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THE STABLE RECONSTRUCTION OF STRONGLY-DECAYING BLOCK SPARSE SIGNALS*

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Abstract In this paper, we reconstruct strongly-decaying block sparse signals by the block generalized orthogonal matching pursuit (BgOMP) algorithm in the l_2-bounded noise case. Under some restraints on the minimum magnitude of the nonzero elements of the strongly-decaying block sparse signal, if the sensing matrix satisfies the the block restricted isometry property (block-RIP), then arbitrary strongly-decaying block sparse signals can be accurately and steadily reconstructed by the BgOMP algorithm in iterations. Furthermore, we conjecture that this condition is sharp.

Key words compressed sensing; strongly-decaying; block sparse signal; block generalized OMP; block-RIP

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1 Introduction

Compressive Sensing (CS), pioneered by Donoho and Candes, Romberg and Tao [1–4], is an efficient data acquisition paradigm. The framework focuses on recovering unknown signals from an undetermined system of linear equations

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{v}. \tag{1.1}$$

Here $\boldsymbol{y} \in \mathbb{R}^m$ is a measurement vector, $\boldsymbol{\Phi} \in \mathbb{R}^{m\times n}(m \ll n)$, is a sensing matrix, $\boldsymbol{v} \in \mathbb{R}^m$ is a l_2-bounded noise vector ($\|\boldsymbol{v}\|_2 \le \epsilon$, for any constant ϵ) and $\boldsymbol{x}$ is an original and K-sparse signal that needs to be reconstructed. Compressed sensing is also known as compression sampling, and is widely used in fields of radar systems [5], signal processing [6], communication [7], medical imaging [8, 9], and so on.

There are various sparse reconstruction methods which have excellent reconstruction performance. In order to analyze the reconstruction performance of sparse reconstruction algorithms, an important and frequently used concept is the block restricted isometry property (block-RIP) [10]. For arbitrary block K-sparse signals, the $\boldsymbol{\Phi}$ has the block-RIP with a parameter δ_{BK} which satisfies that

$$(1-\delta_{BK})\|\boldsymbol{x}\|_2^2 \le \|\boldsymbol{\Phi}\boldsymbol{x}\|_2^2 \le (1+\delta_{BK})\|\boldsymbol{x}\|_2^2, \tag{1.2}$$

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where $0 < \delta_{BK} < 1$, and the smallest constant δ_{BK} is called the block restricted isometry constant (block-RIC). The block-RIP is a special case of the restricted isometry property (RIP) [11, 12].

The BgOMP algorithm first proposed in [13] is a natural extension of the block orthogonal matching pursuit (BOMP) algorithm [14] for reconstructing sparse signals $\boldsymbol{x}$ from (1.1). In this paper, we principally focus on reconstructing the σ-strongly-decaying block K-sparse signal $\boldsymbol{x}$ by the BgOMP algorithm from (1.1). The BgOMP algorithm is also called the block orthogonal multi-matching pursuit (BOMMP); it can be expressed as follows:

Algorithm 1 BgOMP algorithm

Input: measurement vector $\boldsymbol{y}$, sparsity K, sensing matrix $\boldsymbol{\Phi}$, stopping rule.

Initialize: $k = 0$, $\boldsymbol{r}^0 = \boldsymbol{y}$, $\Lambda^0 = \emptyset$.

Iterate until the stopping criterion is met

Step 1: $k = k + 1$,

Step 2: Select N block subscripts $\{\Gamma_i\}|_N$ corresponding to N largest entries consisted in $\{\|\langle \boldsymbol{r}^{k-1}, \boldsymbol{\Phi}[j]\rangle\|_2, j \in S = \{1, 2, \cdots, L\}\}$,

Step 3: $\Lambda^k = \Lambda^{k-1} \bigcup \{\Gamma_1, \Gamma_2 \cdots \Gamma_N\}$,

Step 4: $\hat{\boldsymbol{x}}[\Lambda^k] = \arg \min\limits_{x:\mathrm{supp}(\boldsymbol{x})=\Lambda^k} \|\boldsymbol{y} - \boldsymbol{\Phi x}\|_2$,

Step 5: $\boldsymbol{r}^k = \boldsymbol{y} - \boldsymbol{\Phi}[\Lambda^k]\hat{\boldsymbol{x}}[\Lambda^k]$,

Output: $\hat{\boldsymbol{x}} = \arg\min \|\boldsymbol{y} - \boldsymbol{\Phi x}\|_2$, where $\mathrm{supp}(\boldsymbol{x}) = \Lambda^k$.

It is easy to see that if $N = 1$, the BgOMP algorithm degenerates to the BOMP algorithm.

Now we give a definition of the σ-strongly-decaying block K-sparse signal [15, 16]. For $\boldsymbol{x} \in \mathbb{R}^n$,

$$\boldsymbol{x} = [\underbrace{x_1, x_2, \cdots, x_{d_1}}_{\boldsymbol{x}[1]}, \underbrace{x_{d_1+1}, x_{d_1+2}, \cdots x_{d_1+d_2}}_{\boldsymbol{x}[2]}, \cdots, \underbrace{x_{n-d_L+1}, \cdots, x_n}_{\boldsymbol{x}[L]}]^T,$$

where $\boldsymbol{x}[i]$ represents the i-th block of $\boldsymbol{x}$, and d_i represents the block size for the homologous block. Now we define the l_2/l_0-norm as

$$\|\boldsymbol{x}\|_{2,0} = \sum_{i=1}^{K} I(\|\boldsymbol{x}[i]\|_2 > 0), \tag{1.3}$$

where $I(\cdot)$ is an indicator function; that is,

$$I(\|\boldsymbol{x}[i]\|_2 > 0) = \begin{cases} 1, \|\boldsymbol{x}[i]\|_2 > 0, \\ 0, \|\boldsymbol{x}[i]\|_2 = 0. \end{cases} \tag{1.4}$$

If $\|\boldsymbol{x}\|_{2,0} \le K$, then $\boldsymbol{x}$ is block K-sparse. In this paper, we mainly investigate the reconstruction of a block sparse signal with even block size, i.e., $d_1 = d_2 = \cdots = d_L = d$, so that

$$n = Ld, \boldsymbol{x}[i] = (x_{d(i-1)+1}, x_{d(i-1)+2}, \cdots, x_{di})^T.$$

If the block of $\boldsymbol{x}$ also satisfies that $\|\boldsymbol{x}[i]\|_2 \ge \sigma\|\boldsymbol{x}[i+1]\|_2, 1 \le i \le K-1$, where $\sigma > 1$, then $\boldsymbol{x}$ is a σ-strongly-decaying block K-sparse signal.

When $d = 1$ and $\sigma = 1$, the block K-sparse signals degenerate into ordinary K-sparse signals. There are many examples of strongly-decaying signals, including communication signals, ultrasonic detection, current propagation, etc.

