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VARIATIONAL ANALYSIS FOR THE MAXIMAL TIME FUNCTION IN NORMED SPACES*

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Abstract For a general normed vector space, a special optimal value function called a maximal time function is considered. This covers the farthest distance function as a special case, and has a close relationship with the smallest enclosing ball problem. Some properties of the maximal time function are proven, including the convexity, the lower semicontinuity, and the exact characterizations of its subdifferential formulas.

Key words maximal time function; subdifferential; normal cone; nonsmooth analysis

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1 Introduction

The maximal time problem is widely used in machine learning and support vector machines. The optimal value function of the maximal time problem is called the maximal time function. Let X be a normed space and let $Q \subset X$ be a nonempty, bounded and closed set. The maximal time function $C_{Q|K}$ for the point x to reach the target set Q with the constant dynamic K is defined by

$$C_{Q|K}(x) := \inf \{t \geq 0 : \ Q \subset x + tK\}, \quad \text{for all } x \in X. \tag{1.1}$$

When K is the closed unit ball of X, then the maximal time function (1.1) reduces to the corresponding farthest distance function as follows:

$$M_Q(x) := \sup \{\|x - \omega\| : \omega \in Q\}.$$

There is an essential difference between $C_{Q|K}(x)$ and $M_Q(x)$: the set K defining $C_{Q|K}(x)$ is possibly asymmetric, while the unit ball is always symmetric. The properties of the farthest distance function can be found in References [1–4].

The maximal time function has a close relationship with the smallest enclosing ball problem; it asks for the smallest ball that encloses all of the given balls. Mordukhovich and Nam *et al.*

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[5] proved that the smallest enclosing ball problem can be modeled with maximal time function (1.1) as the following optimization problem:

$$\text{minimize } C(x) \text{ subject to } x \in \Omega.$$

Here $\Omega \subset X$ is a nonempty closed constraint set and

$$C(x) := \max\left\{C_{Q_i|K}(x) : i = 1, \cdots, n\right\}.$$

Hence, it is a significant endeavour to study the properties of the maximal time function for the smallest enclosing ball problems.

Similar to the maximal time function, the minimal time function is defined as

$$T_{Q|K}(x) := \inf\{t \geq 0 : (x + tK) \cap Q \neq \emptyset\};$$

this signifies the minimal time for the point x to reach the target set Q following the dynamic K. General and generalized differentiation properties of the minimal time function have been studied extensively; see, e.g., [6–11] and the references therein. In [6–8], the proximal and the Fréchet subdifferentials of the minimal time function in which K is a bounded, closed and convex set that contains the origin as an interior point were considered. Further extensions to the case, where the origin is not necessarily an interior point of K, were considered in [10]. Without the calmness, subgradients of the minimal time function were obtained in [11, 12]. Moreover, the subdifferential and some other properties regarding the minimal time function with K being unbounded were presented in [9], and the minimal time function associated with a collection of sets were considered in [13].

The minimal time function is the optimal value function of time optimal control problems [14]. Hence, the known results regarding the minimal time function $T_{Q|K}(\cdot)$ are based on time optimal control theory. However, the maximal time function $C_{Q|K}(\cdot)$ does not have this advantages. Until now, there have not been many studies done on the maximal time function. We try to draw some relevant conclusions regarding the maximal time function. In this paper, we obtain the convexity and the lower semicontinuity of the maximal time function and prove that the subdifferential of $C_{Q|K}(\cdot)$ can be characterized in terms of the corresponding normal cones of an enlarged set of Q and the support function of K.

The rest of this paper is organized as follows: In Section 2, we present some related definitions and preliminaries widely used in the sequel. In Section 3, we give our main results about some general properties of the maximal time function. Finally, in Section 4, we establish estimates for the subdifferential of the maximal time function.

2 Preliminary

Let X be a real normed vector space with the norm denoted by $\|\cdot\|$, and let X^* denote the topological dual of X. The canonical paring $\langle\cdot,\cdot\rangle$ is between X^* and X. Suppose that $f : X \to \bar{\mathbb{R}}$ is an extended real-valued function and $\bar{x} \in \operatorname{dom} f := \{x \in X : f(x) < \infty\}$. Now, we recall some definitions and notations, most of them were derived from [15].

