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THE GLOBAL EXISTENCE AND UNIQUENESS OF SMOOTH SOLUTIONS TO A FLUID-PARTICLE INTERACTION MODEL IN THE FLOWING REGIME*

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Abstract This paper is concerned with the Cauchy problem for a 3D fluid-particle interaction model in the so-called flowing regime in $\mathbb{R}^3$. Under the smallness assumption on both the external potential and the initial perturbation of the stationary solution in some Sobolev spaces, the existence and uniqueness of global smooth solutions in H^3 of the system are established by using the careful energy method.

Key words fluid-particle; flowing regime; global existence

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1 Introduction

In this paper, we consider a fluid-particle interaction model, the so-called flowing regime [1], which is in the whole spatial domain $\mathbb{R}^3$ as follows

$$\rho_t + \nabla\cdot(\rho u) = 0, \tag{1.1}$$

$$((\rho+\beta^{-2}\eta)u)_t+\mathrm{div}((\rho+\beta^{-2}\eta)u\otimes u)+\nabla(p_F+\eta)-\mu\triangle u-\lambda\nabla(\nabla\cdot u) = -(\alpha\beta^2\rho+\eta)\nabla\Phi, \tag{1.2}$$

$$\eta_t + \nabla\cdot(\eta u) = 0. \tag{1.3}$$

Here $\rho:(0,\infty)\times\mathbb{R}^3\to\mathbb{R}_+$ is the density of the fluid, u is the fluid velocity field, and the density of the particles in the mixture $\eta:(0,\infty)\times\mathbb{R}^3\to\mathbb{R}_+$ is related to the probability distribution function $f(t,x,\xi)$ in the macroscopic description through the relation

$$\eta(t,x) = \int_{R^3} f(t,x,\xi)\mathrm{d}\xi.$$

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The time independent external potential $\Phi = \Phi(x) : \mathbb{R} \to \mathbb{R}_+$ represents the effects of gravity and buoyancy, p_F is the pressure function, α, β are some related dimensionless parameters [1], and λ and μ are constant viscosity coefficients satisfying the physical condition

$$\mu > 0, \lambda + \frac{2}{3}\mu \geq 0. \tag{1.4}$$

Without loss of generality (the case of general constants α and β can be done similarly), we let α, β be 1 and rewrite the equations (1.1)–(1.3) as

$$\rho_t + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{R}^3, t > 0, \tag{1.5}$$

$$(\rho + \eta)u_t + (\rho + \eta)u \cdot \nabla u + \nabla(p_F + \eta) - \mu \triangle u - \lambda \nabla(\nabla \cdot u) = -(\rho + \eta)\nabla\Phi, \ x \in \mathbb{R}^3, t > 0, \tag{1.6}$$

$$\eta_t + \nabla \cdot (\eta u) = 0, \quad x \in \mathbb{R}^3, t > 0, \tag{1.7}$$

with the initial data

$$(\rho, u, \eta)|_{t=0} = (\rho_0, u_0, \eta_0), \quad x \in \mathbb{R}^3 \tag{1.8}$$

satisfying that

$$(\rho_0, u_0, \eta_0) \to (\rho^\infty, 0, \eta^\infty) \quad \text{as} \quad |x| \to \infty$$

for some constant vector $(\rho^\infty, 0, \eta^\infty)$ with $\rho^\infty > 0, \eta^\infty > 0$. Here, the function $p_F = p_F(\rho)$ denotes the pressure of the fluid, and $p_F(\rho)$ is smooth in a neighborhood of ρ^∞ with $p_F(\rho^\infty) > 0$ and $p'_F(\rho^\infty) > 0$.

The fluid-particle interaction model plays an important role in the sedimentation analysis of disperse suspensions of particles in fluids, which has in many practical applications in biotechnology, medicine, chemical engineering and mineral processes [2–4]. In addition, such interaction systems are also used in combustion theory to model diesel engines and rocket propulsors [5, 6]. The system was derived formally from the Kinetic-Fluid model in fluid-particle transport by Carrilo and Goudon [1]. There are two different scaling limits for the coupling system between the kinetic and the fluid equations: the so-called bubbling and flowing regimes. They correspond to the diffusive approximation of the kinetic equation for the bubbling regime [2, 7] and the strong drag force and strong Brownain motion for the flowing regime. There has been a lot of work on the bubbling regime [7–14], but on the flowing regime there have been few studies.

