Acta Mathematica Scientia, 2024, **44B**(5): 1903–1915
https://doi.org/10.1007/s10473-024-0515-2

Mathematica Acta Scientia
数学物理学报
http://actams.apm.ac.cn

ELLIPTIC EQUATIONS IN DIVERGENCE FORM WITH DISCONTINUOUS COEFFICIENTS IN DOMAINS WITH CORNERS*

Jun CHEN (陈俊) *Xuemei DENG* (邓雪梅)†
Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China
E-mail: *chenjun4@ctgu.edu.cn; dxuemei@ctgu.edu.cn*

Abstract We study equations in divergence form with piecewise C^α coefficients. The domains contain corners and the discontinuity surfaces are attached to the edges of the corners. We obtain piecewise $C^{1,\alpha}$ estimates across the discontinuity surfaces and provide an example to illustrate the issue regarding the regularity at the corners.

Key words elliptic equations; divergence form; discontinuous coefficients; corner regularity

2020 MR Subject Classification 35J15; 35J25

1 Introduction

We consider the elliptic problem

$$\partial_i(a^{ij}\partial_j u) = h + \partial_i g^i \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = \varphi \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a closed $(n-1)$-surface as its boundary, $a^{ij} \in L^\infty(\Omega)$, and the following uniform ellipticity condition is satisfied

$$\lambda|\boldsymbol{\xi}|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\boldsymbol{\xi}|^2, \quad \forall \mathbf{x} \in \Omega, \quad \boldsymbol{\xi} = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n. \tag{1.3}$$

Here λ, Λ are two positive constants.

Without the assumption on the continuity of coefficients a^{ij}, we can only obtain the Hölder continuity of the solution by De Giogi-Nash estimates (cf. [7, Theorem 8.24]). Here arises the question: if the coefficients are piecewise Hölder continuous, can we obtain piecewise Hölder estimates for the gradient of the solution? In [9], Li and Vogelius studied the problems arising from models concerned with materials of fiber-reinforced composite. They showed that under the assumption that the coefficients are piecewise C^α, the solution is piecewise $C^{1,\bar\alpha}$ for some $\bar\alpha \in \left(0, \frac{\alpha}{(\alpha+1)n}\right)$, and their estimates are independent of the distance between the discontinuity surfaces. Hence, they can deal with the case for two touching discontinuity surfaces.

*Received November 20, 2022; revised June 5, 2024. Chen's research was supported by National Natural Science Foundation of China (12061080, 12161087 and 12261093) and the Science and Technology Project of the Education Department of Jiangxi Province (GJJ211601). Deng's research was supported by National Natural Science Foundation of China (11871305).

†Corresponding author

In this paper, we are interested in elliptic problems from conservation laws in aerodynamics. For example, when contact discontinuities happen in a subsonic flow described by steady compressible Euler equations, elliptic equations in divergence form with discontinuous coefficients across the contact discontinuity surfaces or lines can be derived from the Euler system (cf. [1, 4–6, 8]). Very often, there is only one contact discontinuity surface in the domain, which makes the structure of the subdomains simpler than that in [9]. On the other hand, the equations from the conservation laws are, in general, nonlinear. One needs to linearize the equations and design iteration schemes to solve the nonlinear problems. Hence, in each iteration step when solving the linearized equation, loss of regularity as in [9] is not allowed in order to close the iteration arguments; that is, if the coefficients are piecewise C^{α} and the discontinuity surface is $C^{1,\alpha}$, piecewise $C^{1,\alpha}$ estimates are needed, rather than piecewise $C^{1,\bar{\alpha}}$ estimates with $\bar{\alpha} < \alpha$. Furthermore, when we study Mach reflection (cf. [5, 6]) or airfoils with vortex lines (cf. [4]), the contact discontinuity lines are attached to the corners of the domain boundaries (see Figure 1 and Figure 2). This is a different situation from that of [2] and [9], in which discontinuity surfaces neither stretch to the boundary, nor cross with corners. In [2], Bonnetier and Vogelius gave an example of discontinuity surfaces crossing with corners. In their example, the solution is not even $W^{1,\infty}$, thus illustrating so called "corner effect".

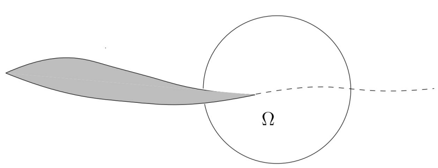

Figure 1 A vortex line attached to an airfoil.

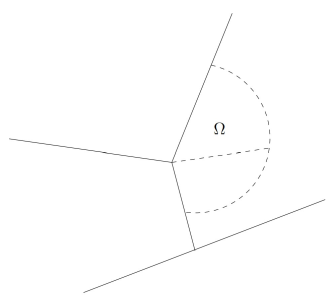

Figure 2 A Mach configuration.

This paper tries to understand how the intersection between the boundary and discontinuity surface affects the regularity of solutions near the boundary corners. We will study bounded domains with corners described as follows:

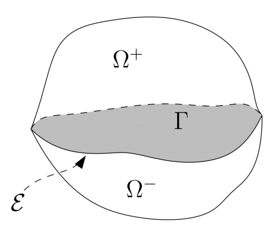

Figure 3 Domain Ω.

Suppose that domain Ω is divided by an $(n-1)$-surface Γ into two disjoint open sets, Ω^+ and Ω^- (see Figure 3), i.e.,

$$\begin{aligned} &\Omega^+ \cap \Omega^- = \emptyset, \Omega^+ \cup \Gamma \cup \Omega^- = \Omega, \\ &\Gamma \cap \partial\Omega = \emptyset, \quad \overline{\Omega^+} \cap \overline{\Omega^-} = \partial\Omega^+ \cap \partial\Omega^- = \overline{\Gamma}. \end{aligned} \tag{1.4}$$

Denote the upper and lower parts of boundary $\partial\Omega$ by

$$(\partial\Omega)^+ := \overline{\partial\Omega^+}\backslash\Gamma, \quad (\partial\Omega)^- := \overline{\partial\Omega^-}\backslash\Gamma$$

and the intersection of $(\partial\Omega)^+$ and $(\partial\Omega)^-$ by

$$\mathcal{E} := (\partial\Omega)^+ \cap (\partial\Omega)^- = \partial\Gamma;$$

this is called the edge of domain Ω and is assumed to be a closed $(n-2)$-surface.