In recent years, there have been a series of significant achievements for accurately and steadily reconstructing block K-sparse signals $\boldsymbol{x}$ by the BgOMP algorithm. In [17], it was proven that if $\boldsymbol{\Phi}$ satisfies the block-RIP with $\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N}+1}}$, then every ordinary block K-sparse signal $\boldsymbol{x}$ can be accurately and steadily reconstructed by the BgOMP algorithm. In [18], it was shown that $\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N}+1}}$ is a sharp sufficient condition. It was also shown that $\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N}+1}}$ together with a condition on the minimum l_2 norm of nonzero blocks of block K-sparse signals is sufficient to ensure that the BgOMP algorithm selects at least one true block index at each iteration until all true block indices are selected in the noisy case. In [19], Chen and Ge found that $\delta_{KN-N+1} + \sqrt{\frac{K}{N}}\theta_{KN-N+1} < 1$ is a sufficient condition for the stable reconstruction of block sparse signals using the BgOMP algorithm under l_2 and l_∞ bounded noise environments.

Some papers are about reconstructing ordinary block K-sparse signals $\boldsymbol{x}$ by the Bgomp algorithm. At present, there are few results about reconstructing the σ-strongly-decaying block K-sparse signal $\boldsymbol{x}$ by the BgOMP algorithm. Consequently, in this paper, we show that the condition with the block-RIC satisfying $\delta_{NK+1} < \frac{\sqrt{2}}{2}$ of order $NK + 1$ is sufficient to perfectly recover any σ-strongly-decaying block K-sparse signals via the BgOMP algorithm in the l_2 noisy case. Moreover, we also conjecture that there exists a matrix $\boldsymbol{\Phi}$ satisfying that $\delta_{NK+1} = \frac{\sqrt{2}}{2}$, which makes the BgOMP algorithm fail in terms of reconstructing some σ-strongly-decaying block K-sparse signals $\boldsymbol{x}$.

The rest of this paper is organized as follows: In Section 2, we introduce some notations and lemmas. In Section 3, we draw conclusions and present evidence. In Section 4, we give some numerical simulation results.

2 Preliminaries

In what follows, we present some notations and useful lemmas that will be used throughout the paper.

Notations We usually use $\mathbb{R}$ to represent the real field. Vectors are in boldface lowercase letters, and matrices are in boldface uppercase letters, e.g., $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{\Phi} \in \mathbb{R}^{m\times n}$. Let S be the support of a vector $\boldsymbol{x}$ such that $\boldsymbol{x}[i], i \in S$ are nonzero vectors for a σ-strongly-decaying block K-sparse signal $\boldsymbol{x}$, so $|S| \le K$, where $|S|$ stand for the cardinality of S. Λ is the index set selected by the BgOMP algorithm iteration. Let $S \setminus \Lambda = \{i \mid i \in S, i \notin \Lambda\}$ and $|S \cap \Lambda| = l$. In this paper, Λ^c with S^c stands for the complementarity of Λ and S, and $\Lambda^c = \{1, 2, 3, \cdots, L\} \setminus \Lambda$, $S^c = \{1, 2, 3, \cdots, L\} \setminus S$ as well. Let $\boldsymbol{\Phi}[\Lambda]$ be a submatrix of $\boldsymbol{\Phi}$ that only comprises the column blocks indexed by the Λ. Similarly, $\boldsymbol{x}[\Lambda]$ is a subvector of Λ that only comprises the atom blocks indexed by the S. For example, if $\Lambda = \{2, 3, 5\}$, then $\boldsymbol{\Phi}[\Lambda] = [\boldsymbol{\Phi}[2], \boldsymbol{\Phi}[3], \boldsymbol{\Phi}[5]]$ and $\boldsymbol{x}[\Lambda] = [\boldsymbol{x}[2]^T, \boldsymbol{x}[3]^T, \boldsymbol{x}[5]^T]^T$. Let $\boldsymbol{\Phi}^T[\Lambda]$ be the transpose matrix of $\boldsymbol{\Phi}[\Lambda]$, where $\boldsymbol{\Phi}[\Lambda]$ is an arbitrary full column rank matrix. Let $\boldsymbol{P}[\Lambda] = \boldsymbol{\Phi}[\Lambda](\boldsymbol{\Phi}^T[\Lambda]\boldsymbol{\Phi}[\Lambda])^{-1}\boldsymbol{\Phi}^T[\Lambda]$ show the projector and let $\boldsymbol{P}^{\perp}[\Lambda] = \boldsymbol{I} - \boldsymbol{P}[\Lambda]$ denote the orthogonal complementary projection on the column space of $\boldsymbol{\Phi}[\Lambda]$. Let $\boldsymbol{I}$ be a unit matrix and let the e_i represent the i-th column of $\boldsymbol{I}$.

l_2/l_p**-norm** ([20, 21]) For any $\boldsymbol{x} \in \mathbb{R}^n$, we define the l_2/l_p-norm (where $p = 1, 2, \infty$) as

$$\|\boldsymbol{x}\|_{2,p} = \|\boldsymbol{\omega}\|_p, \tag{2.1}$$

where $\boldsymbol{\omega} \in \mathbb{R}^L$ and $\boldsymbol{\omega}_i = \|\boldsymbol{x}[i]\|_2$, with any $1 \le i \le L$. It is easy to see that $\|\boldsymbol{x}\|_{2,2} = \|\boldsymbol{x}\|_2$. In addition, if $d = 1$, then $\boldsymbol{x}[i] = \boldsymbol{x}_i$. Thus, we have that $\|\boldsymbol{x}\|_{2,p} = \|\boldsymbol{x}\|_p$, with $p = 1, 2, \infty$.

Lemma 2.1 ([22]) When $K_1 > K_2$, if the matrix $\boldsymbol{\Phi}$ satisfies the block-RIP of orders K_1 and K_2, then $\delta_{K_1} > \delta_{K_2}$.

Lemma 2.2 ([23]) Let $\boldsymbol{\Phi}$ satisfy the block-RIP of order K. Then, for any $\boldsymbol{x} \in \mathbb{R}^m$, $\|\boldsymbol{\Phi}^T[\Lambda]\boldsymbol{x}\|_2^2 \le (1 + \delta_K)\|\boldsymbol{x}\|_2^2$.

Lemma 2.3 ([16]) Let S_1, S_2 satisfy $|S_2 \setminus S_1| \ge 1$ and the block-RIP of order $|S_1 \cup S_2|$. Then, for any vector $\boldsymbol{x} \in \mathbb{R}^{|S_2 \setminus S_1| \times d}$, $(1 - \delta_{|S_1 \cup S_2|})\|\boldsymbol{x}\|_2^2 \le \|\boldsymbol{P}^{\perp}[S_1]\boldsymbol{\Phi}[S_1 \cup S_2]\boldsymbol{x}\|_2^2 \le (1 + \delta_{|S_1 \cup S_2|})\|\boldsymbol{x}\|_2^2$.

3 Main Results

In this section, we give a detailed description of the BgOMP algorithm to reconstruct the σ-strongly-decaying block K-sparse signal.