Because of the structure of the equations (1.5) and (1.7), similar to the statement about the stationary solutions of Navier-Stokes equations in [15] and Navier-Stokes-Smoluchowski equations in [13], there exists a stationary solution $(\rho_*, u_*, \eta_*)(x)$ in a small neighborhood of $(\rho^\infty, 0, \eta^\infty)$ such that

$$\int_{\rho^\infty}^{\rho_*} \frac{p'_F(\zeta)}{\zeta + \eta^\infty} + \Phi = 0, \quad u_*(x) = 0, \quad \eta_*(x) = \eta^\infty, \tag{1.9}$$

and that

$$\|\rho_* - \rho^\infty\|_{H^3} \leq C\|\Phi\|_{H^3} \leq \varepsilon \tag{1.10}$$

for some positive constants C and ε. In this paper, we consider the global stability in time of this kind of the steady state. We should point out that there exists another kind of steady state $(\rho_*, u_*, \eta_*)(x) = (\rho_*(x), 0, \eta_*(x))$, with non-constant functions $\rho_*(x)$ and $\eta_*(x)$, and the similar stability results will be discussed in the future.

2 Preliminary and Main Results

First, we give a reformulation of (1.5)–(1.8), by denoting that

$$\mu_1=\frac{\mu}{\rho^\infty+\eta^\infty},\quad \mu_2=\frac{\mu+\lambda}{\rho^\infty+\eta^\infty},\quad \chi=\sqrt{p'_F(\rho^\infty)},$$
$$\tilde\rho(x,t)=\rho(x,t)-\rho_*(x),\quad \tilde u(x,)=u(x,t),\quad \tilde\eta=\eta-\eta^\infty,$$

and

$$\bar\rho=\rho_*(x)-\rho^\infty.$$

Then the initial value problem (1.5)–(1.8) is reformulated as

$$\tilde\rho_t+\rho^\infty\nabla\cdot\tilde u=\tilde S_1, \tag{2.1}$$

$$\tilde u_t-\mu_1\Delta\tilde u-\mu_2\nabla\mathrm{div}\tilde u+\frac{p'_F(\rho^\infty)}{\rho^\infty+\eta^\infty}\nabla\tilde\rho+\frac{1}{\rho^\infty+\eta^\infty}\nabla\tilde\eta=\tilde S_2, \tag{2.2}$$

$$\tilde\eta_t+\eta^\infty\nabla\cdot\tilde u=\tilde S_3, \tag{2.3}$$

$$(\tilde\rho,\tilde u,\tilde\eta)|_{t=0}=(\rho_0-\rho_*,u_0,\eta_0-\eta^\infty)\to(0,0,0)\quad\text{as}\quad|x|\to\infty, \tag{2.4}$$

where $\tilde S_1=-\nabla\cdot[(\tilde\rho+\bar\rho)\tilde u]$,

$$\begin{aligned}\tilde S_2=&-\tilde u\cdot\nabla\tilde u+\Big(\frac{\mu}{\tilde\rho+\rho_*+\tilde\eta+\eta^\infty}-\mu_1\Big)\Delta\tilde u+\Big(\frac{\lambda}{\tilde\rho+\rho_*+\tilde\eta+\eta^\infty}-\mu_2\Big)\nabla\mathrm{div}\tilde u\\&-\frac{p'_F(\tilde\rho+\rho_*)}{\tilde\rho+\rho_*+\tilde\eta+\eta^\infty}\nabla(\tilde\rho+\rho_*)-\frac{1}{\tilde\rho+\rho_*+\tilde\eta+\eta^\infty}\nabla\tilde\eta\\&+\frac{p'_F(\rho^\infty)}{\rho^\infty+\eta^\infty}\nabla\tilde\rho+\frac{1}{\rho^\infty+\eta^\infty}\nabla\tilde\eta-\nabla\Phi,\end{aligned}$$
$$\tilde S_3=-\nabla\cdot(\tilde\eta\tilde u)$$

To simplify the computations in the proof, we let $\rho^\infty=\eta^\infty$ and denote that

$$\varrho(x,t)=\tilde\rho(x,t),\quad U(x,t)=\frac{\rho^\infty+\eta^\infty}{\sqrt{p'_F(\rho^\infty)}}\tilde u(x,t),\quad z=\frac{1}{\sqrt{p'_F(\rho^\infty)}}\tilde\eta(x,t).$$