Let $(r,\theta,\mathbf{x}') := (r,\theta,x_3,\cdots,x_n)$ be cylindrical coordinates. Assume that $\theta_- < 0 < \theta_+, \theta_+ - \theta_- < 2\pi$. We call

$$W = \{\mathbf{x} \in \mathbb{R}^n : r > 0, \theta_- < \theta < \theta_+\} \tag{1.5}$$

a wedge, and set that

$$W^+ = \{\mathbf{x} \in \mathbb{R}^n : 0 < \theta < \theta_+\}, \quad W^- = \{\mathbf{x} \in \mathbb{R}^n : \theta_- < \theta < 0\}. \tag{1.6}$$

Definition 1.1 (Wedge condition) We say that the edge $\mathcal{E}$ satisfies the wedge condition if, for any $\mathbf{x} \in \mathcal{E}$, there exist $r > 0$, a wedge W, a neighborhood U of $\mathbf{0}$ and a $C^{1,\alpha}$ homeomorphism $\chi : B_r(\mathbf{x}) \to U$ such that

$$\chi(\Omega \cap B_r(\mathbf{x})) = W \cap U, \quad \chi(\Omega^I \cap B_r(\mathbf{x})) = W^I \cap U, \quad I = +,-.$$

Since the loss of regularity of solutions near the edge happens very often, it is convenient to introduce the following weighted Hölder norms: suppose that $\mathcal{D}$ is an open domain in $\mathbb{R}^n$ with a given boundary portion $E \subset \partial\mathcal{D}$. For any $\mathbf{x},\mathbf{y}$ in $\mathcal{D}$, define that

$$\delta_{\mathbf{x}} := \min(\mathrm{dist}(\mathbf{x},E),1), \quad \delta_{\mathbf{x},\mathbf{y}} := \min(\delta_{\mathbf{x}},\delta_{\mathbf{y}}).$$

Let $\alpha \in (0,1)$, $\tau \in \mathbb{R}$ and k be a nonnegative integer. Let $\mathbf{k} = (k_1,\cdots,k_n)$ be an integer-valued vector, where $k_i \geq 0, i = 1,\cdots,n$, $|\mathbf{k}| = k_1 + \cdots + k_n$, and let $D^{\mathbf{k}} = \partial_{x_1}^{k_1}\cdots\partial_{x_n}^{k_n}$. We define that

$$[f]_{k,0;\mathcal{D}}^{(\tau;E)} = \sup_{\substack{\mathbf{x}\in\mathcal{D}\\ |\mathbf{k}|=k}} \left((\delta_{\mathbf{x}})^{\max(k+\tau,0)}|D^{\mathbf{k}}f(\mathbf{x})|\right), \tag{1.7}$$

$$[f]_{k,\alpha;\mathcal{D}}^{(\tau;E)} = \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathcal{D}\\ \mathbf{x}\neq\mathbf{y}\\ |\mathbf{k}|=k}} \left((\delta_{\mathbf{x},\mathbf{y}})^{\max(k+\alpha+\tau,0)}\frac{|D^{\mathbf{k}}f(\mathbf{x}) - D^{\mathbf{k}}f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}\right), \tag{1.8}$$

$$\|f\|_{k,\alpha;\mathcal{D}}^{(\tau;E)} = \sum_{i=0}^{k}[f]_{i,0;\mathcal{D}}^{(\tau;E)} + [f]_{k,\alpha;\mathcal{D}}^{(\tau;E)}. \tag{1.9}$$

Denote that

$$C_{(\tau;E)}^{k,\alpha}(\mathcal{D}) := \{f : \|f\|_{k,\alpha;\mathcal{D}}^{(\tau;E)} < \infty\}.$$

Theorem 1.2 Assume that (1.4) holds, and that $\partial\Omega^+, \partial\Omega^-, \Gamma$ are $C^{1,\alpha}$ surfaces of $n-1$ dimension and edge $\mathcal{E}$ is a $C^{1,\alpha}$ surface of $n-2$ dimensions, satisfying the wedge condition (1.5). Suppose that a^{ij} satisfies the uniform ellipticity condition (1.3), $a^{ij}, g^i \in C^\alpha(\overline{\Omega^+}) \cap C^\alpha(\overline{\Omega^-})$, $h \in L^\infty(\Omega)$ and $\varphi \in C^{1,\alpha}(\overline{\Omega^+}) \cap C^{1,\alpha}(\overline{\Omega^-}) \cap C(\overline{\Omega})$. Then there exist positive constants β and C, depending on $n,\alpha,\lambda,\Lambda,\Omega$ such that there exists a unique solution $u \in C_{(-\beta;\mathcal{E})}^{1,\alpha}(\Omega^+) \cap C_{(-\beta;\mathcal{E})}^{1,\alpha}(\Omega^-) \cap C^\beta(\overline{\Omega})$ to elliptic problem (1.1) (1.2) with the following estimate:

$$\max_{I=+,-}\|u\|_{1,\alpha;\Omega^I}^{(-\beta;\mathcal{E})} \leq C(\max_{I=+,-}\|\varphi\|_{1,\alpha;\Omega^I} + \|h\|_{L^\infty(\Omega)} + \max_{i=1,\cdots,n;I=+,-}\|g^i\|_{0,\alpha;\Omega^I}). \tag{1.10}$$

Remark 1.3 When $n = 3$, the surface Γ is two dimensional with a closed curve as its edge $\mathcal{E}$; in the case when $n = 2$, Γ is a closed curve with two ends points as the edge $\mathcal{E}$.

The organization of the paper is as follows: in Section 2, we establish the Schauder interior estimates across the discontinuity surface through Lemma 2.2. In Section 3, we obtain corner estimates in Lemma 3.1, which, combined with the interior estimates in Section 2, give rise to the global estimates. In Section 4, we provide an example to show that different boundary shapes and data can render solutions with different regularity, such as C^γ or $C^{1,\alpha}$ smoothness at the corners.