Lemma 3.1 For $\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{v}$, let S be the support of the block K-sparse signal $\boldsymbol{x}$, $\Lambda \subset \{1, 2, \cdots, L\}$ and $W \subset S^c$, where $|W| \le N$, $|\Lambda| = kN$ and $0 \le k \le |S \cap \Lambda^k| = l \le |S| - 1$. Let

$$\|\boldsymbol{x}[S \setminus \Lambda]\|_{2,1}^2 \le \rho\|\boldsymbol{x}[S \setminus \Lambda]\|_2^2 \tag{3.1}$$

for some $\rho \ge 1$. Then

$$|\Lambda \cup (S \setminus \Lambda) \cup W| \le NK + 1$$

and

$$\|\boldsymbol{\Phi}^T[S \setminus \Lambda]\boldsymbol{q}[\Lambda]\|_{2,\infty} - \frac{1}{N}\sum_{j \in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2 \ge \frac{(1 - \sqrt{\frac{\rho}{N} + 1}\delta_{NK+1})\|\boldsymbol{x}[S \setminus \Lambda]\|_2}{\sqrt{\rho}}, \tag{3.2}$$

where

$$\boldsymbol{q}[\Lambda] = \boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S \setminus \Lambda]\boldsymbol{x}[S \setminus \Lambda]. \tag{3.3}$$

The proof of this lemma is shown in Appendix.

Lemma 3.2 ([24]) For each $1 \le i \le K$, define that

$$\varphi_i(t) = \frac{(t^i - 1)(t + 1)}{(t^i + 1)(t - 1)} \qquad (t > 1). \tag{3.4}$$

Then

$$1 = \varphi_1(t) < \varphi_2(t) < \cdots < \varphi_K(t). \tag{3.5}$$

Moreover, $\varphi_i(t)$ is strictly monotonically decreasing with t and $1 < \varphi_i(t) < i$ for $2 < i < K$.

By Lemma 3.2, if $1 < z < NK, \varphi_K(t) = z$ has a sole solution t_z. To predigest the notation, we let that

$$\varphi_K^{-1}(z) = \begin{cases} t_z, & 1 < z < K, \\ 1, & z \ge K. \end{cases} \tag{3.6}$$

Lemma 3.3 ([16, 25]) Let $\sigma > 1$ and $\mu \geq \nu \geq 1$ give positive constants and give that

$$\hat{\varphi}_i(t_1, t_2, \cdots, t_i, \mu, \nu) = \frac{(\sum_{j=1}^{i-1} t_j + \mu t_i)^2}{\sum_{j=1}^{i-1} t_j^2 + \nu t_i^2}, \tag{3.7}$$

where

$$t_1 \geq \sigma t_2 \geq \cdots \geq \sigma^{K-1} t_K > 0, \qquad \sum_{j=1}^{0} \cdot = 0 \tag{3.8}$$

for $1 \leq i \leq K$, $\hat{\varphi}_i$ increasing with t_i and

$$\hat{\varphi}_i(t_1, t_2, \cdots, t_i, 1, 1) \leq \varphi_i(\sigma), \tag{3.9}$$

where function φ_i is defined as in Lemma 3.2.

Theorem 3.4 For $\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{v}$, suppose that $\boldsymbol{v}$ satisfies $\|\boldsymbol{v}\|_2 \leq \epsilon$, and $\boldsymbol{\Phi}$ satisfies the block-RIP of order $NK+1$ with

$$\delta_{NK+1} < \frac{\sqrt{2}}{2}, \tag{3.10}$$

assuming that $\boldsymbol{x} \in \mathbb{R}^n$ is a block σ-strongly-decaying K-sparse signal with a satisfying

$$\sigma > \varphi_K^{-1}(\delta_{NK+1}^{-2} - 1). \tag{3.11}$$

Then the BgOMP algorithm with the stopping criterion $\|\boldsymbol{r}^k\|_2 \leq \epsilon$ selects at least one true block index of block K-sparse signals $\boldsymbol{x}$ at each iteration until all true block indices are selected if all the nonzero blocks $\boldsymbol{x}[i]$ satisfy

$$\min_{i \in S} \|\boldsymbol{x}[i]\|_2 > \frac{2\epsilon}{1 - \sqrt{1 + \min(s^{-2} - 1, \frac{|S|}{N})\delta_{NK+1}}}, \tag{3.12}$$

where s satisfies that

$$\delta_{NK+1} < s < \frac{\sqrt{2}}{2}, \ \varphi_K^{-1}[(\delta_{NK+1}^{-2} - 1)N] < \varphi_K^{-1}[(s^{-2} - 1)N] \leq \sigma. \tag{3.13}$$

Thus, the reconstruct error can be bounded according to

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2 \leq \frac{2\epsilon}{\sqrt{1 - \delta_{NK+1}}}. \tag{3.14}$$

Proof First, if $s^{-2} - 1 \geq \frac{|S|}{N}$, then $s \leq \frac{1}{\sqrt{\frac{|S|}{N} + 1}}$. Then according to the assumption, we have that $\delta_{NK+1} < s \leq \frac{1}{\sqrt{\frac{|S|}{N} + 1}}$. Thus, the BgOMP algorithm accurately reconstructs block K-sparse signals in [18].

Second, if $s^{-2} - 1 < \frac{|S|}{N}$, then $(s^{-2} - 1)N < |S|$. By Lemma 3.2, we can easily find out that $\varphi_K(\varphi_K^{-1}((s^{-2} - 1)N)) = (s^{-2} - 1)N$, and according to the characteristics of the block σ-strongly-decaying K-sparse signal, we have that

$$\begin{aligned} \|(\boldsymbol{x}[S \setminus \Lambda^k])[1]\|_2 \geq \sigma \|(\boldsymbol{x}[S \setminus \Lambda^k])[2]\|_2 \geq \sigma^2 \|(\boldsymbol{x}[S \setminus \Lambda^k])[3]\|_2 \geq \cdots \\ \geq \sigma^{|S \setminus \Lambda^k| - 1} \|(\boldsymbol{x}[S \setminus \Lambda^k])[S \setminus \Lambda^k]\|_2. \end{aligned}$$

Next, we obtain that

$$\frac{\|\boldsymbol{x}[S \setminus \Lambda^k]\|_{2,1}^2}{\|\boldsymbol{x}[S \setminus \Lambda^k]\|_2^2} = \hat{\varphi}_{K-l}(\|(\boldsymbol{x}[S \setminus \Lambda^k])[1]\|_2, \|(\boldsymbol{x}[S \setminus \Lambda^k])[2]\|_2, \cdots, \|(\boldsymbol{x}[S \setminus \Lambda^k])[|S \setminus \Lambda^k|]\|_2, 1, 1)$$

$$\begin{aligned}&\le \varphi_{K-l}(\sigma) < \varphi_{K-l}(\varphi_K^{-1}((s^{-2}-1)N))\\&< \varphi_K(\varphi_K^{-1}((s^{-2}-1)N)) = (s^{-2}-1)N.\end{aligned}\tag{3.15}$$

