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MULTIFRACTAL ANALYSIS OF CONVERGENCE EXPONENTS FOR PRODUCTS OF CONSECUTIVE PARTIAL QUOTIENTS IN CONTINUED FRACTIONS

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Abstract For each real number $x \in (0,1)$, let $[a_1(x), a_2(x), \cdots, a_n(x), \cdots]$ denote its continued fraction expansion. We study the convergence exponent defined by

$$\tau(x) := \inf\Big\{s \geq 0 : \sum_{n=1}^{\infty} \big(a_n(x)a_{n+1}(x)\big)^{-s} < \infty\Big\},$$

which reflects the growth rate of the product of two consecutive partial quotients. As a main result, the Hausdorff dimensions of the level sets of $\tau(x)$ are determined.

Key words continued fractions; product of partial quotients; Hausdorff dimension

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1 Introduction

It is a well-known fact that each irrational number $x \in (0,1)$ has a unique continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n(x) + \ddots}}} = [a_1(x), a_2(x), \cdots, a_n(x), \cdots]. \tag{1.1}$$

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Here and in the sequel, for each $n \geq 1$, the n-th partial quotient $a_n(x)$ is a positive integer, and the n-th convergent is the fraction

$$\frac{p_n(x)}{q_n(x)} := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \frac{1}{a_n(x)}}} = [a_1(x), a_2(x), \cdots, a_n(x)]$$

in the lowest term. For more details about continued fractions, we refer to [16, 20] and references therein.

The continued fraction expansion is closely related to the theory of Diophantine approximation. This relationship has its origin in the classic theorem of Dirichlet, which states that, for any $x \in (0,1)$ and $t > 1$, there exist non-trivial solutions (q,p) satisfying that

$$q \in \mathbb{N}, \quad p \in \mathbb{Z}, \quad |qx - p| < 1/t, \quad \text{and } 1 \leq q < t.$$

Given $t_0 \geq 1$, let $\psi : [t_0, \infty) \to \mathbb{R}^+$ be a non-increasing function, and let $D(\psi)$ denote the set of ψ-Dirichlet improvable numbers, i.e., the set of all $x \in (0,1)$ for which there exists $T \geq t_0$ such that for each $t > T$, the system

$$q \in \mathbb{N}, \quad p \in \mathbb{Z}, \quad |qx - p| < \psi(t), \quad \text{and } 1 \leq q < t$$

has non-trivial integer solutions. Davenport and Schmidt [6, Theorem 1] proved that for any $0 < c < 1$, the set $D(c/t)$ is contained in the union of the set of rational numbers and the set of irrational numbers with uniformly bounded partial quotients. Via continued fractions, under the condition that $t\psi(t) < 1$ for all $t \geq t_0$, Kleinbock and Wadleigh [21, Lemma 2.2] obtained a useful criterion to characterize whether or not a real number belongs to $D(\psi)$. More precisely, they showed that

$$\begin{aligned} &\Big\{x \in (0,1) : \ a_n(x)a_{n+1}(x) \geq \big((q_n(x)\psi(q_n(x)))^{-1} - 1\big)^{-1} \text{ for infinitely many } n\Big\} \subseteq D^c(\psi) \\ &\subseteq \Big\{x \in (0,1) : \ a_n(x)a_{n+1}(x) \geq \frac{1}{4}\big((q_n(x)\psi(q_n(x)))^{-1} - 1\big)^{-1} \text{ for infinitely many } n\Big\}. \end{aligned}$$

Furthermore, Kleinbock and Wadleigh [21, Theorem 1.8] proved a zero-one law for the Lebesgue measure of $D^c(\psi)$. Meanwhile, Hussain, Kleinbock, Wadleigh and Wang [15] established a zero-infinity law for the f-dimensional Hausdorff measure of $D^c(\psi)$, where f is the essentially sub-linear dimension function. Very recently, Bos, Hussain and Simmons [4] generalized the above zero-infinity law for all dimension functions under natural non-restrictive conditions.

In recent years, much attention has been paid to the multifractal property of sets which are relevant to the growth rate of the product of two consecutive partial quotients; see, for example, Huang and Wu [13], Huang, Wu and Xu [14], Bakhtawar, Bos and Hussain [2, 3], Zhang [26], Feng and Xu [10], Hu, Hussain and Yu [12], and Bakhtawar and Feng [1]. Aside from these, various exponents related to continued fractions have been investigated under different contexts to list a few of these, we would like to mention the works of Jarník [18], Pollicott and Weiss [23], Kesseböhmer and Stratmann [19], Fan, Liao, Wang and Wu [8], Nicolay and Simons [22], and Jaffard and Martin [17].