Then, by (1.9), (2.1)–(2.4) can be rewritten as

$$\varrho_t+\chi\nabla\cdot U=S_1, \tag{2.5}$$

$$U_t-\mu_1\Delta U-\mu_2\nabla\mathrm{div}U+\nabla z+\chi\nabla\varrho=S_2, \tag{2.6}$$

$$z_t+\nabla\cdot U=S_3, \tag{2.7}$$

$$(\varrho,U,z)|_{t=0}=\Big(\rho_0-\rho_*,\frac{\rho^\infty+\eta^\infty}{\sqrt{p'_F(\rho^\infty)}}u_0,\frac{\eta_0-\eta^\infty}{\sqrt{p'_F(\rho^\infty)}}\Big)\to(0,0,0)\quad\text{as}\quad|x|\to\infty, \tag{2.8}$$

where

$$S_1=-\frac{\mu_1\chi}{\mu}\mathrm{div}[(\varrho+\bar\rho)U], \tag{2.9}$$

$$\begin{aligned}S_2=&-\frac{\mu_1\chi}{\mu}U\cdot\nabla U-\mu_1h(\varrho,\bar\rho,z)\triangle U-\mu_2h(\varrho,\bar\rho,z)\nabla\mathrm{div}U\\&-\frac{\mu}{\mu_1\chi}g_1(\varrho,\bar\rho,z)\nabla\bar\rho-\frac{\mu}{\mu_1\chi}g_2(\varrho,\bar\rho,z)\nabla\varrho+\frac{\mu}{\mu_1}h(\varrho,\bar\rho,z)\nabla z,\end{aligned} \tag{2.10}$$

$$S_3=-\frac{\mu_1\chi}{\mu}\mathrm{div}(zU), \tag{2.11}$$

and

$$h(\varrho,\bar{\rho},z)=\frac{\varrho+\bar{\rho}+\sqrt{p_F'(\rho^\infty)}z+\eta^\infty}{\varrho+\bar{\rho}+\rho^\infty+\sqrt{p_F'(\rho^\infty)}z+\eta^\infty},$$

$$g_1(\varrho,\bar{\rho},z)=\frac{p_F'(\varrho+\bar{\rho})}{\varrho+\bar{\rho}+\rho^\infty+\sqrt{p_F'(\rho^\infty)}z+\eta^\infty}-\frac{p_F'(\bar{\rho}+\rho^\infty)}{\bar{\rho}+\rho^\infty+\eta^\infty},$$

$$g_2(\varrho,\bar{\rho},z)=\frac{p_F'(\varrho+\bar{\rho}+\rho^\infty)}{\varrho+\bar{\rho}+\rho^\infty+\sqrt{p_F'(\rho^\infty)}z+\eta^\infty}-\frac{p_F'(\rho^\infty)}{\rho^\infty+\eta^\infty}.$$

Our main purpose in this paper is to study the global existence of the smooth solution (ρ,u,η) in a small perturbation of the stationary solution $(\rho_*,0,\eta^\infty)$, i.e., the global existence of the perturbed solution (ϱ,U,z). In what follows, we state our main results.

Theorem 2.1 Let $(\varrho_0,U_0,z_0)\in H^3(\mathbb{R}^3)$ and $\Phi\in H^4(\mathbb{R}^3)$. Suppose that the potential function $\Phi(x)$ satisfies that

$$\begin{cases}\|\Phi\|_{H^4}+\|(1+|x|)\nabla\Phi\|_{L^2\cap L^3}\le\varepsilon,\\ \|(\varrho_0,U_0,z_0)\|_{H^3}\le\varepsilon\end{cases}\tag{2.12}$$

for some small constant $\varepsilon>0$. Then the Cauchy problem (2.5)–(2.8) admits a unique global smooth solution $(\varrho,U,z)\in(C^0(0,\infty;H^3(\mathbb{R}^3)))^3$.

3 Global Existence: The Proof of Theorem 2.1

Proposition 3.1 Suppose that the initial data satisfies $(\varrho_0,U_0,z_0)\in H^3(\mathbb{R}^3)$ and (2.12). Then there exists a positive constant $T_1>0$ depending on (ϱ_0,U_0,z_0) such that the initial value problem (2.5)–(2.8) has a unique solution (ϱ,U,z) which satisfies that

$$\varrho,z\in C^0(0,T_1;H^3(\mathbb{R}^3))\cap C^1(0,T_1;H^2(\mathbb{R}^3))$$

$$U\in C^0(0,T_1;H^3(\mathbb{R}^3))\cap C^1(0,T_1;H^1(\mathbb{R}^3))$$

$$\nabla\varrho,\nabla z\in L^2(0,T_1;H^2(\mathbb{R}^3)),\nabla U\in L^2(0,T_1;H^3(\mathbb{R}^3))$$

and

$$\sup_{0\le t\le T_1}\|(\varrho,U,z)\|_{H^3}^2\le C\|(\varrho_0,U_0,z_0)\|_{H^3}^2.$$

Remark 3.2 Proposition 3.1 is a special case of Theorem 1 in [16], so we omit it here.