By the Cauchy-Schwarz inequality, we also have that

$$\|\boldsymbol{x}[S\setminus\Lambda^k]\|_{2,1}^2 \le (|S|-l)\|\boldsymbol{x}[S\setminus\Lambda^k]\|_2^2.\tag{3.16}$$

Combining (3.15) and (3.16)

$$\|\boldsymbol{x}[S\setminus\Lambda^k]\|_{2,1}^2 \le \min\{(s^{-2}-1)N, |S|-l\}\|\boldsymbol{x}[S\setminus\Lambda^k]\|_2^2.\tag{3.17}$$

Let us define that $g(t)=\min((s^{-2}-1)N,t)$. Thus, it is not hard to see that $g(t)$ is a non-decreasing function with $t>0$, and by simple calculation, we have that $0<g(t)\le t$. Then, letting $g(|S\setminus\Lambda^k|)=\min\{(s^{-2}-1)N,|S\setminus\Lambda^k|\}$ gives that

$$\|\boldsymbol{x}[S\setminus\Lambda^k]\|_{2,1}^2 \le g(|S\setminus\Lambda^k|)\|\boldsymbol{x}[S\setminus\Lambda^k]\|_2^2.\tag{3.18}$$

Let us give two definitions: α_j^{k+1} denotes the j-th largest correlation between $\boldsymbol{r}^k$ and $\boldsymbol{\Phi}[j], j\in W$, and β_i^{k+1} denotes the i-th largest correlation between $\boldsymbol{r}^k$ and $\boldsymbol{\Phi}[S\setminus\Lambda^k]$. To prove Theorem 3.1, let us first claim that $\beta_1^{k+1}>\alpha_N^{k+1}$.

On account of

$$\beta_1^{k+1} = \|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{r}^k\|_{2,\infty},\tag{3.19}$$

$$\alpha_N^{k+1} = \min\left[\left|\langle\boldsymbol{\Phi}[j],\boldsymbol{r}^k\rangle\right|, j\in W\right] \le \frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}[j]^T\boldsymbol{r}^k\|_2,\tag{3.20}$$

it suffices to claim that

$$\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{r}^k\|_{2,\infty} > \frac{1}{N}\sum_{j\in W}\|\Phi[j]^T\boldsymbol{r}^k\|_2.\tag{3.21}$$

By the BgOMP algorithm, we have that

$$\begin{aligned}\boldsymbol{r}^k &= \boldsymbol{y}-\boldsymbol{\Phi}[\Lambda^k]\hat{\boldsymbol{x}}[\Lambda^k] = (\boldsymbol{I}-\boldsymbol{\Phi}[\Lambda^k](\boldsymbol{\Phi}^T[\Lambda^k]\boldsymbol{\Phi}[\Lambda^k])^{-1}\boldsymbol{\Phi}^T[\Lambda^k])\boldsymbol{y}\\&= \boldsymbol{P}^\perp[\Lambda^k](\boldsymbol{\Phi x}+\boldsymbol{v}) = \boldsymbol{P}^\perp[\Lambda^k](\boldsymbol{\Phi}[S]\boldsymbol{x}[S]+\boldsymbol{v})\\&= \boldsymbol{P}^\perp[\Lambda^k](\boldsymbol{\Phi}[S\cap\Lambda^k]\boldsymbol{x}[S\cap\Lambda^k]+\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]+\boldsymbol{v})\\&= \boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]+\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{v},\end{aligned}\tag{3.22}$$

where substituting (3.22) into (3.19) and (3.20) gives that

$$\begin{aligned}\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{r}^k\|_{2,\infty} \ge{}& \|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_{2,\infty}\\&-\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{v}\|_{2,\infty},\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}\frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}[j]^T\boldsymbol{r}^k\|_2 &\le \frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_2+\frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{v}\|_2\\&\le \frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_2+\|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{v}\|_{2,\infty}.\end{aligned}\tag{3.24}$$

Thus, we just need to verify that

$$\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{P}^\perp[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_{2,\infty}$$

$$
\begin{aligned}
&-\frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_2 \\
&>\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_{2,\infty}+\|\boldsymbol{\Phi}^T[W]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_{2,\infty}. \qquad (3.25)
\end{aligned}
$$

According to the properties of $g(t)$ above, we have that

$$
\begin{aligned}
\|\boldsymbol{x}[S\setminus\Lambda^k]\|_2 &\geq \sqrt{|S|-l}\min_{i\in S\setminus\Lambda^k}\|\boldsymbol{x}[i]\|_2 \\
&\geq \sqrt{|S|-l}\min_{i\in S}\|\boldsymbol{x}[i]\|_2 \geq \sqrt{g(|S|-l)}\min_{i\in S}\|\boldsymbol{x}[i]\|_2. \qquad (3.26)
\end{aligned}
$$

Therefore, by Lemma 3.1, and through the simple calculation, the left-hand side of (3.25) becomes

$$
\begin{aligned}
&\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_{2,\infty}-\frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_2 \\
\geq\ &\frac{(1-\sqrt{\frac{g(|S|-l)}{N}+1}\delta_{NK+1})\|\boldsymbol{x}[S\setminus\Lambda^k]\|_2}{\sqrt{g(|S|-l)}} \overset{\text{(a)}}{\geq} \frac{(1-\sqrt{\frac{g(|S|)}{N}+1}\delta_{NK+1})\|\boldsymbol{x}[S\setminus\Lambda^k]\|_2}{\sqrt{g(|S|-l)}} \\
\overset{\text{(b)}}{\geq}\ &(1-\sqrt{\frac{g(|S|)}{N}+1}\delta_{NK+1})\min_{i\in S}\|\boldsymbol{x}[i]\|_2. \qquad (3.27)
\end{aligned}
$$

Here (a) comes from the fact that $g(t)$ is a nondecreasing function, and (b) follows from (3.26).