• The function f is convex on a convex set Ω iff, for every x, $y \in \Omega$ and $\lambda \in [0, 1]$, one has that

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

• The function f is said to be lower semicontinuous at $\bar{x}$ iff, for any sequence $\{x_n\}$ that converges to $\bar{x}$, one has that

$$\liminf_{n\to\infty} f(x_n) \geq f(\bar{x}).$$

The function f is called lower semicontinuous iff it is lower semicontinuous at every point of its domain.

• The function f is said to be ℓ-Lipschitz continuous iff, for a given constant $\ell \geq 0$, one has that

$$|f(x) - f(y)| \leq \ell \|x - y\|, \quad \text{for all } x,\ y \in X.$$

• The subdifferential of convex function f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \{\xi \in X^* : \langle \xi, y - \bar{x}\rangle \leq f(y) - f(\bar{x}), \ \text{for all } y \in X\}.$$

• The normal cone of a convex set $\Omega \subset X$ at $x \in \Omega$ is the set

$$N_\Omega(x) := \{\xi \in X^* : \langle \xi, y - x\rangle \leq 0, \ \text{for all } y \in \Omega\}.$$

• The support function of a set $K \subset X$ is defined by

$$\Im_K(\xi) := \sup_{x\in K} \langle \xi, x\rangle.$$

• The asymptotic cone of K at $x \in K$ is defined by

$$K_\infty(x) := \{d \in X :\ x + td \in K, \ \text{for all } t > 0\},$$

where $K \subset X$ is a closed and convex set. An equivalent representation of $K_\infty(x)$ is

$$K_\infty(x) = \bigcap_{t>0} \frac{K - x}{t}.$$

This shows that $K_\infty(x)$ is a cone that contains the origin. Moreover, $K_\infty(x)$ is closed and convex because K is a closed and convex set, and the intersection of closed (convex) sets is closed (convex).

• The Minkowski function generated by K is given by

$$\rho_K(x) := \inf\{t \geq 0 : x \in tK, \ \text{for all } \ x \in X\},$$

where $K \subset X$ is a convex set and $0 \in K$.

The main results of this paper are based on the following results of [5, 9, 15]:

Proposition 2.1 ([15]) The Minkowski function $\rho_K(\cdot)$ is a positively homogenous and subadditive extended real-valued function. Suppose, further, that $0 \in \operatorname{int} K$. Define that

$$\ell := \inf\left\{\frac{1}{r} : B(0;\ r) \subseteq K,\ r > 0\right\}.$$

Then $\rho_K(\cdot)$ is an ℓ-Lipschitz function. In particular, $\rho_K(x) \leq \ell \|x\|$ for all $x \in X$.

Proposition 2.2 ([5]) Suppose that $K \subset X$ is convex and that $0 \in \operatorname{int} K$. Then the maximal time function (1.1) has the following representation:

$$C_{Q|K}(x) = \sup\{\rho_K(\omega - x) : \omega \in Q\}.$$

Moreover, if K is the closed unit ball of X, then

$$C_{Q|K}(x) = \sup\{\|x - \omega)\| : \omega \in Q\}.$$

Proposition 2.3 ([9]) Suppose that K is a closed convex set. Then, for all $x_1,\ x_2 \in K$, one has that

$$K_\infty(x_1) = K_\infty(x_2);$$

that is, the asymptotic cone does not depend on $x \in K$. Thus, the asymptotic cone is denoted as K_∞, for simplicity.

Proposition 2.4 ([9]) Suppose that K is a closed convex set. Then the following are equivalent:

(a) $d \in K_\infty$;

(b) there exists a sequence $\{t_n\} \subseteq [0, \infty[$ such that $t_n \to 0$, and a sequence $\{k_n\} \subseteq K$ with $t_n k_n \to d$.

Proposition 2.5 ([9]) If K is a closed convex set and contains the origin, then

$$K_\infty = \bigcap_{t>0} tK.$$

3 General Properties of the Maximal Time Function

In this section, we study the general properties of maximal time function (1.1), including the convexity, the lower semicontinuity and the Lipschitz continuity. These properties are of independent interest.

3.1 Convexity of the Maximal Time Function

Theorem 3.1 If K is a nonempty convex set, then the maximal time function $C_{Q|K}(\cdot)$ is a convex function.