The present paper is concerned with the multifractal property of sets which are associated with the growth rate of the sequence of products of two or more consecutive partial quotients. Let us begin by recalling (see [24, p26]) the convergence exponent of the sequence

$\{a_n(x)a_{n+1}(x)\}_{n\geq 1}$ given by

$$\tau(x) := \inf\Big\{s \geq 0 : \sum_{n\geq 1}\big(a_n(x)a_{n+1}(x)\big)^{-s} < \infty\Big\}. \tag{1.2}$$

We shall study the multifractal spectrum of $\tau(x)$, or more precisely, the dimension function $\alpha \mapsto \dim_{\rm H}\Gamma(\alpha)$, where

$$\Gamma(\alpha) = \big\{x \in (0,1) : \tau(x) = \alpha\big\},\ 0 \leq \alpha < \infty.$$

Here and in the sequel, the notation $\dim_{\rm H}$ means the Hausdorff dimension. Our first result is the following:

Theorem 1.1 For any $0 \leq \alpha < \infty$, we have

$$\dim_{\rm H}\Gamma(\alpha) = \frac{1}{2}.$$

Note that if the sequence $(a_n(x)a_{n+1}(x))_{n\geq 1}$ is non-decreasing, then the convergence exponent defined by (1.2) can be given by the following formula:

$$\tau(x) = \limsup_{n\to\infty}\frac{\log n}{\log(a_n(x)a_{n+1}(x))}. \tag{1.3}$$

Then, we can proceed to study the multifractal spectrum of $\tau(x)$, i.e., the Hausdorff dimension of the intersection of sets $\Gamma(\alpha)$ and Δ, where

$$\Delta = \Big\{x \in (0,1) :\ a_n(x) \leq a_{n+2}(x),\ \forall\ n \geq 1\Big\}.$$

For any $0 \leq \alpha < \infty$, it follows from (1.3) that

$$\Gamma(\alpha)\cap\Delta = \Big\{x \in \Delta : \liminf_{n\to\infty}\frac{\log(a_n(x)a_{n+1}(x))}{\log n} = \frac{1}{\alpha}\Big\} := E_{\inf}(\Delta, \frac{1}{\alpha}). \tag{1.4}$$

Now we are in a position to state

Theorem 1.2 For any $0 \leq \alpha < \infty$, we have

$$\dim_{\rm H} E_{\inf}(\Delta,\alpha) = \begin{cases} 0, & 0 \leq \alpha < 1; \\ \dfrac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

Before proceeding, we give some remarks.

Remark 1.3 (i) By Theorem 1.2 and (1.4), we obtain

$$\dim_{\rm H}\big(\Gamma(\alpha)\cap\Delta\big) = \begin{cases} \dfrac{1-\alpha}{2}, & 0 \leq \alpha \leq 1; \\ 0, & \alpha > 1. \end{cases}$$

(ii) We point out that $\dim_{\rm H}\Delta = \frac{1}{2}$. In fact,

$$\big\{x \in (0,1) :\ a_n(x) \leq a_{n+1}(x),\ \forall\ n \geq 1\big\} \subseteq \Delta,$$

and from the main theorem of Ramharter [25], we see that

$$\frac{1}{2} = \dim_{\rm H}\big\{x \in (0,1) :\ a_n(x) \leq a_{n+1}(x),\ \forall\ n \geq 1\big\} \leq \dim_{\rm H}\Delta.$$

On the other hand,

$$\Delta \subseteq \big\{x \in (0,1) :\ a_n(x)a_{n+1}(x) \to \infty \text{ as } n \to \infty\big\},$$

and by the result of Zhang [26, Proposition 3.1], we have

$$\dim_{\mathrm{H}} \Delta \le \dim_{\mathrm{H}} \big\{x \in (0,1) : \ a_n(x)a_{n+1}(x) \to \infty \text{ as } n \to \infty\big\} = \frac{1}{2}.$$

Throughout this paper, the following notations are commonly used:

- $|\mathrm{I}|$ is the length of the interval I;
- $\lfloor x \rfloor$ is the greatest integer not exceeding x;
- $\sharp A$ is the cardinality of the finite set A;
- $\mathcal{H}^s(E)$ is the s-dimensional Hausdorff measure of the set E.

The rest of this paper is organized as follows: in Section 2, we provide some basic results of continued fractions and useful lemmas for calculating the Hausdorff dimension. The proofs of the main results are given in Section 3. Generalizations to the product of more than two consecutive partial quotients are discussed in Section 4.