$i_0\in S\setminus\Lambda^k$ and $j_0\in W\subset S^c$ can be used to get that

$$
\begin{aligned}
&\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_{2,\infty}=\|\boldsymbol{\Phi}^T[i_0]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2, \\
&\|\boldsymbol{\Phi}^T[W]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_{2,\infty}=\|\boldsymbol{\Phi}^T[j_0]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2. \qquad (3.28)
\end{aligned}
$$

Then, the left-hand side of (3.25) becomes

$$
\begin{aligned}
&\|\boldsymbol{\Phi}^T[S\setminus\Lambda^k]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_{2,\infty}+\|\boldsymbol{\Phi}^T[W]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_{2,\infty} \\
=\ &\|\boldsymbol{\Phi}^T[i_0]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2+\|\Phi^T[j_0]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2 \\
=\ &\|\boldsymbol{\Phi}^T[i_0\cup j_0]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_{2,1}\overset{\text{(a)}}{\leq}\sqrt{2}\|\boldsymbol{\Phi}^T[i_0\cup j_0]\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2 \\
\overset{\text{(b)}}{\leq}\ &\sqrt{2(1+\delta_{NK+1})}\|\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2\overset{\text{(c)}}{\leq}\sqrt{2(1+\delta_{NK+1})}\epsilon. \qquad (3.29)
\end{aligned}
$$

Here the (a) and (b) are obtained from the Cauchy-Schwarz inequality and Lemma 2.2, respectively. (c) follows from

$$
\|\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2\leq\|\boldsymbol{P}^{\perp}[\Lambda^k]\|_2\|\boldsymbol{v}\|_2\leq\|\boldsymbol{v}\|_2\leq\epsilon.
$$

By combining (3.27) and (3.29), we can find that (3.25) is guaranteed by

$$
(1-\sqrt{\frac{g(|S|)}{N}+1}\delta_{NK+1})\min_{i\in S}\|\boldsymbol{x}[i]\|_2>\sqrt{2(1+\delta_{NK+1})}\epsilon. \qquad (3.30)
$$

From $s^{-2}-1<\frac{|S|}{N}$, we can get that $g(|S|)=(s^{-2}-1)N<|S|$. Then, it is easy to see that $\frac{1}{\sqrt{\frac{g(|S|)}{N}+1}}=\frac{1}{\sqrt{s^{-2}}}=s$.

Based on this assumption, it can be concluded that

$$
\delta_{NK+1}<\frac{1}{\sqrt{\frac{g(|S|)}{N}+1}}<\frac{\sqrt{2}}{2},
$$

and thus that

$$\min_{i\in S}\|\boldsymbol{x}[i]\|_2 > \frac{\sqrt{2(1+\delta_{NK+1})}\epsilon}{1-\sqrt{\frac{g(|S|)}{N}+1}\delta_{NK+1}}. \tag{3.31}$$

Therefore, if (3.12) holds, the BgOMP algorithm has selected at least one correct block index in the $(k+1)$-th iteration.

Next, it needs to be verified that the BgOMP algorithm selects all correct block indexes under the stop criterion of $\|\boldsymbol{r}^k\|_2 \le \epsilon$. This can be discussed in two situations.

When $S\backslash\Lambda^k = \varnothing$, that is, after k iterations, all correct block indexes are selected. According to formula (3.22), $\boldsymbol{r}^k = \boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}$, so that

$$\|\boldsymbol{r}^k\|_2 = \|\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2 \le \|\boldsymbol{P}^{\perp}[\Lambda^k]\|_2\|\boldsymbol{v}\|_2 \le \epsilon \tag{3.32}$$

When $S\setminus\Lambda^k \neq \varnothing$, that is, after k iterations, there are some correct block indexes that have not been selected. Similarly,

$$\begin{aligned}\|\boldsymbol{r}^k\|_2 &\ge \|\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_2 - \|\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{v}\|_2\\ &\ge \|\boldsymbol{P}^{\perp}[\Lambda^k]\boldsymbol{\Phi}[S\setminus\Lambda^k]\boldsymbol{x}[S\setminus\Lambda^k]\|_2 - \epsilon\\ &\ge \sqrt{1-\delta_{NK+1}}\|\boldsymbol{x}[S\setminus\Lambda^k]\|_2 - \epsilon\\ &> \sqrt{1-\delta_{NK+1}}\min_{i\in S}\|\boldsymbol{x}[i]\|_2 - \epsilon > \epsilon.\end{aligned} \tag{3.33}$$

Therefore, under the stop criterion, the BgOMP algorithm can stably reconstruct block σ-strongly-decaying K-sparse signals.

With the fourth step of the BgOMP algorithm, we can get that

$$\begin{aligned}\|\boldsymbol{x}-\hat{\boldsymbol{x}}\|_2 &\le \frac{1}{\sqrt{1-\delta_{NK+1}}}\|\boldsymbol{\Phi}(\boldsymbol{x}-\hat{\boldsymbol{x}})-\boldsymbol{y}+\boldsymbol{y}\|_2\\ &\le \frac{1}{\sqrt{1-\delta_{NK+1}}}\|\boldsymbol{\Phi}\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{y}-\boldsymbol{\Phi}\hat{\boldsymbol{x}}\|_2\\ &\le \frac{1}{\sqrt{1-\delta_{NK+1}}}(\|\boldsymbol{\Phi}\boldsymbol{x}-\boldsymbol{y}\|_2+\|\boldsymbol{y}-\boldsymbol{\Phi}\hat{\boldsymbol{x}}\|_2)\\ &\le \frac{1}{\sqrt{1-\delta_{NK+1}}}(\|\boldsymbol{v}\|_2+\|\boldsymbol{r}\|_2) \le \frac{2\epsilon}{\sqrt{1-\delta_{NK+1}}}.\end{aligned}$$

Hence, we can verify that Theorem 3.1 is accurate. □

Remark 3.5 In the special case of $N=1$, the conclusion of Theorem 3.1 will regress into the BOMP algorithm [16]. If $d=1$, this conclusion will degenerate into the relevant conclusions of the gOMP algorithm.

Our results show that the BgOMP algorithm recovery of the σ-strongly-decaying block K-sparse signal in the l_2-bounded noise is guaranteed if $\delta_{NK+1} < \frac{\sqrt{2}}{2}$. Inspired by the BOMP algorithm, it is then natural to ask the question: is the block-RIP condition sharp?

Conjecture 3.6 For any given positive integers $d \ge 1$, there is a σ-strongly-decaying block K-sparse signal $\boldsymbol{x}$ and a matrix $\boldsymbol{\Phi}$ satisfying that

$$\delta_{NK+1} = \frac{\sqrt{2}}{2} \tag{3.34}$$

such that the BgOMP algorithm fails to recover.

Next, we will provide a special counterexample to further explain the aforementioned conjecture.

We set that $N=2, K=4$ and $\mathbf{\Phi}(d) \in \mathbb{R}^{d(NK+1)\times d(NK+1)}$ for any positive integer d.

$$\mathbf{\Phi}^T(d)\mathbf{\Phi}(d) = \begin{pmatrix} \frac{1}{2}I_d & 0 & 0 & -\frac{1}{6}I_d & 0 & \cdots & 0 & -\frac{1}{6}I_d & -\frac{1}{6}I_d \\ 0 & \frac{1}{2}I_d & 0 & -\frac{1}{6}I_d & 0 & \cdots & 0 & -\frac{1}{6}I_d & -\frac{1}{6}I_d \\ 0 & 0 & \frac{1}{2}I_d & -\frac{1}{6}I_d & 0 & \cdots & 0 & -\frac{1}{6}I_d & -\frac{1}{6}I_d \\ -\frac{1}{6}I_d & -\frac{1}{6}I_d & -\frac{1}{6}I_d & \frac{3}{2}I_d & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & \frac{1}{2}I_d & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \frac{1}{2}I_d & 0 & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \frac{1}{2}I_d & 0 & 0 \\ -\frac{1}{6}I_d & -\frac{1}{6}I_d & -\frac{1}{6}I_d & 0 & 0 & \cdots & 0 & \frac{3}{2}I_d & 0 \\ -\frac{1}{6}I_d & -\frac{1}{6}I_d & -\frac{1}{6}I_d & 0 & 0 & \cdots & 0 & 0 & \frac{3}{2}I_d \end{pmatrix}.$$