Proof Let $x,\ y$ be in the domain of the maximal time function, and let $0 \le \lambda \le 1$. Denote $x_\lambda := \lambda x + (1-\lambda)\, y$, $r_1 := C_{Q|K}(x)$, and $r_2 := C_{Q|K}(y)$. From the definition of the maximal time function, we have, for any $\varepsilon > 0$, that there exists, t_i $(i = 1, 2)$ with

$$r_i \le t_i < r_i + \varepsilon,\ Q \subset x + t_1 K \ \text{ and } \ Q \subset y + t_2 K.$$

Then, for any $\omega \in Q$, there exist k_1 and $k_2 \in K$ such that

$$\omega = x + t_1 k_1 = y + t_2 k_2.$$

Since K is convex, one has that

$$\begin{aligned}\omega &= \lambda\omega + (1-\lambda)\,\omega = \lambda\,(x + t_1 k_1) + (1-\lambda)\,(y + t_2 k_2)\\ &= \lambda x + \lambda t_1 k_1 + (1-\lambda)\, y + (1-\lambda)\, t_2 k_2 = x_\lambda + \lambda t_1 k_1 + (1-\lambda)\, t_2 k_2\\ &\in x_\lambda + \lambda t_1 K + (1-\lambda)\, t_2 K \subset x_\lambda + [\lambda t_1 + (1-\lambda)\, t_2]\, K.\end{aligned}$$

It follows that $Q \subset (x_\lambda + [\lambda t_1 + (1-\lambda)\, t_2]\, K)$, since ω is arbitrary. By the definition of $C_{Q|K}(x_\lambda)$, one has that

$$C_{Q|K}(x_\lambda) \le \lambda t_1 + (1-\lambda)\, t_2 \le \lambda r_1 + (1-\lambda)\, r_2 + \varepsilon.$$

As $\varepsilon \to 0^+$, we have that

$$C_{Q|K}(x_\lambda) \le \lambda C_{Q|K}(x) + (1-\lambda)\, C_{Q|K}(y)\,.$$

The proof of Theorem 3.1 is now complete. □

3.2 Lower Semicontinuity of the Maximal Time Function

Theorem 3.2 If K is a nonempty, bounded and weakly closed set, then the function $C_{Q|K}$ is lower semicontinuous.

Proof For any $x \in \operatorname{dom} C_{Q|K}$ and a sequence $\{x_n\}$ that converges to x, we will prove that

$$\liminf_{n\to\infty} C_{Q|K}(x_n) \geq C_{Q|K}(x).$$

The inequality holds clearly if $\liminf\limits_{n\to\infty} C_{Q|K}(x_n) = \infty$. Therefore we only need to consider the case where $\liminf\limits_{n\to\infty} C_{Q|K}(x_n) = \gamma \in [0,\infty)$, and to show that $\gamma \geq C_{Q|K}(x)$. Without loss of generality, by the definition of maximal time function (1.1), there exists a sequence $\{t_n\} \subseteq [0,\infty)$ such that

$$C_{Q|K}(x_n) \leq t_n < C_{Q|K}(x_n) + \frac{1}{n} \quad \text{and} \quad Q \subset (x_n + t_n K), \quad \text{for all } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, one has that

$$\omega \in x_n + t_n K, \quad \text{for all } \omega \in Q. \tag{3.1}$$

Now, consider two cases: $\gamma > 0$ and $\gamma = 0$.

If $\gamma > 0$, then $t_n > 0$. Since K is a bounded and weakly closed set, one has that

$$\frac{\omega - x_n}{t_n} \xrightarrow{\text{weakly}} \frac{\omega - x}{\gamma} \in K.$$

It follows from the arbitrary of ω that $Q \subset (x + \gamma K)$. Hence $C_{Q|K}(x) \leq \gamma$.

If $\gamma = 0$, then $t_n \to 0$ as $n \to \infty$. From (3.1), we can see that, for any $w \in Q$, there exists $f_w^n \in K$ such that

$$w = x_n + t_n f_w^n, \quad \text{for all } n \in \mathbb{N}.$$

As $n \to \infty$, this implies that $w = x$; that is, $Q = \{x\}$. Hence, $Q \subset x + \frac{1}{n}K$. This implies that $0 \leq C_{Q|K}(x) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we can get that $C_{Q|K}(x) = 0$. Therefore, $C_{Q|K}(x) \leq \gamma$. □

Without the assumption that K is bounded, we can also show that $C_{Q|K}$ is lower semicontinuous. Let us start with the following theorem.