2 Interior Estimates

To obtain the interior estimates, we need a proposition which is a simplified version of Proposition 3.2 in [9]. In order to state the proposition, we first introduce the following weighted L^p norm in a given domain $\mathcal{D}$ for any $s > 0$ and $p \in (1, \infty)$:

$$\|f\|_{Y^{s,p}(\mathcal{D})} := \sup_{0<r\le 1} r^{1-s} \left(\fint_{r\mathcal{D}} |f|^p \right)^{1/p}.$$

Let $B_r(\mathbf{x})$ be the open ball centered at $\mathbf{x}$ with radius r, and let B_r be the open ball centered at the origin $\mathbf{0}$ with radius r and

$$D_r := B_r \cap \{x_n = 0\}, \qquad B_r^+ := B_r \cap \{x_n > 0\}, \qquad B_r^- := B_r \cap \{x_n < 0\}.$$

Proposition 2.1 Suppose that a^{ij} and $\bar{a}^{ij}$ satisfy the uniform ellipticity condition (1.3), and that $\bar{a}^{ij}, \overline{G}^i, \overline{H}$ are constants in both B_4^+ and B_4^-. Suppose that $g^i \in L^q(B_4), h \in L^{q/2}(B_4)$ for some $q > n$. Let $\bar{\alpha} \in (0,1)$ and $u \in H^1(B_4)$ be a solution to

$$\partial_i(a^{ij}\partial_j u) = h + \partial_i g^i \tag{2.1}$$

in B_4 with

$$\|u\|_{L^\infty(B_4)} \le 1. \tag{2.2}$$

Then there exist constants $\varepsilon_0 > 0$ and $C > 0$, depending on $n, q, \bar{\alpha}, , \lambda, \Lambda$, such that, it holds that

$$\max_{i,j=1,\cdots,n} \|a^{ij} - \bar{a}^{ij}\|_{Y^{1+\bar{\alpha},q}(B_4)} \le \varepsilon_0, \tag{2.3}$$

$$\max_{i=1,\cdots,n} \|g^i - \overline{G}^i\|_{Y^{1+\bar{\alpha},q}(B_4)} + \|h - \overline{H}\|_{Y^{\bar{\alpha},q/2}(B_4)} \le \varepsilon_0, \tag{2.4}$$

$$\max_{i=1,\cdots,n} \|\overline{G}^i\|_{L^\infty(B_4)} + \|\overline{H}\|_{L^\infty(B_4)} \le 1, \tag{2.5}$$

then there exists a function p, continuous in B_1 and piecewise linear in $B_1^+ \cup B_1^-$, with coefficients bounded by C and which satisfies that

$$\partial_i(\bar{a}^{ij}\partial_j p) = \overline{H} + \partial_i \overline{G}^i \quad \text{ in } B_1,$$

and

$$|u(\mathbf{x}) - p(\mathbf{x})| \le C|\mathbf{x}|^{1+\bar{\alpha}}, \quad \mathbf{x} \in B_1. \tag{2.6}$$

We refer to [9] for details of the proof.

Lemma 2.2 Suppose that a^{ij} satisfies the uniform ellipticity condition (1.3) and that $a^{ij}, g^i \in C^\alpha(\overline{B_2^+}) \cap C^\alpha(\overline{B_2^-}), h \in L^\infty(B_2)$ and

$$\max_{I=+,-;i,j=1,\cdots,n} \|a^{ij}\|_{0,\alpha;B_2^I} \leq \Lambda,$$

where Λ is the same constant as in condition (1.3). Let $u \in H^1(B_2)$ be a solution to (2.1). Then there exists a constant C, depending on $n, \alpha, \lambda, \Lambda$, such that

$$\max_{I=+,-} \|u\|_{1,\alpha;B_1^I} \leq C(\|u\|_{0,0;B_2} + \|h\|_{L^\infty(B_2)} + \max_{i=1,\cdots,n;I=+,-} \|g^i\|_{0,\alpha;B_2^I}). \tag{2.7}$$

Proof We will first obtain a $C^{1,\alpha}$ estimate for u restricted on the interface $D_{\frac{3}{2}}$.

For any given point $\mathbf{x}_0 \in D_{\frac{3}{2}}$, consider $B_{4d_0}(\mathbf{x}_0)$, where d_0 is sufficiently small and to be determined later. Set that

$$C_0 := \|u\|_{0,0;B_2} + \|h\|_{L^\infty(B_2)} + \max_{i=1,\cdots,n;I=+,-} \|g^i\|_{0,\alpha;B_2^I}. \tag{2.8}$$

We rescale domain $B_{4d_0}(\mathbf{x}_0)$ to B_4 by coordinate transformation $\mathbf{y} = (\mathbf{x} - \mathbf{x}_0)/d_0$ and set that

$$\hat{u}(\mathbf{y}) = u(\mathbf{x}_0 + d_0\mathbf{y})/C_0, \qquad \hat{a}^{ij}(\mathbf{y}) = a^{ij}(\mathbf{x}_0 + d_0\mathbf{y}), \tag{2.9}$$

$$\hat{g}^i(\mathbf{y}) = d_0 g^i(\mathbf{x}_0 + d_0\mathbf{y})/C_0, \quad \hat{h}(\mathbf{y}) = d_0^2 h(\mathbf{x}_0 + d_0\mathbf{y})/C_0. \tag{2.10}$$

Then $\hat{u}$ satisfies that

$$\partial_i(\hat{a}^{ij}\partial_j\hat{u}) = \hat{h} + \partial_i\hat{g}^i \quad \text{in} \quad B_4. \tag{2.11}$$

Define that

$$\bar{a}^{ij}(\mathbf{x}) = \hat{a}^{ij}(\mathbf{0}\pm) = a^{ij}(\mathbf{x}_0\pm), \quad \overline{H}(\mathbf{x}) \equiv 0, \quad \overline{G}^i(\mathbf{x}) = \hat{g}^i(\mathbf{0}\pm) \tag{2.12}$$

for $\mathbf{x} \in B_4^\pm$. We will verify, by choosing d_0 sufficiently small, that conditions (2.2)–(2.5) in Proposition 2.1 are satisfied, where $u = \hat{u}, a^{ij} = \hat{a}^{ij}, g^i = \hat{g}^i, h = \hat{h}, \bar{\alpha} = \alpha$.

By the definitions (2.8)–(2.10) of C_0, $\hat{u}$, $\hat{g}^i$ and $\hat{h}$, it is obvious that conditions (2.2) and (2.5) hold, provided that $d_0 < 1$.

Then we verify condition (2.3) as follows: since a^{ij} is C^α in each half ball $B_2^\pm$ with C^α norms bounded by Λ, we have, for $\mathbf{x} \in B_4^\pm$,

$$|\hat{a}^{ij}(\mathbf{x}) - \bar{a}^{ij}(\mathbf{x})| = |a^{ij}(\mathbf{x}_0 + d_0\mathbf{x}) - a^{ij}(\mathbf{x}_0\pm)| \leq \Lambda|d_0\mathbf{x}|^\alpha,$$

leading to

$$\begin{aligned}
\|\hat{a}^{ij} - \bar{a}^{ij}\|_{Y^{1+\alpha,q}(B_4)} &= \sup_{0<r\leq 1} r^{-\alpha} \left(⨍_{rB_4} |\hat{a}^{ij} - \bar{a}^{ij}|^q \right)^{1/q} \\
&\leq \sup_{0<r\leq 1} r^{-\alpha} \left(\frac{1}{|B_{4r}|} \int_{B_{4r}} \Lambda^q |d_0\mathbf{x}|^{\alpha q} \mathrm{d}\,\mathbf{x} \right)^{1/q} \\
&\leq \Lambda d_0^\alpha.
\end{aligned}$$

Hence, condition (2.3) holds for sufficiently small d_0, depending on $\Lambda, \alpha, \varepsilon_0$. The same arguments apply to the estimates leading to condition (2.4).