By a simple calculation, we get that

$$|\mathbf{\Phi}^T(d)\mathbf{\Phi}(d) - \lambda I_{d(NK+1)}| = (\frac{1}{2}-\lambda)^{4d}(\frac{3}{2}-\lambda)^{3d}(\lambda^2 - 2\lambda + \frac{1}{2})^d.$$

Hence, it is clear that $\frac{1}{2}$, $\frac{3}{2}$, $1-\frac{\sqrt{2}}{2}$ and $1+\frac{\sqrt{2}}{2}$ are eigenvalues of $\mathbf{\Phi}^T(d)\mathbf{\Phi}(d)$. Thus, we easily claim that, for each $\boldsymbol{x} \in \mathbb{R}^{d(NK+1)}$,

$$(1-\frac{\sqrt{2}}{2})\|\boldsymbol{x}\|_2^2 \leq \boldsymbol{x}^T\mathbf{\Phi}^T\mathbf{\Phi}\boldsymbol{x} \leq (1+\frac{\sqrt{2}}{2})\|\boldsymbol{x}\|_2^2; \tag{3.35}$$

i.e.,

$$(1-\frac{\sqrt{2}}{2})\|\boldsymbol{x}\|_2^2 \leq \|\mathbf{\Phi}\boldsymbol{x}\|_2^2 \leq (1+\frac{\sqrt{2}}{2})\|\boldsymbol{x}\|_2^2.$$

It follows from the above inequality that we have that $\delta_{NK+1} \leq \frac{\sqrt{2}}{2}$. Next, we let $\boldsymbol{h} \in \mathbb{R}^{d(NK+1)}$ be the eigenvector of $\mathbf{\Phi}^T(1)\mathbf{\Phi}(1)$, corresponding to the eigenvalue $1+\frac{\sqrt{2}}{2}$, and $\boldsymbol{x} \in \mathbb{R}^{d(NK+1)}$ with $\boldsymbol{x}[i] = h_i\boldsymbol{e}_1$ ($\boldsymbol{e}_1 \in \mathbb{R}^d$ be the first coordinate unit vector) for $1 \leq i \leq NK+1$. Then we gain that

$$\boldsymbol{x}^T\mathbf{\Phi}^T\mathbf{\Phi}\boldsymbol{x} = \boldsymbol{h}^T\mathbf{\Phi}(1)\mathbf{\Phi}(1)\boldsymbol{h} = (1+\frac{\sqrt{2}}{2})\|\boldsymbol{h}\|_2^2 = (1+\frac{\sqrt{2}}{2})\|\boldsymbol{x}\|_2^2. \tag{3.36}$$

Therefore, $\mathbf{\Phi}$ satisfies the block-RIP with $\delta_{NK+1} = \frac{\sqrt{2}}{2}$.

Consider the σ-strongly-decaying block K-sparse signal

$$\boldsymbol{x}(d) = (\boldsymbol{e}_1^T, \sigma^{-1}\boldsymbol{e}_1^T, \sigma^{-2}\boldsymbol{e}_1^T, \sigma^{-3}\boldsymbol{e}_1^T, 0, \cdots 0)^T \in \mathbb{R}^{d(NK+1)};$$

i.e., $S = \mathrm{supp}(\boldsymbol{x}) = \{1,2,3,4\}$ and $\sigma = \frac{50}{41}$. For the first iteration, we have that

$$\|\mathbf{\Phi}^T[i]\boldsymbol{r}^0\|_2 = \|\mathbf{\Phi}^T[i]\mathbf{\Phi}\boldsymbol{x}\|_2 = \begin{cases} 0.4081, & i=1, \\ 0.3181, & i=2, \\ 0.2439, & i=3, \\ 0.4117, & i=4, \\ 0, & i \in \{K+1,\cdots,NK+1-N\}, \\ 0.4154, & i \in \{NK+2-N, NK+1\}. \end{cases} \tag{3.37}$$

Then it follows from the definitions of β_1^1 and α_N^1 that $\beta_1^1 = 0.4117$, $\alpha_N^1 = 0.4154$ and $\beta_1^1 < \alpha_N^1$. This indicates that if $N = 2, K = 4$, then the BgOMP algorithm may fail to recover the σ-strongly-decaying block K-sparse signal $\boldsymbol{x}$. Thus, it is reasonable to conjecture that the sufficient condition given in Theorem 3.1 may be sharp.

Due to the complexity of the BgOMP algorithm, the constructed matrix will be exceptionally complex for general N, and this is what we are committed to achieving in the future.

4 Numerical Simulation

This section provides some numerical simulation results to visually analyze the performance of the BgOMP algorithm.

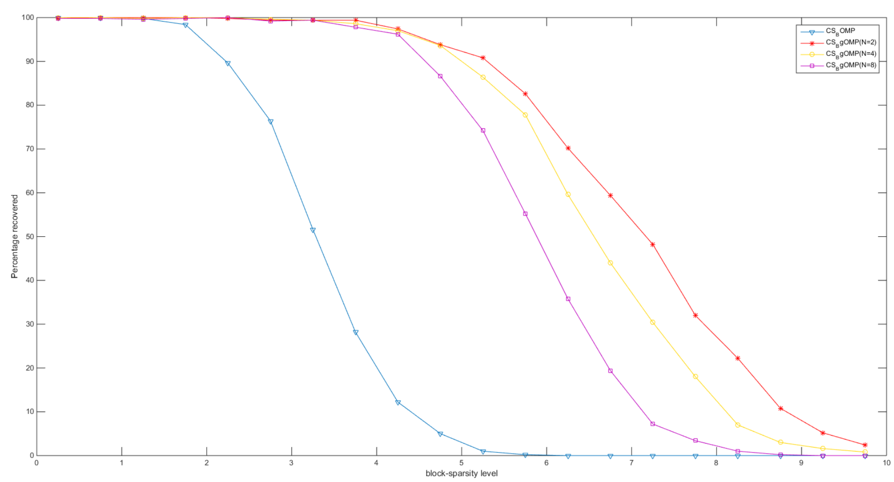

Figure 1 Performance of BgOMP and BOMP algorithms recovering a σ-strongly-decaying block K-sparse signal under the l_2-bounded noise with $d = 4, m = 80, n = 256, \sigma = 1.1$.