In what follows, we will establish some *a priori* estimates of the solutions globally in time. With the help of the local existence theory and those estimates, the global existence of solutions will be obtained by employing the standard continuity argument. To begin with, we make *a priori* assumption that

$$\sup_{0\le t\le T}\|(\varrho,U,z)\|_{H^3}\le\delta\tag{3.1}$$

for some $T\in(0,T^*)$, where T^* represents the maximal time of the existence of the solutions, and the constant δ is sufficiently small. Using the Sobolev imbedding inequality, we are able to obtain that

$$|h(\varrho,\bar{\rho},z)|,\quad|g_1(\varrho,\bar{\rho},z)|\preceq|\varrho|,\quad|g_2(\varrho,\bar{\rho},z)|\preceq|\varrho|+|\bar{\rho}|,\tag{3.2}$$

and

$$|\partial_\varrho^\alpha\partial_{\bar\rho}^\beta\partial_z^\gamma h|, |\partial_\varrho^\alpha\partial_{\bar\rho}^\beta\partial_z^\gamma g_1|, |\partial_\varrho^\alpha\partial_{\bar\rho}^\beta\partial_z^\gamma g_2| \le C \quad \text{with} \quad |\alpha|+|\beta|+|\gamma| \ge 1. \tag{3.3}$$

Here $\cdot \preceq \cdot$ represents that $\cdot \le C\cdot$ for some known constant $C>0$.

With the *a priori* assumption (3.1), we obtain the following estimates, which can ensure the global existence of the solution:

Lemma 3.3 Under the assumptions of Proposition 3.1 and (3.1), it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho,U,z)\|_{L^2}^2 + C_1\|\nabla U\|_{L^2}^2 \preceq C(\delta+\varepsilon)(\|\nabla\varrho\|_{L^2}^2 + \|\nabla U\|_{H^1}^2 + \|\nabla z\|_{L^2}^2), \tag{3.4}$$

where C_1 and C are constants.

Proof Multiplying (2.3), (2.4), (2.5) by ϱ, U and z, respectively, and then integrating by parts over $\mathbb{R}^3$, we have, from the sum of the resulting equalities, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho,U,z)\|_{L^2}^2 + \mu_1\|\nabla U\|_{L^2}^2 + \mu_2\|\mathrm{div}U\|_{L^2}^2 = \langle\varrho,S_1\rangle + \langle U,S_2\rangle + \langle z,S_3\rangle \tag{3.5}$$

To estimate the three terms on the right-hand side of (3.5), we use the Hölder's inequality, Lemma 4.3 in [13], (1.9), (2.12), and Young's inequality, to get that

$$\begin{aligned}\langle\varrho,S_1\rangle &\preceq \|\varrho\|_{L^6}\|\nabla\varrho\|_{L^2}\|U\|_{L^3} + \|\varrho\|_{L^6}\|\varrho\|_{L^3}\|\nabla U\|_{L^2}\\ &\quad + \|\varrho\|_{L^6}\|(1+|x|)\nabla\bar\rho\|_{L^3}\|\frac{U}{1+|x|}\|_{L^2} + \|\varrho\|_{L^6}\|\bar\rho\|_{L^3}\|\nabla U\|_{L^2}\\ &\preceq (\delta+\varepsilon)(\|\nabla\varrho\|_{L^2}^2 + \|\nabla U\|_{L^2}^2),\end{aligned} \tag{3.6}$$

where we have also used the Hardy inequality

$$\|\frac{U}{1+|x|}\|_{L^2} \preceq \|\nabla U\|_{L^2}.$$

Similarly, we get that

$$\langle U,S_2\rangle \preceq (\delta+\varepsilon)(\|\nabla\varrho\|_{L^2}^2 + \|\nabla U\|_{H^1}^2 + \|\nabla z\|_{L^2}^2) \tag{3.7}$$

$$\langle U,S_3\rangle \preceq (\delta+\varepsilon)(\|\nabla U\|_{L^2}^2 + \|\nabla z\|_{L^2}^2). \tag{3.8}$$