Theorem 3.3 Suppose that K is a nonempty, closed, convex set that contains the origin. Then

$$C_{Q|K}(x) = 0 \text{ if and only if } Q \subset x + K_\infty.$$

Proof Assuming that $C_{Q|K}(x) = 0$, for every $n \in \mathbb{N}$, there exists $t_n \geq 0$ such that

$$Q \subset x + t_n K \text{ and } \lim_{n\to\infty} t_n = 0.$$

Hence, we can find $k_n \in K$ for every $n \in \mathbb{N}$ and $\omega \in Q$ with $t_n k_n = \omega - x$. Therefore,

$$t_n k_n \to \omega - x \text{ as } n \to \infty.$$

It follows from Proposition 2.4 that $\omega \in x + K_\infty$. Hence, $Q \subset x + K_\infty$, by the arbitrariness of ω.

Conversely, for any x satisfying $Q \subset x + K_\infty$, there exists $d \in K_\infty$ such that

$$\omega = x + d, \quad \text{for all } \omega \in Q.$$

Since $0 \in K$ and $d \in K_\infty$, Proposition 2.3 and the definition of K_∞ imply that $n(\omega - x) = nd \in K$; that is

$$\omega - x \in \frac{1}{n}K, \text{ for all } n \in \mathbb{N}.$$

Hence, $Q \subset x + \frac{1}{n}K$, by the arbitrariness of ω. This implies that $0 \le C_{Q|K}(x) \le \frac{1}{n}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we can get that $C_{Q|K}(x) = 0$. The proof of theorem is complete. □

Theorem 3.4 Suppose that K is a nonempty, closed, convex set that contains the origin. Then the function $C_{Q|K}(\cdot)$ is lower semicontinuous.

Proof For any $x \in \operatorname{dom} C_{Q|K}$ and a sequence $\{x_n\}$ that converges to x, we will prove that

$$\liminf_{n\to\infty} C_{Q|K}(x_n) \ge C_{Q|K}(x).$$

It is obvious that the inequality holds when $\liminf_{n\to\infty} C_{Q|K}(x_n) = \infty$. Thus we only need to consider the case where $\liminf_{n\to\infty} C_{Q|K}(x_n) = \gamma \in [0,\infty)$ and $r \ge C_{Q|K}(x)$. It follows from the definition of maximal time function (1.1) that there exists a sequence $\{t_n\} \subseteq [0,\infty)$ such that

$$C_{Q|K}(x_n) \le t_n < C_{Q|K}(x_n) + \frac{1}{n} \quad \text{and} \quad Q \subset (x_n + t_n K), \quad \text{for all } n \in \mathbb{N}.$$

Therefore, for every $n \in \mathbb{N}$, one has that

$$\omega \in x_n + t_n K, \quad \text{for all } \omega \in Q.$$

Consider two cases: $\gamma > 0$ and $\gamma = 0$. First, for the case $\gamma > 0$ and closed K, we can obtain that

$$\frac{\omega - x_n}{t_n} \to \frac{\omega - x}{\gamma} \in K \text{ as } n \longrightarrow \infty.$$

The arbitrariness of ω implies that $Q \subset (x + \gamma K)$. Hence, $C_{Q|K}(x) \le \gamma$.

Now, consider the case where $\gamma = 0$. For every $n \in \mathbb{N}$, there exists $k_n \in K$ such that

$$\omega = x_n + t_n k_n, \quad \text{for all } \omega \in Q.$$

In this case, the sequence $\{t_n\}$ converges to 0, and

$$t_n k_n \to \omega - x, \text{ as } n \longrightarrow \infty.$$

It follows from Proposition 2.4 that $\omega - x \in K_\infty$. This implies that $Q \subset x + K_\infty$, since ω is arbitrary. Employing Theorem 3.3, we have that $C_{Q|K}(x) = 0$. Hence, $\gamma \ge C_{Q|K}(x)$. The proof of Theorem 3.4 is complete. □

Remark 3.5 Theorems 3.2 and 3.4 show that the maximal time function is lower semicontinuous. The assumptions of K are absolutely different.

3.3 Lipschitz Continuity of the Maximal Time Function

Theorem 3.6 Let ℓ be defined as in Proposition 2.1. Suppose that K is nonempty, convex set, and that $0 \in \operatorname{int} K$. Then the maximal time function $C_{Q|K}(\cdot)$ is ℓ-Lipschitz.