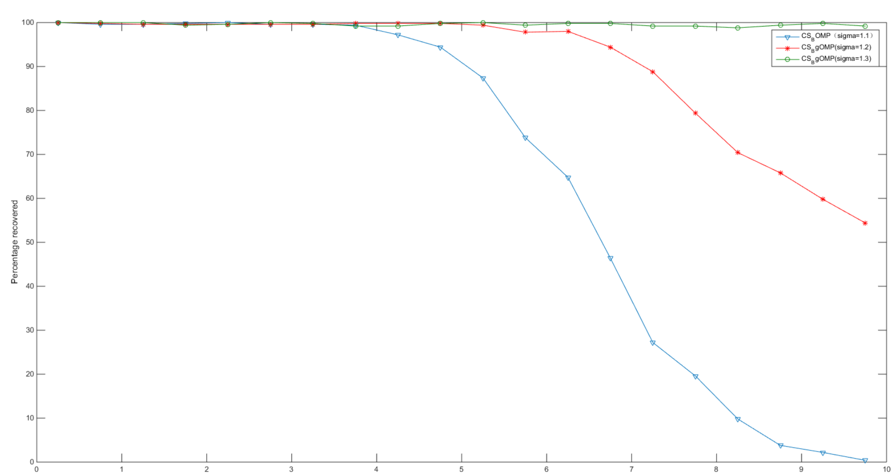

Figure 2 Performance of BgOMP ($N = 4$) algorithm recovering a σ-strongly-decaying block K-sparse signal under the l_2-bounded noise with $d = 4, m = 80, n = 256$.

Figure 1 shows that the BgOMP algorithm performs better in reconstructing a strongly-decaying signal compared to the BOMP algorithm. The different values of N can lead to differences in the performance of the BgOMP algorithm in recovering a strongly-decaying signal. When the block sparsity is $K > 4$, the larger the N, the worse the reconstruction performance of the BgOMP algorithm. Figure 2 shows that the greater the degree of strongly-decaying signal, the better the recovery performance of the BgOMP algorithm. Therefore, the BgOMP algorithm is better suited for recovering a high-dimensional strongly-decaying signal.

Conflict of Interest The authors declare no conflict of interest.

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Appendix Proof of Lemma 3.1

This proof includes two parts. In part one, we prove that

$$\|\boldsymbol{q}[\Lambda]\|_2^2 \leq \sqrt{\rho}\|\boldsymbol{x}[S\setminus\Lambda]\|_2\|\boldsymbol{\Phi}^T[S\setminus\Lambda]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda]\|_{2,\infty}. \tag{A.1}$$

In the second part, we prove that

$$\|\boldsymbol{q}[\Lambda]\|_2^2 - \sqrt{\frac{\rho}{N}}\|\boldsymbol{x}[S\setminus\Lambda]\|_2\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2 \geq (1-\sqrt{\frac{\rho}{N}+1}\delta_{NK+1})\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2. \tag{A.2}$$

First of all, let us prove (A.1). According to the idempotent property of orthogonal operators,

$$(\boldsymbol{P}^{\perp}[\Lambda])^T\boldsymbol{P}^{\perp}[\Lambda] = \boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{P}^{\perp}[\Lambda] = \boldsymbol{P}^{\perp}[\Lambda].$$

For each $i\in S\setminus\Lambda$, it is easy to observe that

$$\|\boldsymbol{\Phi}^T[S\setminus\Lambda]\boldsymbol{q}[\Lambda]\|_{2,\infty} \geq \|\boldsymbol{\Phi}^T[i]\boldsymbol{q}[\Lambda]\|_2.$$

In what follows, we have that

$$\begin{aligned}
&\sqrt{\rho}\|\boldsymbol{x}[S\setminus\Lambda]\|_2\|\boldsymbol{\Phi}^T[S\setminus\Lambda]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda]\|_{2,\infty}\\
&\geq \|\boldsymbol{x}[S\setminus\Lambda]\|_{2,1}\|\boldsymbol{\Phi}^T[S\setminus\Lambda]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda]\|_{2,\infty}\\
&= (\sum_{i\in[S\setminus\Lambda]}\|\boldsymbol{x}[i]\|_2)\|\boldsymbol{\Phi}^T[S\setminus\Lambda]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda]\|_{2,\infty}\\
&\geq \sum_{i\in[S\setminus\Lambda]}(\|\boldsymbol{x}[i]\|_2\|\boldsymbol{\Phi}^T[i]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda]\|_2)\\
&\geq \sum_{i\in[S\setminus\Lambda]}(\|\boldsymbol{x}^T[i]\boldsymbol{\Phi}^T[i]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda]\|_2)\\
&= \sum_{i\in[S\setminus\Lambda]}(\boldsymbol{x}^T[i]\boldsymbol{\Phi}^T[i](\boldsymbol{P}^{\perp}[\Lambda])^T\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda])\\
&= \|\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S\setminus\Lambda]\boldsymbol{x}[S\setminus\Lambda]\|_2^2 = \|\boldsymbol{q}[\Lambda]\|_2^2.
\end{aligned} \tag{A.3}$$

So far, the first part is fully verified. Then we define that

$$\alpha = -\frac{\sqrt{\frac{\rho}{N}+1}-1}{\sqrt{\frac{\rho}{N}}},$$

so that

$$\frac{2\alpha}{1-\alpha^2} = -\sqrt{\frac{\rho}{N}}, \frac{1+\alpha^2}{1-\alpha^2} = \sqrt{\frac{\rho}{N}+1}. \tag{A.4}$$

To avoid complex notation, let us define $\boldsymbol{e}\in\mathbb{R}^{Nd}$ as

$$\boldsymbol{e} = (e_{11}, e_{12}, \cdots, e_{1d}, e_{21}, \cdots, e_{2d}, \cdots, e_{N1}, e_{N2}\cdots, e_{Nd})^T.$$

According to the foregoing, $W \subset S^c$ when $1 \le j \le N, 1 \le i \le d$, so we have that

$$e_{ij} = \frac{(\boldsymbol{\Phi}^T[j])_i \boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S \setminus \Lambda]\boldsymbol{x}[S \setminus \Lambda]}{\|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S \setminus \Lambda]\boldsymbol{x}[S \setminus \Lambda]\|_2}. \tag{A.5}$$

Therefore, it is pretty obvious that

$$\boldsymbol{e}^T[j]\boldsymbol{\Phi}^T[j]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S \setminus \Lambda]\boldsymbol{x}[S \setminus \Lambda] = \|\boldsymbol{\Phi}^T[j]\boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S \setminus \Lambda]\boldsymbol{x}[S \setminus \Lambda]\|_2.$$

Expanding the above conclusions from local to global and combining them with (3.3), we can further obtain that

$$\boldsymbol{e}^T\boldsymbol{\Phi}^T[W]\boldsymbol{q}[\Lambda] = \sum_{j\in W} \|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2. \tag{A.6}$$

