)^2(4\alpha^2 - 12\alpha + 13), \quad z \in \mathbb{B} \setminus \{0\}.$$

If $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, then

$$\begin{aligned} & \left\| \frac{1}{3!} D^3G(0) \left(z^2, \frac{D^3G(0)(z^3)}{3!} \right) - \frac{1}{2!} D^2G(0) \left(z, \frac{D^2G(0)(z^2)}{2!} \right) \right\| \\ & \leq (1-\alpha)^2((3-2\alpha)^2||z||^5 + 4||z||^3), \quad z \in \mathbb{U}^n. \end{aligned}$$

All the estimates are sharp.

Corollary 3.3 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and $G(z) = zg(z) \in \mathcal{SS}_\gamma^*(\mathbb{B})$. Then for $\gamma \in [1/3, 1]$, the following holds:

$$\left| \left(\frac{l_z(D^2G(0)(z^2))}{2!||z||^2} \right)^2 - \left(\frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \right)^2 \right| \leq 9\gamma^4 + 4\gamma^2, \quad z \in \mathbb{B} \setminus \{0\}, l_z \in T_z.$$

If $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, then

$$\left\| \frac{1}{3!} D^3G(0) \left(z^2, \frac{D^3G(0)(z^3)}{3!} \right) - \frac{1}{2!} D^2G(0) \left(z, \frac{D^2G(0)(z^2)}{2!} \right) \right\| \leq 9||z||^5\gamma^4 + 4||z||^3\gamma^2, \quad z \in \mathbb{U}^n.$$

All the estimates are sharp.

Corollary 3.4 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and $G(z) = zg(z) \in \mathcal{S}^*(\mathbb{B})$. Then the following holds:

$$|2b_2^2b_3 - b_3^2 - 2b_2^2 + 1| \leq 24,$$

where

$$b_3 = \frac{l_z(D^3G(0)(z^3))}{3!||z||^3}, \quad b_2 = \frac{l_z(D^2G(0)(z^2))}{2!||z||^2}, \quad l_z \in T_z.$$

The estimate is sharp.

Corollary 3.5 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and $G(z) = zg(z) \in \mathcal{S}_\alpha^*(\mathbb{B})$. Then for $\alpha \in [0, 2/3]$, the following holds:

$$|2b_2^2b_3 - b_3^2 - 2b_2^2 + 1| \leq 12\alpha^4 - 52\alpha^3 + 91\alpha^2 - 74\alpha + 24,$$

where

$$b_3 = \frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \quad \text{and} \quad b_2 = \frac{l_z(D^2G(0)(z^2))}{2!||z||^2}, \quad l_z \in T_z.$$

The estimate is sharp.

Corollary 3.6 Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$ and $G(z) = zg(z) \in \mathcal{SS}_\gamma^*(\mathbb{B})$. Then for $\gamma \in [1/3, 1]$, the following holds:

$$|2b_2^2b_3 - b_3^2 - 2b_2^2 + 1| \leq 15\gamma^4 + 8\gamma^2 + 1,$$

where

$$b_3 = \frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \quad \text{and} \quad b_2 = \frac{l_z(D^2G(0)(z^2))}{2!||z||^2}, \quad l_z \in T_z.$$

The estimate is sharp.