Since the estimates above are independent of q, we may choose that $q = 2n$. Thus, by Proposition 2.1, there exists a positive constant C, depending on $n, \alpha, \lambda, \Lambda$, and a continuous, piecewise linear function $p(\mathbf{x})$, which is linear in both B_2^+ and B_2^- and whose coefficients are uniformly bounded by C, satisfying that

$$\partial_i(\bar{a}^{ij}\partial_j p) = \partial_i \overline{G}^i \quad \text{in} \quad B_1,$$

with the estimate

$$|\hat{u}(\mathbf{x}) - p(\mathbf{x})| \leq C|\mathbf{x}|^{1+\alpha}, \quad \forall\, \mathbf{x} \in B_1. \tag{2.13}$$

Estimate (2.13) directly implies that

$$\hat{u}(0) = p(0), \tag{2.14}$$

$$D\hat{u}(0+) = Dp(0+), \quad D\hat{u}(0-) = Dp(0-), \tag{2.15}$$

$$|D\hat{u}(0+)| \leq C, \qquad |D\hat{u}(0-)| \leq C. \tag{2.16}$$

We will obtain $C^{1,\alpha}$ estimates for u up to the interface D_1 in each subdomain B_1^+ and B_1^-. In order to achieve this, we will first consider domain B_2^+ and obtain the $C^{1,\alpha}$ estimates for $u(\hat{\mathbf{x}}, 0+)$ on $D_{\frac{3}{2}}$, where $\hat{\mathbf{x}} = (x_1, \cdots, x_{n-1})$. Denoting that

$$v(\mathbf{x}) := u(\mathbf{x}) - u(\mathbf{x}_0) - Du(\mathbf{x}_0+)\cdot(\mathbf{x} - \mathbf{x}_0),$$

we know that v satisfies that

$$\partial_i(a^{ij}\partial_j v) = h + \partial_i \bar{g}^i \quad \text{in} \quad B_2^+, \tag{2.17}$$

where

$$\bar{g}^i = g^i - \partial_j u(\mathbf{x}_0+) a^{ij}.$$

For any point $\mathbf{x}_0 \in D_{\frac{3}{2}}, \mathbf{x} \in B_{d_0}^+(\mathbf{x}_0)$, let

$$\mathbf{x} = \mathbf{x}_0 + d_0\mathbf{y}.$$

Then $\mathbf{y} \in B_1^+$ and estimates (2.13)–(2.16) imply that the following holds

$$|v(\mathbf{x})| = C_0|\hat{u}(\mathbf{y}) - p(\mathbf{y})| \leq CC_0|\mathbf{y}|^{1+\alpha} = \frac{CC_0}{d_0^{1+\alpha}}|\mathbf{x} - \mathbf{x}_0|^{1+\alpha}. \tag{2.18}$$

Suppose another point $\mathbf{y}_0 \in D_{\frac{3}{2}}$ with

$$d := \operatorname{dist}(\mathbf{x}_0, \mathbf{y}_0) < \frac{1}{4}d_0.$$

Set that

$$\bar{\mathbf{x}} = \mathbf{x}_0 + (0, \cdots, 0, 2d), \quad Q_1 = B_d(\bar{\mathbf{x}}), \quad Q_2 = B_{2d}(\bar{\mathbf{x}}).$$

By combining the interior Hölder estimate for solutions of Poisson's equations (see [7, Theorem 4.15 and estimate (4.45)]) and a standard perturbation argument (see the proof in [7, Theorem 6.2]), we obtain the following Schauder interior estimate:

$$\|v\|'_{1,\alpha;Q_1} \leq C\left(\|v\|_{0,0;Q_2} + d^2\|h\|_{0,0;Q_2} + d\max_{i=1,\cdots,n}\|\bar{g}^i - \bar{g}^i(\mathbf{x}_0+)\|'_{0,\alpha;Q_2}\right). \tag{2.19}$$

Here the $\|\cdot\|'$ norm is defined as follows: let $\mathcal{D}$ be a domain and let u be a function defined in $\mathcal{D}$, $d = \operatorname{diam}\mathcal{D}$,

$$\|u\|'_{k;\mathcal{D}} = \sum_{j=0}^{k} d^j [u]_{j,0;\mathcal{D}},$$

$$\|u\|'_{k,\alpha;\mathcal{D}} = \|u\|'_{k;\mathcal{D}} + d^{k+\alpha}[u]_{k,\alpha;\mathcal{D}}.$$

Obviously, $Q_2 \subset B_1(\mathbf{x}_0)$, and so estimates (2.16), (2.18) and (2.19) imply that

$$|Dv(\bar{\mathbf{x}})| = |Du(\bar{\mathbf{x}}) - Du(\mathbf{x}_0+)| \leq CC_0 d^\alpha. \tag{2.20}$$

The same argument can be applied to $\mathbf{y}_0$. That is, set that

$$\hat{v}(\mathbf{x}) := u(\mathbf{x}) - u(\mathbf{y}_0) - Du(\mathbf{y}_0+) \cdot (\mathbf{x} - \mathbf{y}_0).$$

A similar Schauder interior estimate to (2.19) gives rise to

$$|D\hat{v}(\bar{\mathbf{x}})| = |Du(\bar{\mathbf{x}}) - Du(\mathbf{y}_0+)| \leq CC_0 d^\alpha. \tag{2.21}$$

Estimates (2.20) and (2.21) lead to

$$|Du(\mathbf{x}_0+) - Du(\mathbf{y}_0+)| \leq CC_0 d^\alpha = CC_0|\mathbf{x}_0 - \mathbf{y}_0|^\alpha. \tag{2.22}$$

Set that $\phi(\hat{\mathbf{x}}) = u(\hat{\mathbf{x}}, 0+)|_{D_2}$. Then estimate (2.22) implies the estimate for the boundary data ϕ:

$$|\phi|_{1,\alpha;D_{\frac{3}{2}}} \leq CC_0. \tag{2.23}$$

Then we have the following boundary estimate:

$$\begin{aligned}\|u\|_{1,\alpha;B_1^+} &\leq C\left(\|u\|_{0,0;B_2^+} + \|\phi\|_{1,\alpha;D_{\frac{3}{2}}} + \|h\|_{0,0;B_2^+} + \max_{i=1,\cdots,n}\|g^i\|_{0,\alpha;B_2^+}\right) \\ &\leq CC_0.\end{aligned} \tag{2.24}$$

Applying the same argument to the lower domain B_2^- and combining with estimate (2.24) gives the interior estimate (2.7). □

3 Boundary and Global Estimates

Let

$$\begin{aligned} &W_r = W \cap B_r, && W_r^I = W^I \cap B_r, \\ &T_r = \partial W \cap B_r, && T_r^I = \partial W \cap B_r^I, \\ &\mathcal{E}^0 = \{\mathbf{x} \in \mathbb{R}^n : r = 0\}, && \mathcal{E}_r^0 = \mathcal{E}_0 \cap B_r, \end{aligned}$$

where $I = +, -$ and $W, W^\pm$ are defined by (1.5) and (1.6).