Thus, we have completed the proof of Lemma 3.3. □

Lemma 3.4 Under the assumptions of Proposition 3.1 and (3.1), it holds that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k(\varrho,U,z)\|_{L^2}^2 + C_2\|\nabla U^{k+1}\|_{L^2}^2\\ &\preceq C(\delta+\varepsilon)(\|\nabla^k\varrho\|_{L^2}^2 + \|\nabla U\|_{H^2}^2 + \|\nabla z\|_{H^1}^2 + \|\nabla\varrho\|_{H^1}^2 + \|\nabla^k z\|_{L^2}^2).\end{aligned} \tag{3.9}$$

Proof Multiplying $\nabla^k(2.3), \nabla^k(2.4), \nabla^k(2.5)$ by $\nabla^k\varrho, \nabla^k U$ and $\nabla^k z$, respectively, and then integrating by parts over $\mathbb{R}^3$, we have, from the sum of the resulting equalities, that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k(\varrho,U,z)\|_{L^2}^2 + \mu_1\|\nabla^{k+1}U\|_{L^2}^2 + \mu_2\|\nabla^k\mathrm{div}U\|_{L^2}^2\\ &= \langle\nabla^k\varrho,\nabla^k S_1\rangle + \langle\nabla^k U,\nabla^k S_2\rangle + \langle\nabla^k z,\nabla^k S_3\rangle\end{aligned} \tag{3.10}$$

We are going to estimate the terms on the right hand side of the above equality. For the first term on the right hand side of (3.10), we get that

$$\langle\nabla^k\varrho,\nabla^k S_1\rangle \preceq \int_{\mathbb{R}^3}\nabla^k(\varrho\mathrm{div}U)\nabla^k\varrho\mathrm{d}x + \int_{\mathbb{R}^3}\nabla^k(\nabla\varrho\cdot U)\nabla^k\varrho\mathrm{d}x$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}\nabla^k(\bar{\rho}\mathrm{div}U)\nabla^k\varrho\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^k(\nabla\bar{\rho}\cdot U)\nabla^k\varrho\mathrm{d}x\\
&=I_1+I_2+I_3+I_4.
\end{aligned}\tag{3.11}
$$

From Hölder's inequality, (3.1) and Lemma 4.5 in [13], we have that

$$I_1\le\delta(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^k\mathrm{div}U\|_{L^2}^2),\tag{3.12}$$

$$I_2\le\delta(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^{k+1}U\|_{L^2}^2),\tag{3.13}$$

$$I_3\le\varepsilon(\|\nabla^k\varrho\|_{L^2}^2+C\varepsilon\Sigma_{2\le l\le k}\|\nabla^lU\|_{L^2}^2),\tag{3.14}$$

$$I_4\le\varepsilon(\|\nabla^k\varrho\|_{L^2}^2+C\varepsilon\Sigma_{1\le l\le k-1}\|\nabla^lU\|_{L^2}^2).\tag{3.15}$$

For the second term on the right hand side of (3.10), we obtain that

$$
\begin{aligned}
\langle\nabla^kU,\nabla^kS_2\rangle\preceq&\int_{\mathbb{R}^3}\nabla^k[(U\cdot\nabla)U]\cdot\nabla^kU\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^k(h\Delta U)\cdot\nabla^kU\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\nabla^k(h\nabla\mathrm{div}U)\cdot\nabla^kU\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^k(h\nabla z)\cdot\nabla^kU\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\nabla^k(g_1\nabla\bar{\rho})\cdot\nabla^kU\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^k(g_2\nabla\varrho)\cdot\nabla^kU\mathrm{d}x=\sum_{i=1}^7M_i.
\end{aligned}\tag{3.16}
$$

For M_1, from Hölder's inequality, the Sobolev inequality, the Gagliardo-Nirenberg inequality and (3.1), we have that

$$M_1\le\delta\|\nabla^{k+1}U\|_{L^2}^2.\tag{3.17}$$

Similarly to (3.17), we get

$$
\begin{aligned}
M_2\approx&\int_{\mathbb{R}^3}\nabla^{k-1}(h\Delta U)\cdot\nabla^{k+1}U\mathrm{d}x\\
\preceq&(\|\nabla^{k-1}h\|_{L^6}\|\Delta U\|_{L^3}+\|h\|_{L^\infty}\|\nabla^{k+1}U\|_{L^2})\|\nabla^{k+1}U\|_{L^2}\\
\preceq&(\delta+\varepsilon)(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^kz\|_{L^2}^2+\|\nabla^2U\|_{H^1}^2+\|\nabla^{k+1}U\|_{L^2}^2),
\end{aligned}\tag{3.18}
$$

and

$$M_3\preceq(\delta+\varepsilon)(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^kz\|_{L^2}^2+\|\nabla^2U\|_{H^1}^2+\|\nabla^{k+1}U\|_{L^2}^2).\tag{3.19}$$