Proof Let $x,\ y \in X$ and $n \in \mathbb{N}$. It follows from Proposition 2.2 that there exists $x_n \in Q$ such that

$$\rho_K(x_n - x) > C_{Q|K}(x) - \frac{1}{n} \tag{3.2}$$

and

$$C_{Q|K}(y) \geq \rho_K(x_n - y). \tag{3.3}$$

Applying (3.2) and (3.3), we have that

$$C_{Q|K}(x) - C_{Q|K}(y) \leq \rho_K(x_n - x) + \frac{1}{n} - \rho_K(x_n - y) \leq \rho_K(y - x) + \frac{1}{n}.$$

This, together with Proposition 2.1, yields that

$$\left|C_{Q|K}(x) - C_{Q|K}(y)\right| \leq \left|\rho_K(y - x) + \frac{1}{n}\right| \leq \ell \|y - x\| + \frac{1}{n} = \ell \|x - y\| + \frac{1}{n}.$$

The conclusion is verified by letting $n \to \infty$. $\square$

Remark 3.7 We should point out that the Lipschitz continuity of the maximal time function were proven in Reference [5]. Here, we show it in a different and more detailed way.

4 Subdifferentials of the Maximal Time Function

In this section, we discuss the properties of the subdifferentials of the maximal time function where K is convex. In this case, the maximal time function is also convex, by Theorem 3.1. Now we show that subdifferential of the maximal time function $C_{Q|K}(\cdot)$ can be described by corresponding notions of normal cones of sublevel sets of $C_{Q|K}(\cdot)$, and the support function of K.

For $r > 0$, the r-sublevel set of $C_{Q|K}(\cdot)$ is defined as follows:

$$Q_r := \left\{x \in X : \ C_{Q|K}(x) \leq r\right\}.$$

If K is a nonempty and convex set, it is easy to see that the r-sublevel set Q_r is a convex set. Now we show the inequality between the maximal time function and its r-sublevel set.

Proposition 4.1 Suppose that $r > 0$, that K is a nonempty, convex set, and that $C_{Q|K}(x) < \infty$. Then

$$C_{Q|K}(x) \leq C_{Q_r|K}(x) + r.$$

Proof According to the definition of $C_{Q_r|K}(x)$, we can see that, for any $\varepsilon > 0$, there exists $t \in \left[C_{Q_r|K}(x), C_{Q_r|K}(x) + \varepsilon\right)$ such that

$$Q_r \subset (x + tK).$$

For any $u \in Q_r$, there exists $k \in K$ such that

$$u = x + tk. \tag{4.1}$$

Since $u \in Q_r$, we can see that $C_{Q|K}(u) \leq r$. Therefore, there exists $s \in [r, r + \varepsilon)$ such that $Q \subset (u + sK)$. Then we can find $k' \in K$ with

$$\omega = u + sk', \quad \text{for all } \ \omega \in Q.$$

It follows from (4.1) and the convexity of K that

$$\omega = u + sk' = x + tk + sk' \in (x + tK + sK) \subset (x + (t + s)K).$$

Hence, $Q \subset x + (t + s)K$, thanks to the arbitrariness of ω. Using the definition of $C_{Q|K}(x)$, one has that

$$C_{Q|K}(x) \leq t + s \leq C_{Q_r|K}(x) + r + 2\varepsilon.$$

The conclusion is verified by letting $\varepsilon \to 0^+$. □

To show our main results about the subdifferentials of the maximal time function, we start with the following lemma:

Lemma 4.2 Suppose that K is a nonempty and convex set. Then, for $t \geq 0$ and any $k \in K$, one has that

$$C_{Q|K}(x - tk) - t \leq C_{Q|K}(x) \leq C_{Q|K}(x + tk) + t.$$

Proof For $x \in X,\ t \geq 0$, and $k \in K$, we first prove the inequality $C_{Q|K}(x - tk) - t \leq C_{Q|K}(x)$. By the definition of maximal time function (1.1), we can see that, for any $\varepsilon > 0$, there exists $t' \geq 0$ such that

$$C_{Q|K}(x) \leq t' < C_{Q|K}(x) + \varepsilon \ \text{ and } \ Q \subset (x + t'K).$$

Then, for any $t \geq 0$ and $k \in K$, the convexity of K implies that

$$Q \subset (x - tk + tk + t'K) \subset (x - tk + (t + t')K).$$

Therefore,

$$C_{Q|K}(x - tK) \leq t + t' \leq t + C_{Q|K}(x) + \varepsilon.$$

The conclusion follows by letting $\varepsilon \to 0^+$.