Lemma 3.1 Suppose that a^{ij} satisfy the uniform ellipticity condition (1.3), a^{ij}, $g^i \in C^\alpha(\overline{W_2^+}) \cap C^\alpha(\overline{W_2^-})$, $h \in L^\infty(W_2)$, $\varphi \in C^{1,\alpha}(\overline{T_2^+}) \cup C^{1,\alpha}(\overline{T_2^-}) \cap C(\overline{T_2})$. Let $u \in H^1(W_2)$ be a solution to

$$\partial_i(a^{ij}\partial_j u) = h + \partial_i g^i \quad \text{in} \quad W_2, \tag{3.1}$$

$$u|_{T_2} = \varphi. \tag{3.2}$$

Then there exist constants $\beta \in (0,1)$ and $C > 0$, depending on $n, \lambda, \Lambda, \theta_+, \theta_-$, such that

$$\begin{aligned}&\max_{I=+,-}\|u\|^{(-\beta;\mathcal{E}_1^0)}_{1,\alpha;W_1^I} \\ &\leq C(\|u\|_{0,0;W_2} + \max_{I=+,-}\|\varphi\|_{1,\alpha;T_2^I} + \|h\|_{L^\infty(W_2)} + \max_{i=1,\cdots,n;I=+,-}\|g^i\|_{0,\alpha;W_2^I}).\end{aligned} \tag{3.3}$$

Proof Denote that

$$C^* := \|u\|_{0,0;W_2} + \max_{I=+,-} \|\varphi\|_{1,\alpha;T_2^I} + \|h\|_{L^\infty(W_2)} + \max_{i=1,\cdots,n;I=+,-} \|g^i\|_{0,\alpha;W_2^I}.$$

It is easy to see that φ is Lipschitz on T_2 with the following estimate:

$$\|\varphi\|_{0,1;T_2} \le C \max_{I=+,-} \|\varphi\|_{1,\alpha;T_2^I}. \tag{3.4}$$

Here C is a constant depending on θ_+, θ_-. Hence, by the boundary estimates of De Giogi-Nash ([7, Theorem 8.29]), there exists $\beta \in (0,1)$, depending on $n, \alpha, \lambda, \Lambda, \theta_+, \theta_-$, such that $u \in C^\beta(W_{\frac{3}{2}})$ and

$$\|u\|_{0,\beta;W_{\frac{3}{2}}} \le C(\|u\|_{0,0;W_2} + \|\varphi\|_{0,1;T_2} + \|h\|_{L^n(W_2)} + \max_{i=1,\cdots,n} \|g^i\|_{L^{2n}(W_2)}) \le CC^*, \tag{3.5}$$

where C is a constant depending on $n, \lambda, \Lambda, \theta_+, \theta_-$. Denote that

$$\varphi^*(\mathbf{x}') := \varphi|_{\mathcal{E}^0} = \varphi(0, 0, \mathbf{x}').$$

Since φ is $C^{1,\alpha}$ up to boundary $\mathcal{E}_2^0$ on $\overline{T_2^+}$, it follows that φ^* is $C^{1,\alpha}$ on $\{\mathbf{x}' : |\mathbf{x}'| < 2\}$. Set that

$$v(\mathbf{x}) = u(\mathbf{x}) - \varphi^*(\mathbf{x}'),$$

and then v satisfies that

$$\begin{aligned} &\partial_i(a^{ij}\partial_j v) = h + \partial_i \bar{g}^i, \quad \text{in } \ W_2, \\ &v|_{T_2} = \varphi - \varphi^*, \end{aligned}$$

where $\bar{g}^i := g^i - a^{ij}\partial_j \varphi^*$. In particular, $v|_{\mathcal{E}} = 0$, and by estimate (3.5), we have that

$$|v(\mathbf{x})| = |v(r, \theta, \mathbf{x}')| \le \|u\|_{0,\beta;T_{\frac{3}{2}}} r^\beta \le CC^* r^\beta \tag{3.6}$$

for any $\mathbf{x} \in W_{\frac{3}{2}}$. Then we use the Schauder interior and boundary estimates, together with interior estimate (2.7) across the discontinuity surface, to obtain weighted Schauder estimates up to the corner as follows: set that

$$\theta^* := \frac{1}{8} \min\{\theta_+, -\theta_-\}.$$

We divide W_1 into three domains to suit different types of estimates. Any point $\mathbf{x}_0 = (r_0, \theta_0, \mathbf{x}_0') \in W_1$ falls into one of the following three cases:

Case 1 $-\theta^* < \theta_0 < \theta^*$;

Case 2 $\theta^* \le \theta_0 \le \theta_+ - \theta^*$ or $\theta_- + \theta^* \le \theta_0 \le -\theta^*$;

Case 3 $\theta_+ - \theta^* < \theta_0 < \theta_+$ or $\theta_- < \theta_0 < \theta_- + \theta^*$.