Similarly, we have

$$M_4\preceq(\delta+\varepsilon)(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^kz\|_{L^2}^2+\|\nabla^2U\|_{H^1}^2+\|\nabla^{k+1}U\|_{L^2}^2),\tag{3.20}$$

$$M_5\preceq(\delta+\varepsilon)(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^{k+1}U\|_{L^2}^2),\tag{3.21}$$

$$M_6\preceq(\delta+\varepsilon)(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^{k+1}U\|_{L^2}^2).\tag{3.22}$$

This completes the proof of Lemma 3.4. □

Lemma 3.5 Under the assumptions of Proposition 3.1 and (3.1), it holds that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle\nabla^k\nabla\varrho(t),\nabla^kU\rangle+C_3\|\nabla^{k+1}\varrho\|_{L^2}^2\preceq(\|\nabla^{k+2}U\|_{L^2}^2+\|\nabla^{k+1}z\|_{L^2}^2)\\
&+(\delta+\varepsilon)(\|\nabla^2U\|_{H^1}^2+\|\nabla^{k+1}U\|_{H^1}^2+\|\nabla z\|_{H^1}^2+\|\nabla U\|_{H^1}^2+\|\nabla\varrho\|_{H^1}^2)
\end{aligned}\tag{3.23}
$$

for $k=0,1,2$.

Proof Applying ∇^k to (2.6) and then taking the L^2 inner product with $\nabla\nabla^k\varrho$, we have that

$$\chi\int_{\mathbb{R}^3}|\nabla\nabla^k\varrho|^2\mathrm{d}x \le -\int_{\mathbb{R}^3}\nabla^k\partial_t U\cdot\nabla\nabla^k\varrho\mathrm{d}x + C\|\nabla^{k+2}U\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} - \int_{\mathbb{R}^3}\nabla\nabla^k z\cdot\nabla\nabla^k\varrho\mathrm{d}x + \|\nabla^k S_2\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}. \tag{3.24}$$

With (2.5), the first term on the right hand side of (3.24) is estimated as follows:

$$-\int_{\mathbb{R}^3}\nabla^k\partial_t U\cdot\nabla\nabla^k\varrho\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^k U\cdot\nabla\nabla^k\varrho\mathrm{d}x + \chi\|\nabla^k\mathrm{div}U\|_{L^2}^2 + \frac{\mu_1\chi}{\mu}\int_{\mathbb{R}^3}\nabla^k\mathrm{div}U\cdot\nabla^k\mathrm{div}[(\varrho+\bar\rho)U]\mathrm{d}x. \tag{3.25}$$

Using Hölder's inequality, (1.10), (2.12), (3.1) and Lemma 4.5 in [13], we have that

$$\frac{\mu_1\chi}{\mu}\int_{\mathbb{R}^3}\nabla^k\mathrm{div}U\cdot\nabla^k\mathrm{div}[(\varrho+\bar\rho)U]\mathrm{d}x \preceq (\delta+\varepsilon)(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}U\|_{L^2}^2 + \|\nabla U\|_{H^1}^2). \tag{3.26}$$

The second term and the third term on the right hand side of (3.24) can be estimated as

$$\|\nabla^{k+2}U\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} \le \varepsilon\|\nabla^{k+1}\varrho\|_{L^2}^2 + \frac{C}{\varepsilon}\|\nabla^{k+2}U\|_{L^2}^2, \tag{3.27}$$

and

$$-\int_{\mathbb{R}^3}\nabla\nabla^k z\cdot\nabla\nabla^k\varrho\mathrm{d}x \le \varepsilon\|\nabla^{k+1}\varrho\|_{L^2}^2 + \frac{C}{\varepsilon}\|\nabla\nabla^k z\|_{L^2}^2. \tag{3.28}$$