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References

- [1] Ahuja O P, Khatter K, Ravichandran V. Toeplitz determinants associated with Ma-Minda classes of starlike and convex functions. Iran J Sci Technol Trans A Sci, 2021, **45**(6): 2021–2027
- [2] Ali M F, Thomas D K, Vasudevarao A. Toeplitz determinants whose elements are the coefficients of analytic and univalent functions. Bull Aust Math Soc, 2018, **97**(2): 253–264
- [3] Bracci F, Graham I, Hamada H, Kohr G. Variation of Loewner chains, extreme and support points in the class S^0 in higher dimensions. Constr Approx, 2016, **43**(2): 231–251
- [4] Cartan H. Sur la possibilité d'étendre aux fonctions de plusieurs variables complexes la théorie des fonctions univalentes// Lecons sur les Fonctions Univalentes ou Multivalentes. Paris: Gauthier-Villars, 1933
- [5] Giri S, Kumar S S. Hermitian-Toeplitz determinants for certain univalent functions. Anal Math Phy, 2023, **13**(2): Art 37
- [6] Gong S. Convex and Starlike Mappings in Several Complex Variables. Beijing: Science Press, 1998
- [7] Graham I, Hamada H, Honda T, et al. distortion and coefficient bounds for Carathéodory families in \mathbb{C}^n and complex Banach spaces. J Math Anal Appl, 2014, **416**(1): 449–469
- [8] Graham I, Hamada H, Kohr G. Parametric representation of univalent mappings in several complex variables. Canad J Math, 2002, **54**(2): 324–351
- [9] Graham I, Kohr G. Geometric Function Theory in One and Higher Dimensions. Monographs and Textbooks in Pure and Applied Mathematics, 255. New York: Marcel Dekker, 2003

- [10] Hamada H, Honda T. Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables. *Chinese Ann Math Ser B*, 2008, **29**(4): 353–368
- [11] Hamada H, Honda T, Kohr G. Growth theorems and coefficients bounds for univalent holomorphic mappings which have parametric representation. *J Math Anal Appl*, 2006, **317**(1): 302–319
- [12] Hamada H, Kohr G, Kohr M. The Fekete-Szegő problem for starlike mappings and nonlinear resolvents of the Carathéodory family on the unit balls of complex Banach spaces. *Anal Math Phys*, 2021, **11**(3): Art 115
- [13] Hamada H, Kohr G, Liczberski P. Starlike mappings of order α on the unit ball in complex Banach spaces. *Glas Mat Ser III*, 2001, **36**: 39–48
- [14] Janowski W. Some extremal problems for certain families of analytic functions I. *Ann Polon Math*, 1973, **28**: 297–326
- [15] Kohr G. On some best bounds for coefficients of several subclasses of biholomorphic mappings in \mathbb{C}^n . *Complex Variables Theory Appl*, 1998, **36**(3): 261–284
- [16] Kohr G, Liczberski P. On strongly starlikeness of order alpha in several complex variables. *Glas Mat Ser III*, 1998, **33**: 185–198
- [17] Liu X, Liu T. The sharp estimates of all homogeneous expansions for a class of quasi-convex mappings on the unit polydisk in \mathbb{C}^n . *Chinese Ann Math Ser B*, 2011, **32**(2): 241–252
- [18] Xu Q, Liu T. On the Fekete and Szegő problem for the class of starlike mappings in several complex variables. *Abstr Appl Anal* 2014, **2014**: Art 807026
- [19] Xu Q, Liu T. On the coefficient inequalities for a class of holomorphic mappings. *Complex Var Elliptic Equ*, 2020, **65**(9): 1474–1487
- [20] Xu Q, Liu T. On coefficient estimates for a class of holomorphic mappings. *Sci China Ser A*, 2009, **52**(4): 677–686
- [21] Xu Q, Liu T, Liu X. The sharp estimates of homogeneous expansions for the generalized class of close-to-quasi-convex mappings. *J Math Anal Appl*, 2012, **389**(2): 781–791
- [22] Xu Q, Liu T, Liu X. Fekete and Szegő problem in one and higher dimensions. *Sci China Math*, 2018, **61**(10): 1775–1788
- [23] Xu Q, Yang T, Liu T, Xu H. Fekete and Szegő problem for a subclass of quasi-convex mappings in several complex variables. *Front Math China*, 2015, **10**(6): 1461–1472
- [24] Ye K, Lim L H. Every matrix is a product of Toeplitz matrices. *Found Comput Math*, 2016, **16**(3): 577–598