Let

$$d := 2r_0 \sin\theta^*.$$

In Case 1, let $\hat{\mathbf{x}}$ be the projection point of $\mathbf{x}_0$ onto D_1, i.e.,

$$\hat{\mathbf{x}} = (r_0 \cos\theta_0, 0, \mathbf{x}_0')$$

in cylindrical coordinates. Obviously, $\mathbf{x}_0 \in B_d(\hat{\mathbf{x}})$. By coordinate transformation

$$\mathbf{x} = \hat{\mathbf{x}} + d\mathbf{y},$$

we rescale $B_d(\hat{\mathbf{x}})$ into B_1. Set that

$$\hat{v}(\mathbf{y}) = v(\hat{\mathbf{x}} + d\mathbf{y}), \quad \hat{a}^{ij}(\mathbf{y}) = a^{ij}(\hat{\mathbf{x}} + d\mathbf{y}), \tag{3.7}$$

$$\hat{g}^i(\mathbf{y}) = d\bar{g}^i(\hat{\mathbf{x}} + d\mathbf{y}), \quad \hat{h}(\mathbf{y}) = d^2 h(\hat{\mathbf{x}} + d\mathbf{y}). \tag{3.8}$$

Then $\hat{v}$ satisfies that

$$\partial_i(\hat{a}^{ij}\partial_j \hat{v}) = \hat{h} + \partial_i \hat{g}^i$$

in B_2. Hence, estimate (2.7) in Lemma 2.2 leads to

$$\max_{I=+,-} \|\hat{v}\|_{1,\alpha;B_1^I} \le C(\|\hat{v}\|_{0,0;B_2} + \|\hat{h}\|_{L^\infty(B_2)} + \max_{i=1,\cdots,n;I=+,-} \|\hat{g}^i\|_{0,\alpha;B_2^I}).$$

Scaling domain B_1 back to $B_d(\hat{\mathbf{x}})$ and setting that

$$Q_r = B_r(\hat{\mathbf{x}}), \quad Q_r^I = B_r^I(\hat{\mathbf{x}}), \quad I = +,-, r > 0,$$

we obtain that

$$\max_{I=+,-} \|v\|'_{1,\alpha;Q_1^I} \le C\left(\|v\|_{0,0;Q_2} + d^2\|h\|_{0,0;Q_2} + d \max_{i=1,\cdots,n,I=+,-} \|\bar{g}^i\|'_{0,\alpha;Q_2^I}\right).$$

Therefore, by estimate (3.6) and the definition of $\bar{g}^i$, we have that

$$\max_{I=+,-} \|v\|'_{1,\alpha;Q_1^I} \le CC^* d^\beta. \tag{3.9}$$

In Case 2, set that $Q_r^I = B_r(\mathbf{x}_0)$, where I=+,−, if $B_r(\mathbf{x}_0) \subset W^\pm$, and use the same arguments as for estimate (2.19). We then obtain the following Schauder interior estimate:

$$\|v\|'_{1,\alpha;Q_1^I} \le C\left(\|v\|_{0,0;Q_2^I} + d^2\|h\|'_{0,0;Q_2^I} + d \max_{i=1,\cdots,n} \|\bar{g}^i\|'_{0,\alpha;Q_2^I}\right) \le CC^* d^\beta. \tag{3.10}$$

In Case 3, we project $\mathbf{x}_0$ onto the boundary ∂W and denote the projection point by $\hat{\mathbf{x}}$. Set that

$$Q_r^I = B_r(\hat{\mathbf{x}}) \cap W^I, \quad \hat{T}_r^I = B_r(\hat{\mathbf{x}}) \cap \partial W^I, \quad I = +,-.$$

The Schauder boundary estimates give rise to

$$\begin{aligned} \|v\|'_{1,\alpha;Q_1^I} &\le C(\|v\|_{0,0;Q_2^I} + \|\varphi\|'_{1,\alpha;\hat{T}_2^I} + d^2\|h\|_{L^\infty(Q_2^I)} + d \max_{i=1,\cdots,n;} \|g^i\|_{0,\alpha;Q_2^I}) \\ &\le CC^* d^\beta. \end{aligned} \tag{3.11}$$

We will use estimates (3.9)–(3.11) above to obtain the corner estimate (3.3). The definitions of θ^* and d above imply that

$$d \le r_0 = \delta_{\mathbf{x}_0} \le Cd.$$

We first estimate $[v]^{(-\beta)}_{1,\alpha;W_1^+}$ as follows: for any $\mathbf{x}_0 = (r_0, \theta_0, \mathbf{x}'_0), \mathbf{x} = (r_1, \theta_1, \mathbf{x}')$ in W_1^+, assuming that $r_0 \le r_1$, we have that

$$d \le \delta_{\mathbf{x}_0,\mathbf{x}} = r_0 \le Cd.$$

Thus, estimates (3.9)–(3.11) imply that

$$\begin{aligned} [v]^{(-\beta)}_{1,\alpha;W_1^+} &= \sup_{\substack{\mathbf{x},\mathbf{x}_0 \in W_1^+ \\ \mathbf{x} \ne \mathbf{x}_0}} \left((\delta_{\mathbf{x},\mathbf{x}_0})^{\max(1+\alpha-\beta,0)} \frac{|Dv(\mathbf{x}) - Dv(\mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|^\alpha}\right) \\ &= \sup_{\substack{\mathbf{x},\mathbf{x}_0 \in W_1^+ \\ 0<|\mathbf{x}-\mathbf{x}_0|<d}} \left(r_0^{1+\alpha-\beta} \frac{|Dv(\mathbf{x}) - Dv(\mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|^\alpha}\right) \end{aligned}$$

$$
\begin{aligned}
&+\sup_{\substack{\mathbf{x},\mathbf{x}_0\in W_1^+\\|\mathbf{x}-\mathbf{x}_0|\geq d}}\left(r_0^{1+\alpha-\beta}\frac{|Dv(\mathbf{x})-Dv(\mathbf{x}_0)|}{|\mathbf{x}-\mathbf{x}_0|^\alpha}\right)\\
\leq{}& C\sup_{\mathbf{x}_0\in W_1^+}\left(d^{-\beta}\|v\|'_{1,\alpha;Q_1^+}+d^{1-\beta}\|Dv\|_{L^\infty(Q_1^+)}\right)\\
\leq{}& CC^*.
\end{aligned}
$$

Similarly, we have the estimate in W_1^-:

$$[v]^{(-\beta)}_{1,\alpha;W_1^-}\leq CC^*.$$

It is easy to obtain the estimates for $[v]^{(-\beta)}_{0,0;W_1^I}$ and $[v]^{(-\beta)}_{1,0;W_1^I}$:

$$[v]^{(-\beta)}_{0,0;W_1^I}+[v]^{(-\beta)}_{1,0;W_1^I}\leq CC^*.$$

These lead to the corner estimate (3.3). □

We now use the interior (Lemma 2.2) and corner estimates (Lemma 3.1) to obtain the global estimate in Theorem 1.2.