The fourth term of the right hand side of (3.24) can be estimated as follows:

$$\|\nabla^k[(U\cdot\nabla)U]\|_{L^2} \preceq \delta\|\nabla^{k+1}U\|_{L^2}, \tag{3.29}$$

and

$$\|\nabla^k(h\triangle U)\|_{L^2} + \|\nabla^k(h\nabla\mathrm{div}U)\|_{L^2} \preceq (\delta+\varepsilon)(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}z\|_{L^2} + \|\nabla^{k+2}U\|_{L^2} + \|\nabla^2 U\|_{H^1}), \tag{3.30}$$

and

$$\|\nabla^k(h\nabla z)\|_{L^2} + \|\nabla^k(g_1\nabla\bar\rho)\|_{L^2} + \|\nabla^k(g_2\nabla\varrho)\|_{L^2} \preceq (\delta+\varepsilon)(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}z\|_{L^2} + \|\nabla z\|_{H^1} + \|\nabla\varrho\|_{H^1}). \tag{3.31}$$

Thus, we have completed the proof of Lemma 3.5. □

Lemma 3.6 Under the assumptions of Proposition 3.1 and (3.1), it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\nabla^k\nabla z(t),\nabla^k U\rangle + C_4\|\nabla^{k+1}z\|_{L^2}^2 \preceq (\|\nabla^{k+2}U\|_{L^2}^2 + \|\nabla^{k+1}z\|_{L^2}^2) + (\delta+\varepsilon)(\|\nabla^2 U\|_{H^1}^2 + \|\nabla^{k+1}U\|_{H^1}^2 + \|\nabla z\|_{H^1}^2 + \|\nabla U\|_{H^1}^2 + \|\nabla\varrho\|_{H^1}^2) \tag{3.32}$$

for $k=0,1,2$.

Proof Applying ∇^k to (2.6) and then taking the L^2 inner product with $\nabla\nabla^k z$, we have that

$$\int_{\mathbb{R}^3}|\nabla\nabla^k z|^2\mathrm{d}x \le -\int_{\mathbb{R}^3}\nabla^k\partial_t U\cdot\nabla\nabla^k z\mathrm{d}x + C\|\nabla^{k+2}U\|_{L^2}\|\nabla^{k+1}z\|_{L^2}$$

$$-\chi\int_{\mathbb{R}^3}\nabla\nabla^k\varrho\cdot\nabla\nabla^k z\mathrm{d}x+\|\nabla^k S_2\|_{L^2}\|\nabla^{k+1}z\|_{L^2}. \tag{3.33}$$

With (2.7), the first term on the right hand side of (3.33) is estimated as follows:

$$\begin{aligned}-\int_{\mathbb{R}^3}\nabla^k\partial_t U\cdot\nabla\nabla^k z\mathrm{d}x=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^k U\cdot\nabla\nabla^k z\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^k\mathrm{div}U\cdot\nabla^k\nabla\cdot U\mathrm{d}x\\&+\frac{\mu_1\chi}{\mu}\int_{\mathbb{R}^3}\nabla^k\mathrm{div}U\cdot\nabla^k\mathrm{div}(zU)\mathrm{d}x.\end{aligned} \tag{3.34}$$

Using the Hölder's inequality, (1.10), (2.12), (3.1) and Lemma 4.5 in [13], we have that

$$\frac{\mu_1\chi}{\mu}\int_{\mathbb{R}^3}\nabla^k\mathrm{div}U\cdot\nabla^k\mathrm{div}(zU)\mathrm{d}x\preceq\delta(\|\nabla^{k+1}z\|_{L^2}^2+\|\nabla^{k+1}U\|_{L^2}^2+\|\nabla U\|_{H^1}^2). \tag{3.35}$$

The second term and the third term on the right hand side of (3.33) can be estimated as

$$\|\nabla^{k+2}U\|_{L^2}\|\nabla^{k+1}z\|_{L^2}\le\varepsilon\|\nabla^{k+1}z\|_{L^2}^2+\frac{C}{\varepsilon}\|\nabla^{k+2}U\|_{L^2}^2, \tag{3.36}$$

and

$$-\int_{\mathbb{R}^3}\nabla\nabla^k z\cdot\nabla\nabla^k\varrho\mathrm{d}x\le\varepsilon\|\nabla^{k+1}\varrho\|_{L^2}^2+\frac{C}{\varepsilon}\|\nabla\nabla^k z\|_{L^2}^2. \tag{3.37}$$