2 Preliminaries

2.1 Basic Properties of Continued Fractions

For any $n \ge 1$ and n-tuple $(a_1, \cdots, a_n) \in \mathbb{N}^n$, we call

$$I_n(x) = I_n(a_1, \cdots, a_n) := \big\{x \in (0,1) : \ a_1(x) = a_1, \cdots, a_n(x) = a_n\big\}$$

a basic interval of order n containing x. Notice that all points in $I_n(a_1, \cdots, a_n)$ admit a continued fraction expansion beginning by $a_1, a_2, \cdots, a_n$, and thus they have the same $p_n(x)$ and $q_n(x)$. As long as there is no confusion, we write $p_n(a_1, \cdots, a_n) = p_n = p_n(x)$ and $q_n(a_1, \cdots, a_n) = q_n = q_n(x)$, for simplicity. It is well known (see [16, p6]) that p_n and q_n satisfy the recursive formula

$$\begin{cases} p_{-1} = 1, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2} \ (n \ge 1); \\ q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \ (n \ge 1). \end{cases} \tag{2.1}$$

Proposition 2.1 ([16, p18]) For any n-tuple $(a_1, \cdots, a_n) \in \mathbb{N}^n$, $I_n(a_1, \cdots, a_n)$ is the interval with the endpoints p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$. As a result,

$$|I_n(a_1, \cdots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}.$$

By the second of formula (2.1), we deduce from Proposition 2.1 that

$$\left(2^n \prod_{k=1}^{n} a_k\right)^{-2} \le |I_n(a_1, \cdots, a_n)| \le \left(\prod_{k=1}^{n} a_k\right)^{-2}. \tag{2.2}$$

2.2 Some Useful Lemmas

The following lemma established in [5, 8] is called 3-regular property, and it describes the relation between the basic interval $I_n(a_1, \cdots, a_n)$ and the ball $B(x, |I_n(a_1, \cdots, a_n)|)$:

Lemma 2.2 ([8, Lemma 2.5]) If $|I_{n+1}(x)| \le \rho < |I_n(x)|$ for $x \in (0,1)$ and $n \ge 2$, then

$$I_{n+1}(x) \subseteq B(x, \rho) \subseteq I_{n-2}(x).$$

The next lemma provides a general formula for obtaining the lower bound estimate of the Hausdorff dimension of some sets in continued fractions.

Lemma 2.3 ([8, Lemma 3.2]) Let $\{t_n\}_{n\geq 1}$ be a sequence of positive integers tending to infinity with $t_n \geq 3$ for all $n \geq 1$. Then, for any positive number $N \geq 2$,

$$\dim_{\rm H}\big\{x\in(0,1): t_n \leq a_n(x) < Nt_n, \forall\ n\geq 1\big\} = \liminf_{n\to\infty}\frac{\log(t_1t_2\cdots t_n)}{2\log(t_1t_2\cdots t_n)+\log t_{n+1}}.$$

The following lemma is known as the mass distribution principle:

Lemma 2.4 ([7, Proposition 2.3]) Let $E \subseteq (0,1)$ be a Borel set and let μ be a finite measure with $\mu(E) > 0$. Then, if

$$\liminf_{\rho\to 0}\frac{\log\mu(B(x,\rho))}{\log\rho} \geq s \ \text{ for all } x\in E,$$

we have $\dim_{\rm H} E \geq s$.

To conclude this section, we give two combinatorial formulas which will be used in the sequel.

Lemma 2.5 For positive integers ℓ and n, let

$$D(\ell,n) = \big\{(a_1,\cdots,a_n)\in\mathbb{N}^n : 1\leq a_1\leq\cdots\leq a_n\leq\ell\big\}.$$

Then

$$\sharp D(\ell,n) = \binom{n+\ell-1}{n}.$$

Proof For each $(a_1,a_2,\cdots,a_n)\in D(\ell,n)$, define

$$f(a_1,a_2,\cdots,a_n) = (a_1,a_2+1,\cdots,a_n+n-1).$$

Then f is a bijection between $D(\ell,n)$ and

$$\tilde{D}(\ell,n) = \big\{(r_1,r_2,\cdots,r_n)\in\mathbb{N}^n : 1\leq r_1<r_2<\cdots<r_n\leq \ell+n-1\big\}.$$

It follows that $\sharp D(\ell,n)$ is equal to $\sharp\tilde{D}(\ell,n)$, which is $\binom{n+\ell-1}{n}$, as desired. □

Based on Lemma 2.5, we can obtain the following result:

Lemma 2.6 For positive integers ℓ and n, let

$$F(\ell,n+1) = \big\{(a_1,\cdots,a_{n+1})\in\mathbb{N}^{n+1} : 1\leq a_1a_2\leq\cdots\leq a_na_{n+1}\leq\ell\big\}.$$