Proof of Theorem 1.2 Denote that

$$\hat{C}:=\max_{I=+,-}\|\varphi\|_{1,\alpha;\Omega^I}+\|h\|_{L^\infty(\Omega)}+\max_{i=1,\cdots,n;I=+,-}\|g^i\|_{0,\alpha;\Omega^I}.$$

By the wedge condition (1.5) at $\mathcal{E}$, there exists a constant $r^*>0$ such that, for any $\mathbf{x}_0\in\mathcal{E}$, $\exists\,\chi\in C^{1,\alpha}(B_{r^*}(\mathbf{x}_0))$, which is an isomorphism from $B_{r^*}(\mathbf{x}_0)$ to $\chi(B_{r^*}(\mathbf{x}_0))$ satisfying $W_2\subset\chi(B_{r^*}(\mathbf{x}_0)\cap\Omega)$. Let $r_*>0$ be the radius such that $B_{r_*}(\mathbf{x}_0)\cap\Omega\subset\chi^{-1}(W_1)$.

For any $\mathbf{x}\in\Omega$, if $\operatorname{dist}(\mathbf{x},\mathcal{E})<r_*$, we can find some point $\mathbf{x}_0\in\mathcal{E}$ such that $|\mathbf{x}-\mathbf{x}_0|=\operatorname{dist}(\mathbf{x},\mathcal{E})$. Then $\chi(B_{r_*}(\mathbf{x}_0)\cap\Omega)\subset W_1$, and corner estimate (3.3) gives rise to

$$
\begin{aligned}
\max_{I=+,-}\|u\|^{(-\beta;\mathcal{E})}_{1,\alpha;B_{r_*}(\mathbf{x}_0)\cap\Omega^I}\leq{}& C(\|u\|_{0,0;B_{r^*}(\mathbf{x}_0)}+\max_{I=+,-}\|\varphi\|_{1,\alpha;B_{r^*}(\mathbf{x}_0)\cap(\partial\Omega)^I}\\
&+\|h\|_{L^\infty(B_{r^*}(\mathbf{x}_0))}+\max_{i=1,\cdots,n;I=+,-}\|g^i\|_{0,\alpha;B_{r^*}(\mathbf{x}_0)\cap(\partial\Omega)^I})\\
\leq{}& C(\|u\|_{0,0;\Omega}+\hat{C}).
\end{aligned}
$$

By the weak maximum principle (cf. [7, Theorem 8.16]), we know that

$$\|u\|_{0,0;\Omega}\leq C\hat{C}.$$

Therefore, we conclude that

$$\max_{I=+,-}\|u\|^{(-\beta;\mathcal{E})}_{1,\alpha;B_{r_*}(\mathbf{x}_0)\cap\Omega^I}\leq C\hat{C}. \tag{3.12}$$

If $\operatorname{dist}(\mathbf{x},\mathcal{E})\geq r_*$ and $\operatorname{dist}(\mathbf{x},\Gamma)<r_{**}:=\frac{1}{4}r_*\sin\theta_*$, we can find $\mathbf{x}_0\in\Gamma$ such that $\|\mathbf{x}-\mathbf{x}_0\|=\operatorname{dist}(\mathbf{x},\Gamma)$. We apply the interior estimate (2.7) to obtain that

$$\max_{I=+,-}\|u\|_{1,\alpha;B_{r_{**}}(\mathbf{x}_0)\cap\Omega^I}\leq C\hat{C}. \tag{3.13}$$

For the rest of $\mathbf{x}$, the classical Schauder interior and boundary estimates apply, and we have that

$$\|u\|_{1,\alpha;B_{r_{**}/2}(\mathbf{x})\cap\Omega}\leq C\hat{C}. \tag{3.14}$$

Estimates (3.12), (3.13) and (3.14) combined together give rise to the global estimate (1.10). Thus we have completed the proof of Theorem 1.2. □

4 An Example to Illustrate the Corner Issue

We obtained the C^β estimate up to the edge in the proof of Theorem 1.2. For the purpose of application to nonlinear problems, sometimes we do need the $C^{1,\alpha}$ estimates up to the corners. In this section, we will provide a simple example to show that even the coefficients are piecewise constant; the regularity at the corner is a complicated issue depending on the boundary shape and the jump of the coefficients.

In general, the regularity of the solution up to the corner depends on both the boundary shape near the corner and the coefficients of the elliptic equations. We do not elaborate the general situation and only focus on the equation with a simple jump on coefficients. We will investigate how the angles between the boundary and discontinuity surface affect the regularity of the solutions near the corners.

Let

$$\Omega = W \cap B_1, \quad \Omega^+ = W \cap B_1^+, \quad \Omega^- = W \cap B_1^-, \tag{4.1}$$

where W is defined in (1.5) and the angles between the boundary and discontinuity line are θ_+, θ_-; this will affect the regularity of solutions at the corner $\mathbf{0}$.

Consider the elliptic equation

$$\operatorname{div}(a(\mathbf{x})\nabla u) = 0 \quad \text{in} \quad \Omega, \tag{4.2}$$

where

$$a(\mathbf{x}) = \begin{cases} a_0, \, \mathbf{x} \in \Omega^+, \\ 1, \quad \mathbf{x} \in \Omega^-, \end{cases}$$

and where a_0 is a positive constant to be determined later.

Remark 4.1 The equations with this type of coefficient were studied in [2]. Caffarelli and his collaborators studied interface transmission problems in [3]; these have essentially the same type of coefficient as described above. In [1, 4–6], the background states are also solutions to the elliptic equations with piecewise constant coefficients.

We construct solution u to (4.2) in the following form:

$$u(\mathbf{x}) = \begin{cases} u^+(\mathbf{x}) = r^\gamma(A\sin\gamma\theta + B\cos\gamma\theta), & \mathbf{x} \in \Omega^+, \\ u^-(\mathbf{x}) = r^\gamma(C\sin\gamma\theta + D\cos\gamma\theta), & \mathbf{x} \in \Omega^-. \end{cases} \tag{4.3}$$

Here γ, A, B, C, D are constants and $\gamma > 0$. Obviously, u satisfies that

$$\Delta u = 0$$

in each subdomain Ω^+ or Ω^-. To be a weak solution to (4.2) in the whole domain Ω, two conditions should be satisfied on the discontinuity line $\{(r,\theta) : r > 0, \theta = 0\}$: one is the continuity of u across the discontinuity line; the other is the jump condition $Du^+ \cdot \mathbf{n} = Du^- \cdot \mathbf{n}$ on the discontinuity line, where $\mathbf{n}$ is the normal direction on the line. Thus we have that

$$u^+|_{\theta=0} = u^-|_{\theta=0}, \tag{4.4}$$

$$\partial_\theta u^+|_{\theta=0} = \partial_\theta u^-|_{\theta=0}. \tag{4.5}$$

Conditions (4.4) and (4.5) imply that

$$B = D, \qquad a_0 A = C.$$

Without loss of generality, we may take that

$$B = D = 1.$$

We want $u = 0$ on the boundary $\{\theta = \theta_+\}$ and $\{\theta = \theta_-\}$, so we solve that

$$\begin{cases} u^+(r, \theta_+) = 0, \\ u^-(r, \theta_-) = 0 \end{cases}$$

for A, a_0, and obtain that

$$A = -\frac{\cos \gamma\theta_+}{\sin \gamma\theta_+}, \quad a_0 = \frac{\sin \gamma\theta_+ \cos \gamma\theta_-}{\cos \gamma\theta_+ \sin \gamma\theta_-}.$$

Now we take that

$$\gamma = \frac{4}{5}, \qquad \theta_+ = \frac{3}{4}\pi, \qquad \theta_- = -\frac{1}{4}\pi.$$

Then the wedge wall becomes a straight segment on which $u = 0$. This shows that even if both the boundary and the boundary data are smooth, we still have a non-smooth solution, due to the discontinuity of the coefficient $a(\mathbf{x})$. In this example the solution is $C^{\frac{4}{5}}$ up to the corner $\mathbf{0}$.