$\|\nabla^k S_2\|_{L^2}$ on the fourth term of the right hand side of (3.33) can be estimated as follows:

$$\|\nabla^k[(U\cdot\nabla)U]\|_{L^2}\preceq\delta\|\nabla^{k+1}U\|_{L^2}, \tag{3.38}$$

and

$$\begin{aligned}&\|\nabla^k(h\triangle U)\|_{L^2}+\|\nabla^k(h\nabla\mathrm{div}U)\|_{L^2}\\\preceq&\,(\delta+\varepsilon)(\|\nabla^{k+1}\varrho\|_{L^2}+\|\nabla^{k+1}z\|_{L^2}+\|\nabla^{k+2}U\|_{L^2}+\|\nabla^2U\|_{H^1}),\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}&\|\nabla^k(h\nabla z)\|_{L^2}+\|\nabla^k(g_1\nabla\bar\rho)\|_{L^2}+\|\nabla^k(g_2\nabla\varrho)\|_{L^2}\\\preceq&\,(\delta+\varepsilon)(\|\nabla^{k+1}\varrho\|_{L^2}+\|\nabla^{k+1}z\|_{L^2}+\|\nabla z\|_{H^1}+\|\nabla\varrho\|_{H^1}).\end{aligned} \tag{3.40}$$

This completes the proof of Lemma 3.6. □

Now we are in a position to close the *a priori* assumption (3.1). From the Lemmas 3.3–3.6, for a fixed small constant $\varepsilon_1>0$, we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{0\le k\le3}\|\nabla^k(\varrho,U,z)\|_{L^2}^2+\varepsilon_1\sum_{0\le k\le2}<\nabla^k\nabla\varrho,\nabla^kU>+\varepsilon_1\sum_{0\le k\le2}<\nabla^k\nabla z,\nabla^kU>\right\}\\&+\sum_{0\le k\le3}\|\nabla^{k+1}U\|_{L^2}^2+\varepsilon_1\sum_{0\le k\le2}\|\nabla^{k+1}\varrho\|_{L^2}^2+\varepsilon_1\sum_{0\le k\le2}\|\nabla^{k+1}z\|_{L^2}^2\\\le&\,(\delta+\varepsilon)(\|\nabla\varrho\|_{L^2}^2+\|\nabla z\|_{L^2}^2+\|\nabla U\|_{H^1}^2)\\&+C(\delta+\varepsilon)\sum_{1\le k\le3}(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla z\|_{H^1}^2+\|\nabla\varrho\|_{H^1}^2+\|\nabla U\|_{H^2}^2+\|\nabla^kz\|_{L^2}^2)\\&+C\varepsilon_1\sum_{0\le k\le2}[(\delta+\varepsilon)(\|\nabla\varrho\|_{H^1}^2+\|\nabla z\|_{H^1}^2+\|\nabla^{k+1}U\|_{H^1}^2+\|\nabla^2U\|_{H^2}^2)\\&+C(\|\nabla^{k+1}U\|_{L^2}^2+\|\nabla^{k+1}z\|_{L^2}^2)].\end{aligned} \tag{3.41}$$

With (3.41) and the smallness of ε and δ, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t)+\varepsilon_1\|\nabla\varrho\|_{H^2}^2+\varepsilon_1\|\nabla z\|_{H^2}^2+\|\nabla U\|_{H^3}^2$$

$$\leq C \sum_{0\leq k\leq 3} [(\delta+\varepsilon)\varepsilon_1\|\nabla z\|_{H^1}^2 + \varepsilon_1\|\nabla^{k+1} z\|_{L^2}^2], \tag{3.42}$$

where

$$\begin{aligned} A(t) &= \sum_{0\leq k\leq 3} \|\nabla^k(\varrho, U, z)(t)\|_{L^2}^2 + \varepsilon_1 \sum_{0\leq k\leq 2} \langle \nabla^k\nabla\varrho, \nabla^k U\rangle + \varepsilon_1 \sum_{0\leq k\leq 2} \langle \nabla^k \nabla z, \nabla^k U\rangle \\ &= O(\|(\varrho, U, z)(t)\|_{H^3}^2). \end{aligned}$$

From (3.42), we have that

$$\|(\varrho, U, z)\|_{H^3}^2 + \int_0^t (\varepsilon_1\|\nabla\varrho\|_{H^2}^2 + \varepsilon_1\|\nabla z\|_{H^2}^2 + \|\nabla U\|_{H^3}^2)\mathrm{d}s \leq C\|(\varrho_0, U_0, z_0)\|_{H^3}^2 \leq C\varepsilon^2. \tag{3.43}$$

Here we choose that $\delta > \frac{3}{2}\sqrt{C}\varepsilon$. Then

$$\|(\varrho, U, z)\|_{H^3} < \frac{2\delta}{3}, \tag{3.44}$$

which is the desired estimate for proving the maximal time for the existence of T^*. The proof of Theorem 2.1 is completed.

Conflict of Interest The authors declare no conflict of interest.

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