Now we prove the second inequality. Applying the first inequality, one can easily get that

$$C_{Q|K}(x) = C_{Q|K}(x + tk - tk) \leq C_{Q|K}(x + tk) + t.$$

This finishes the proof. □

Lemma 4.3 If K is a nonempty, affine set, and $\alpha > \beta > 0$, then $\alpha K - \beta K \subset (\alpha - \beta)K$.

Proof For any $k_1, k_2 \in K$, one can see that

$$\frac{\alpha}{\alpha - \beta}k_1 + (1 - \frac{\alpha}{\alpha - \beta})k_2 \in K.$$

Hence, there exists $k_3 \in K$ such that

$$\alpha k_1 - \beta k_2 = (\alpha - \beta)k_3 \in (\alpha - \beta)K.$$

Then the arbitrariness of k_1 and k_2 imply that Lemma 4.3 holds. □

Theorem 4.4 Let $r := C_{Q|K}(x) < \infty$. Suppose that K is a nonempty and affine set. Then

$$\partial C_{Q|K}(x) = N_{Q_r}(x) \cap \{\xi \in X^* : \Im_K(-\xi) = 1\},$$

where $\Im_K(\xi) := \sup\limits_{x \in K} \langle \xi, x \rangle$.

Proof Since K is affine, one can see that K is convex. Theorem 3.1 implies that the maximal time function $C_{Q|K}(x)$ is a convex function. For any $\xi \in \partial C_{Q|K}(x)$, the definition of Clark subdifferential implies that

$$\langle \xi, y - x \rangle \leq C_{Q|K}(y) - C_{Q|K}(x), \ \text{ for all } \ y \in X. \tag{4.2}$$

Since $C_{Q|K}(y) \leq r$ on Q_r, this implies that

$$\langle \xi, y - x \rangle \leq C_{Q|K}(y) - C_{Q|K}(x) = C_{Q|K}(y) - r \leq 0, \ \text{ for all } \ y \in Q_r. \tag{4.3}$$

Thus, $\xi \in N_{Q_r}(x)$. Now, let us show that $\Im_K(-\xi) = 1$.

For any $k \in K$, $t > 0$ and $y := x - tk$, it follows from (4.2) and Lemma 4.2 that

$$\langle \xi, y - x \rangle = \langle \xi, (x - tk) - x \rangle \leq C_{Q|K}(x - tk) - C_{Q|K}(x) \leq t. \tag{4.4}$$

Dividing both sides of (4.4) by t, we can obtain that $\langle \xi, -k \rangle \leq 1$ for all $k \in K$. This implies that

$$\Im_K(-\xi) \leq 1. \tag{4.5}$$

The definition of $r := C_{Q|K}(x)$ implies that, for any $\varepsilon \in (0, r)$, there exists $t > 0$ such that

$$t \in \left[r, r + \varepsilon^2\right) \text{ and } Q \subset (x + tK). \tag{4.6}$$

From (4.6) and Lemma 4.3, it follows that

$$Q \subset (x + tK) \subset (x + \varepsilon k + tK - \varepsilon k) \subset x + \varepsilon k + (t - \varepsilon) K.$$

Hence, $C_{Q|K}(x + \varepsilon k) \leq t - \varepsilon$. Applying (4.2) and (4.6), for $y = x + \varepsilon k$, one has

$$\langle \xi, y - x \rangle = \langle \xi, x + \varepsilon k - x \rangle \leq C_{Q|K}(x + \varepsilon k) - C_{Q|K}(x) \leq t - \varepsilon - r \leq \varepsilon^2 - \varepsilon.$$

This yields that $\Im_K(-\xi) \geq 1$, by letting $\varepsilon \to 0^+$.

Therefore, from (4.3) and (4.5), we can get that $\partial C_{Q|K}(x) \subset N_{Q_r}(x) \cap \{\xi \in X^* : \Im_K(-\xi) = 1\}$.