Proposition 4.2 Suppose that domain Ω is defined by (4.1), $\theta_+ \in (0, \frac{\pi}{2}), \theta_- \in (-\frac{\pi}{2}, 0)$ and $u|_{\partial W \cap B^I_{1/2}} \in C^{1,\alpha}(\overline{\partial W \cap B^I_{1/2}})$, where $I = +, -$ and $\alpha \in (0,1)$ depends on θ_+, θ_-, a_0. Then the solution is $u \in C^{1,\alpha}(\overline{B^+_{1/2}}) \cap C^{1,\alpha}(\overline{B^-_{1/2}})$.

Proof Denote the tangential directions at $\mathbf{0}$ along the upper and lower boundaries as

$$\boldsymbol{\tau}^+ = (\cos \theta_+, \sin \theta_+), \quad \boldsymbol{\tau}^- = (\cos \theta_-, \sin \theta_-).$$

We will find a piecewise linear function p as a solution to (4.2) such that the tangential derivatives of p and u are equal at $\mathbf{0}$. Set that

$$c^+ = D_{\boldsymbol{\tau}^+} u(0,0), \quad c^- = D_{\boldsymbol{\tau}^-} u(0,0),$$

and

$$p(\mathbf{x}) = \begin{cases} p^+(\mathbf{x}) = a^* x_1 + b^+ x_2, \, \mathbf{x} \in \Omega^+, \\ p^-(\mathbf{x}) = a^* x_1 + b^- x_2, \, \mathbf{x} \in \Omega^-. \end{cases}$$

Then we have that

$$D_{\boldsymbol{\tau}^+} p(\mathbf{x}) = \cos \theta_+ a^* + \sin \theta_+ b^+ = c^+, \tag{4.6}$$

$$D_{\boldsymbol{\tau}^-} p(\mathbf{x}) = \cos \theta_- a^* + \sin \theta_- b^- = c^-. \tag{4.7}$$

Since p is a solution to (4.2), the following jump condition on the discontinuity line should be satisfied:

$$a_0 \partial_{x_2} p^+(\mathbf{x}) - \partial_{x_2} p^-(\mathbf{x}) = a_0 b^+ - b^- = 0. \tag{4.8}$$

It is obvious that the linear system (4.6)–(4.8) is uniquely solvable for a^*, b^+, b^- if and only if

$$\cos \theta_+ \sin \theta_- a_0 - \sin \theta_+ \cos \theta_- \neq 0.$$

Thus the assumptions $a_0 > 0, \theta_+ \in (0, \frac{\pi}{2}), \theta_- \in (-\frac{\pi}{2}, 0)$ guarantee the solvability of a^*, b^+, b^-.

Let

$$v(\mathbf{x}) = u(\mathbf{x}) - u(0,0) - p(\mathbf{x}). \tag{4.9}$$

Then v is also a solution to (4.2) and

$$|v(\mathbf{x})|_{\partial W \cap B_1}| \leq Cr^{1+\alpha}.$$

Define a barrier function w by

$$w(\mathbf{x}) = Cr^{1+\alpha} \cos((1 + \alpha + \tau_0)\theta), \tag{4.10}$$

where

$$0 < \alpha, \tau_0 < 1, \quad 1 + \alpha + \tau_0 < \min\left\{\frac{\pi}{2\theta_+}, -\frac{\pi}{2\theta_-}\right\}.$$

By the comparison principle, we conclude that

$$|v(\mathbf{x})| \leq Cr^{1+\alpha}. \tag{4.11}$$

Use similar scaling to that in the proof of Lemma 3.1, estimate (4.11) and interior estimate (2.7) lead to $C^{1,\alpha}$ regularity up to corner $\mathbf{0}$. □

Remark 4.3 In the proof of Proposition 4.2, the construction of the barrier function w is crucial to the $C^{1,\alpha}$ regularity at $\mathbf{0}$. We can choose proper α to obtain the barrier function w, due to the assumption that $\theta_+ \in (0, \frac{\pi}{2}), \theta_- \in (-\frac{\pi}{2}, 0)$. For other coefficients, the angle ranges may vary in order to obtain $C^{1,\alpha}$ regularity. Therefore, it becomes a case by case investigation, and it is difficult to provide a universal criterion to guarantee $C^{1,\alpha}$ regularity at corners.

Conflict of Interest The authors declare no conflict of interest.

References

[1] Bae M. Stability of contact discontinuity for steady Euler system in infinite duct. Z Angew Math Phys, 2013, **64**: 917–936

[2] Bonnetier E, Vogelius M. An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section. SIAM J Math Anal, 2000, **31**: 651–677

[3] Caffarelli L, Soria-Carro M, Stinga P. Regularity for $C^{1,\alpha}$ interface transmission problems. Arch Ration Mech Anal, 2021, **240**(1): 265–294

[4] Chen J, Xin Z, Zang A. Subsonic flows past a profile with a vortex line at the trailing edge. SIAM J Math Anal, 2022, **54**(1): 912–939

[5] Chen S X. Stability of a Mach configuration. Comm Pure Appl Math, 2006, **59**: 1–35

[6] Chen S X. Mach configuration in pseudo-stationary compressible flow. J Amer Math Soc, 2008, **21**(1): 63–100

[7] Gilbarg D, Trudinger N. Elliptic Partial Differential Equations of Second Order. 2nd ed. Berlin: Springer-Verlag, 1983

[8] Huang F, Wang T, Wang Y. On multi-dimensional sonic-subsonic flow. Acta Math Sci, 2011, **31B**(6): 2131–2140

[9] Li Y Y, Vogelius M. Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. Arch Ration Mech Anal, 2000, **153**: 91–151