Subsequently, we get that

$$\boldsymbol{B} = \boldsymbol{P}^{\perp}[\Lambda]\left[\boldsymbol{\Phi}[S \setminus \Lambda] \quad \boldsymbol{\Phi}[W]\right]. \tag{A.7}$$

$$\begin{aligned} \boldsymbol{u} &= [\boldsymbol{x}[S \setminus \Lambda] \quad 0]^T \in \mathbb{R}^{(|S\setminus\Lambda|+N)d}, \\ \boldsymbol{w} &= [0 \quad \frac{\alpha\|\boldsymbol{x}[S \setminus \Lambda]\|_2\boldsymbol{e}}{\sqrt{N}}]^T \in \mathbb{R}^{(|S\setminus\Lambda|+N)d}. \end{aligned} \tag{A.8}$$

It is just a simple calculation to get that

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{P}^{\perp}[\Lambda]\boldsymbol{\Phi}[S \setminus \Lambda]\boldsymbol{x}[S \setminus \Lambda] = \boldsymbol{q}[\Lambda], \tag{A.9}$$

$$\begin{aligned} \|\boldsymbol{u}+\boldsymbol{w}\|_2^2 &= (1+\alpha^2)\|\boldsymbol{x}[S \setminus \Lambda]\|_2^2, \\ \|\alpha^2\boldsymbol{u}-\boldsymbol{w}\|_2^2 &= \alpha^2(1+\alpha^2)\|\boldsymbol{x}[S \setminus \Lambda]\|_2^2. \end{aligned} \tag{A.10}$$

In addition, we have that

$$\begin{aligned} \boldsymbol{w^T B^T B u} &\overset{(a)}{=} \frac{\alpha\|\boldsymbol{x}[S \setminus \Lambda]\|_2\boldsymbol{e}^T}{\sqrt{N}}\boldsymbol{\Phi}^T[W](\boldsymbol{P}^{\perp}[\Lambda])^T\boldsymbol{q}[\Lambda] \\ &= \frac{\alpha\|\boldsymbol{x}[S \setminus \Lambda]\|_2\boldsymbol{e}^T}{\sqrt{N}}\boldsymbol{\Phi}^T[W]\boldsymbol{q}[\Lambda] \\ &= \frac{\alpha\|\boldsymbol{x}[S \setminus \Lambda]\|_2}{\sqrt{N}}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2. \end{aligned} \tag{A.11}$$

Here (a) comes from the operation of the l_2-norm and inner product. Therefore, we have that

$$\begin{aligned} &\|\boldsymbol{B(u+w)}\|_2^2 - \|\boldsymbol{B}(\alpha^2\boldsymbol{u}-\boldsymbol{w})\|_2^2 \\ &= (1-\alpha^4)\|\boldsymbol{Bu}\|_2^2 + 2(1+\alpha^2)\boldsymbol{w^T B^T B u} \\ &= (1-\alpha^4)(\|\boldsymbol{Bu}\|_2^2 + \frac{2}{1-\alpha^2}\boldsymbol{w^T B^T B u}) \\ &= (1-\alpha^4)(\|\boldsymbol{Bu}\|_2^2 + \frac{2\alpha}{1-\alpha^2}\frac{\|\boldsymbol{x}[S \setminus \Lambda]\|_2}{\sqrt{N}}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2) \\ &= (1-\alpha^4)(\|\boldsymbol{Bu}\|_2^2 - \frac{\sqrt{\rho}}{N}\|\boldsymbol{x}[S \setminus \Lambda]\|_2\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2). \end{aligned} \tag{A.12}$$

On the flip side,

$$\begin{aligned} &\|\boldsymbol{B(u+w)}\|_2^2 - \|\boldsymbol{B}(\alpha^2\boldsymbol{u}-\boldsymbol{w})\|_2^2 \\ &\overset{(a)}{\ge} (1-\delta_{N(k+1)+|S|-l})\|\boldsymbol{u}+\boldsymbol{w}\|_2^2 - (1+\delta_{N(k+1)+|S|-l})\|\alpha^2\boldsymbol{u}-\boldsymbol{w}\|_2^2 \end{aligned}$$

$$
\begin{aligned}
&\overset{(b)}{=} (1-\delta_{N(k+1)+|S|-l})(1+\alpha^2)\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2-(1+\delta_{N(k+1)+|S|-l})\alpha^2(1+\alpha^2)\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2\\
&=(1+\alpha^2)\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2[(1-\delta_{N(k+1)+|S|-l})-\alpha^2(1+\delta_{N(k+1)+|S|-l})]\\
&=(1-\alpha^4)\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2(1-\frac{1+\alpha^2}{1-\alpha^2}\delta_{N(k+1)+|S|-l})\\
&\overset{(c)}{\geq} (1-\alpha^4)\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2(1-\frac{1+\alpha^2}{1-\alpha^2}\delta_{NK+1})\\
&=(1-\alpha^4)\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2(1-\sqrt{\frac{\rho}{N}+1}\delta_{NK+1}),
\end{aligned}
\tag{A.13}
$$

where (a) follows from Lemma 2.3 and (b) and (c) are from (A.10) and the inequality

$$
\begin{aligned}
|\Lambda\cup(S\setminus\Lambda)\cup W| &\leq Nk+|S|-l+N\leq(N-1)k+|S|+N\\
&\leq(N-1)(|S|-1)+|S|+N=N|S|+1\leq NK+1.
\end{aligned}
$$

Then, by (A.9), (A.12), (A.13) and the fact that $1-\alpha^4>0$, we get that

$$
\|\boldsymbol{q}[\Lambda]\|_2^2-\frac{\sqrt{\rho}}{N}\|\boldsymbol{x}[S\setminus\Lambda]\|_2\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2\geq\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2(1-\sqrt{\frac{\rho}{N}+1}\delta_{NK+1}).
$$

Next, the beginning of the expression is replaced with (A.3), so that

$$
\begin{aligned}
&\sqrt{\rho}\|\boldsymbol{x}[S\setminus\Lambda]\|_2\|\boldsymbol{\Phi}^T[S\setminus\Lambda]\boldsymbol{q}[\Lambda]\|_{2,\infty}-\frac{\sqrt{\rho}}{N}\|\boldsymbol{x}[S\setminus\Lambda]\|_2\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2\\
&\geq\|\boldsymbol{x}[S\setminus\Lambda]\|_2^2(1-\sqrt{\frac{\rho}{N}+1}\delta_{NK+1}).
\end{aligned}
$$

Removing $\sqrt{\rho}\|\boldsymbol{x}[S\setminus\Lambda]\|_2$ from both sides of the inequality simultaneously, we have that

$$
\|\boldsymbol{\Phi}^T[S\setminus\Lambda]\boldsymbol{q}[\Lambda]\|_{2,\infty}-\frac{1}{N}\sum_{j\in W}\|\boldsymbol{\Phi}^T[j]\boldsymbol{q}[\Lambda]\|_2\geq\frac{(1-\sqrt{\frac{\rho}{N}+1}\delta_{NK+1})\|\boldsymbol{x}[S\setminus\Lambda]\|_2}{\sqrt{\rho}}.
$$

In summary, the narrative of Lemma 3.1 is accurate.