Now we show that $N_{Q_r}(x) \cap \{\xi \in X^* : \Im_K(-\xi) = 1\} \subset \partial C_{Q|K}(x)$. For any $\xi \in N_{Q_r}(x)$ satisfying that $\Im_K(-\xi) = 1$, one has that

$$\langle \xi, y - x \rangle \leq 0, \quad \text{for all } y \in Q_r. \tag{4.7}$$

For any $\varepsilon > 0$, there exists $k \in K$ such that

$$\langle \xi, -k \rangle > 1 - \varepsilon. \tag{4.8}$$

This proves that $\xi \in \partial C_{Q|K}(x)$; that is,

$$\langle \xi, y - x \rangle \leq C_{Q|K}(y) - C_{Q|K}(x), \quad \text{for all } y \in X.$$

Let $q := C_{Q|K}(y)$. The discussion that follows is divided into three cases.

(i) If $y \in X$ and $q = r$, then $y \in Q_r$. From (4.7), we have

$$\langle \xi, y - x \rangle \leq 0 = C_{Q|K}(y) - C_{Q|K}(x).$$

This implies that $\xi \in \partial C_{Q|K}(x)$.

(ii) If $y \in X$ and $q < r$, then it follows from Lemma 4.2 that

$$C_{Q|K}(y - (r - q) k) \leq C_{Q|K}(y) + r - q = r.$$

Then $y - (r - q) k \in Q_r$. From (4.7), one can see that

$$\langle \xi, y - (r - q) k - x \rangle \leq 0.$$

This, together with (4.8), implies that

$$\langle \xi, y - x \rangle \leq \langle \xi, k \rangle (r - q) \leq (1 - \varepsilon)(q - r) = (1 - \varepsilon)\left(C_{Q|K}(y) - C_{Q|K}(x)\right).$$

Letting $\varepsilon \to 0^+$, one can get that $\xi \in \partial C_{Q|K}(x)$.

(iii) If $y \in X$ and $q > r$, then, by the definition of $C_{Q|K}(y)$, for any $\varepsilon \in (0,\ q - r)$, there exists q_ε such that

$$q_\varepsilon \in [q, q + \varepsilon) \text{ and } Q \subset (y + q_\varepsilon K). \tag{4.9}$$

Fix any $k \in K$ and denote that $z := y + (q_\varepsilon - r)\, k$. Lemma 4.3 implies that

$$Q \subset (y + q_\varepsilon K) \subset [y + (q_\varepsilon - r)\, k + q_\varepsilon K - (q_\varepsilon - r)\, k] \subset z + rK.$$

Thus $C_{Q|K}(z) \le r$. This verifies $z \in Q_r$. It follows from (4.7) that

$$\langle \xi, z - x \rangle \le 0. \tag{4.10}$$

From (4.7), (4.9) and (4.10), we have that

$$\begin{aligned} C_{Q|K}(y) - C_{Q|K}(x) - \langle \xi, y - x \rangle &= q - r - \langle \xi, y - z \rangle - \langle \xi, z - x \rangle \\ &= q - r + (r - q_\varepsilon)\langle -\xi, k \rangle - \langle \xi, z - x \rangle \\ &\ge q - r + r - q_\varepsilon \\ &\ge -\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain the desired conclusion: $\xi \in \partial C_{Q|K}(x)$. Therefore, the proof of Theorem 4.4 is complete. □

Define $\bar{Q}$ as follows:

$$\bar{Q} := \{x \in X : Q \subset x + K_\infty\}.$$

By Theorem 3.3, one can see that if $x \in \bar{Q}$, then $C_{Q|K}(x) = 0$. Now, we establish a subdifferential estimate of $x \in \bar{Q}$.

Theorem 4.5 Let $x \in \bar{Q}$. Suppose that K is a nonempty, affine set that contains the origin. Then

$$\partial C_{Q|K}(x) \subset N_{\bar{Q}}(x) \cap \{\xi \in X^* : \Im_K(-\xi) = 1\},$$

where $\Im_K(\xi) := \sup\limits_{x \in K} \langle \xi, x \rangle$.

Proof For any $\xi \in \partial C_{Q|K}(x)$, the definition of the Clark subdifferential implies that

$$\langle \xi, y - x \rangle \le C_{Q|K}(y) - C_{Q|K}(x), \quad \text{for all } \; y \in X. \tag{4.11}$$

Since $C_{Q|K}(x) = 0$ on $\bar{Q}$, this implies that

$$\langle \xi, y - x \rangle \le C_{Q|K}(y) - C_{Q|K}(x) = 0, \quad \text{for all } \; y \in \bar{Q}. \tag{4.12}$$

Thus $\xi \in N_{\bar{Q}}(x)$. Now, let us show that $\Im_K(-\xi) = 1$.

For any $k \in K$, $t > 0$ and $y := x - tk$, it follows from (4.2) and Lemma 4.2 that

$$\langle \xi, y - x \rangle = \langle \xi, (x - tk) - x \rangle \le C_{Q|K}(x - tk) - C_{Q|K}(x) \le t.$$

Dividing both sides of the above inequality by t, we can obtain that $\langle \xi, -k \rangle \le 1$ for all $k \in K$. This implies that

$$\Im_K(-\xi) \le 1.$$

One the other hand, the definition of $C_{Q|K}(x) = 0$ implies that, for any $\sigma > 0$, there exists $t > 0$ such that

$$0 < t < \sigma \; \text{ and } \; Q \subset (x + tK). \tag{4.13}$$

Let $v \in K$ and $0 < \varepsilon < t$. From (4.13) and Lemma 4.3, it follows that

$$Q \subset (x + tK) \subset (x + \varepsilon k + tK - \varepsilon k) \subset x + \varepsilon k + (t - \varepsilon)\, K.$$

Hence, $C_{Q|K}(x+\varepsilon k) \leq t-\varepsilon$. Applying (4.11) and (4.13), for $y = x+\varepsilon k$, we have that

$$\langle \xi, y-x\rangle = \langle \xi, x+\varepsilon k - x\rangle \leq C_{Q|K}(x+\varepsilon k) - C_{Q|K}(x) \leq t-\varepsilon \leq \varepsilon^2 - \varepsilon.$$

This yields that $\Im_K(-\xi) \geq 1$, by letting $\sigma \to 0^+$. $\square$

Remark 4.6 The results of Theorems 4.4 and 4.5 are due to the assumption on the affine property of the set K. This just covers some special cases such as the hyperplane. It would be interesting to consider the general case where K is the closed unit ball.

Conflict of Interest The authors declare no conflicts of interest.

References

[1] Deville R, Zizler V E. Farthest points in W*-compact sets. Bull Aust Math Soc, 1988, **38**(3): 433–439

[2] Lau K S. Farthest points in weakly compact sets. Israel Journal of Mathematics, 1975, **22**(2): 168–174

[3] Ni R X, Li C. Well posedness of furthest point and simultaneously furthest point problems in Banach spaces. Acta Mathematica Sinica Chinese Series, 2000, **43**: 421–426

[4] Westphal U, Schwartz T. Farthest points and monotone operators. Bull Aust Math Soc, 1998, **58**(1): 75–92

[5] Mordukhovich B S, Nam N M, Villalobos C. The smallest enclosing ball problem and the smallest intersecting ball problem: existence and uniqueness of solutions. Optim Lett, 2013, **7**(5): 839–853

[6] Colombo G, Wolenski P R. The subgradient formula for the minimal time function in the case of constant dynamics in Hilbert space. J Glob Optim, 2004, **28**: 269–282

[7] Colombo G, Wolenski P R. Variational analysis for a class of minimal time functions in Hilbert spaces. J Convex Anal, 2004, **11**(2): 335–361

[8] He Y R, Ng K F. Subdifferentials of a minimum time function in Banach spaces. J Math Anal Appl, 2006, **321**(2): 896–910

[9] Nam N M, Villalobos C M, An N T. Minimal time functions and the smallest intersecting ball problem with unbounded dynamics. J Optim Theory Appl, 2012, **154**(3): 768–791

[10] Jiang Y, He Y R. Subdifferentials of a minimal time function in normed spaces. J Math Anal Appl, 2009, **358**: 410–418

[11] Sun S Q, He Y R. Exact characterization for subdifferentials of a special optimal value function. Optim Lett, 2018, **12**(3): 519–534

[12] Nam N M, Cuong D V. Subgradients of minimal time functions without calmness. Journal of Convex Analysis, 2019, **26**(1): 189–200

[13] Nguyen L V, Qin X L. The minimal time function associated with a collection of sets. ESAIM-Control Optimisation and Calculus of Variations, 2020, **26**: Art 93

[14] Lu W P, Liu H B. Sampled-data time optimal control for heat equation with potential in R^n. Acta Mathematica Scientia, 2023, **43A**(4): 1269–1283

[15] Clarke F H. Optimization and Nonsmooth Analysis. New York: Wiley, 1983