Then

$$\sharp F(\ell,n+1) = \begin{cases}\displaystyle\sum_{u=1}^{\ell}\binom{k-2+u}{k-1}\binom{k+\lfloor\frac{\ell}{u}\rfloor}{k+1}, & n=2k;\\ \displaystyle\sum_{u=1}^{\ell}\binom{k-2+u}{k-1}\binom{k-1+\lfloor\frac{\ell}{u}\rfloor}{k}, & n=2k-1.\end{cases}$$

Proof We shall only provide a proof for the case in which $n=2k$. For any $(a_1,\cdots,a_{2k+1}) \in\mathbb{N}^{2k+1}$, the inequality $1\leq a_1a_2\leq a_2a_3\leq\cdots\leq a_{2k}a_{2k+1}\leq\ell$ is equivalent to the conditions that

$$a_1\leq a_3\leq\cdots\leq a_{2k+1},\quad a_2\leq a_4\leq\cdots\leq a_{2k}\quad\text{and}\quad 1\leq a_{2k}a_{2k+1}\leq\ell.$$

Notice that, for each $1\leq u\leq\ell$, if $a_{2k}=u$, then $1\leq a_{2k+1}:=v\leq\lfloor\ell/u\rfloor$, and we have

$$\sharp F(\ell,n+1) = \sharp F(\ell,2k+1) = \sum_{u=1}^{\ell}\Big(\sharp D_2(u,k-1)\cdot\sum_{v=1}^{\lfloor\frac{\ell}{u}\rfloor}\sharp D_1(v,k)\Big), \tag{2.3}$$

where

$$D_1(v,k)=\Big\{(a_1,\cdots,a_{2k-1})\in\mathbb{N}^k: 1\le a_1\le a_3\le\cdots\le a_{2k-1}\le v\Big\}$$

and

$$D_2(u,k-1)=\Big\{(a_2,\cdots,a_{2k-2})\in\mathbb{N}^{k-1}: 1\le a_2\le a_4\le\cdots\le a_{2k-2}\le u\Big\}.$$

By Lemma 2.5,

$$\sharp D_1(v,k)=\binom{k+v-1}{v-1}\ \text{ and }\ \sharp D_2(u,k-1)=\binom{k-2+u}{k-1}.$$

Therefore,

$$\begin{aligned}\sharp F(\ell,n+1)=\sharp F(\ell,2k+1)&=\sum_{u=1}^{\ell}\left\{\binom{k-2+u}{k-1}\cdot\sum_{v=1}^{\lfloor\frac{\ell}{u}\rfloor}\binom{k+v-1}{v-1}\right\}\\&=\sum_{u=1}^{\ell}\left\{\binom{k-2+u}{k-1}\cdot\sum_{i=0}^{\lfloor\frac{\ell}{u}\rfloor-1}\binom{k+i}{i}\right\}\\&=\sum_{u=1}^{\ell}\binom{k-2+u}{k-1}\binom{k+\lfloor\frac{\ell}{u}\rfloor}{k+1}.\end{aligned}$$

The last equality follows by the identity

$$\sum_{i=0}^{j-1}\binom{k+i}{i}=\binom{k+j}{k+1},\quad j\ge 1.$$

□

3 Proofs of the Main Results

This section is devoted to proving our main results. Our method is inspired by those of Good [11], Fan, Liao, Wang and Wu [8], and Fang, Ma, Song and Wu [9].

3.1 Proof of Theorem 1.1

Recall that $\Gamma(\alpha)=\big\{x\in(0,1):\tau(x)=\alpha\big\}$. For any $x\in\Gamma(\alpha)$ and $\varepsilon>0$, by the definition of $\tau(x)$, we have that $\sum\limits_{n=1}^{\infty}(a_n(x)a_{n+1}(x))^{-(\alpha+\varepsilon)}<\infty$, which implies that

$$\Gamma(\alpha)\subseteq\big\{x\in(0,1):\ a_n(x)a_{n+1}(x)\to\infty \text{ as } n\to\infty\big\}.$$

From a result of Zhang [26, Proposition 3.1], we have

$$\dim_{\mathrm H}\Gamma(\alpha)\le\dim_{\mathrm H}\big\{x\in(0,1):\ a_n(x)a_{n+1}(x)\to\infty \text{ as } n\to\infty\big\}=\frac{1}{2}.$$

To bound $\dim_{\mathrm H}\Gamma(\alpha)$ from below, we shall construct subsets of $\Gamma(\alpha)$ as follows:

(i) for the case $\alpha=0$, it is easy to verify that

$$\Big\{x\in(0,1):2^n\le a_n(x)<2^{n+1},\forall\, n\ge1\Big\}\subseteq\Gamma(\alpha);$$

(ii) for the case $\alpha\in(0,\infty)$, it is also easy to check that

$$\Big\{x\in(0,1):n^{1/(2\alpha)}\le a_n(x)<2n^{1/(2\alpha)},\forall\, n\ge1\Big\}\subseteq\Gamma(\alpha).$$

By Lemma 2.3, the above subsets of $\Gamma(\alpha)$ are of Hausdorff dimension 1/2, and thus

$$\dim_{\mathrm H}\Gamma(\alpha)\ge\frac{1}{2}.$$

3.2 Proof of Theorem 1.2

Recall that

$$E_{\inf}(\Delta,\alpha)=\left\{x\in\Delta:\ \liminf_{n\to\infty}\frac{\log(a_n(x)a_{n+1}(x))}{\log n}=\alpha\right\}.$$

We shall consider the cases $0\le\alpha<1$ and $\alpha\ge 1$ separately.

3.2.1 Case $0\le\alpha<1$

We shall prove that $\dim_{\rm H} E_{\inf}(\Delta,\alpha)=0$. It suffices to show that, for any $s\in(0,1-\alpha)$, the s-dimensional Hausdorff measure is $\mathcal{H}^s(E_{\inf}(\Delta,\alpha))=0$. For this purpose, let

$$\mathcal{C}_{n+1}(\alpha,s):=\big\{(a_1,\cdots,a_{n+1})\in\mathbb{N}^{n+1}:1\le a_1a_2\le\cdots\le a_na_{n+1}\le n^{\alpha+s}\big\}.$$

Then

$$E_{\inf}(\Delta,\alpha)\subseteq\bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{(a_1,\cdots,a_{n+1})\in\mathcal{C}_{n+1}(\alpha,s)}I_{n+1}(a_1,\cdots,a_{n+1}).$$

This implies that, for any $j\ge 1$,

$$E_{\inf}(\Delta,\alpha)\subseteq\bigcup_{n=j}^{\infty}\bigcup_{(a_1,\cdots,a_{n+1})\in\mathcal{C}_{n+1}(\alpha,s)}I_{n+1}(a_1,\cdots,a_{n+1}).$$

Therefore,

$$\begin{aligned}\mathcal{H}^s(E_{\inf}(\Delta,\alpha))&\le\liminf_{j\to\infty}\sum_{n=j}^{\infty}\sum_{(a_1,\cdots,a_{n+1})\in\mathcal{C}_{n+1}(\alpha,s)}|I_{n+1}(a_1,\cdots,a_{n+1})|^s\\&\le\liminf_{j\to\infty}\sum_{n=j}^{\infty}\big(\sharp\mathcal{C}_{n+1}(\alpha,s)\cdot|I_{n+1}(a_1,\cdots,a_{n+1})|^s\big).\end{aligned}\tag{3.1}$$

For the moment, we assume that $n=2k$. It follows from Lemma 2.6 that

$$\begin{aligned}\sharp\mathcal{C}_{n+1}(\alpha,s)&=\sum_{u=1}^{\ell}\binom{k-2+u}{k-1}\binom{k+\lfloor\frac{\ell}{u}\rfloor}{k+1}=\sum_{u=1}^{\lfloor n^{\alpha+s}\rfloor}\binom{\frac{n}{2}-2+u}{\frac{n}{2}-1}\cdot\binom{\frac{n}{2}+\lfloor\frac{n^{\alpha+s}}{u}\rfloor}{\frac{n}{2}+1}\\&\le\sum_{u=1}^{\lfloor n^{\alpha+s}\rfloor}\binom{\frac{n}{2}-2+\lfloor n^{\alpha+s}\rfloor}{\frac{n}{2}-1}\cdot\binom{\frac{n}{2}+\lfloor n^{\alpha+s}\rfloor}{\frac{n}{2}+1}\\&\le n^{\alpha+s}\Big((\frac{n}{2}+1)\cdots(\frac{n}{2}+\lfloor n^{\alpha+s}\rfloor)\Big)^2\le n^{\alpha+s}\big(\frac{n}{2}+n^{\alpha+s}\big)^{2n^{\alpha+s}}\\&\le n^{\alpha+s}\mathrm{e}^{2n^{\alpha+s}(1+\log n)}.\end{aligned}\tag{3.2}$$

As for the case $n=2k+1$, a similar calculation yields the same inequality. In addition, the second formula of (2.1) indicates that

$$q_n\ge\frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n-\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n\ge\frac{1}{2\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n.$$

Combining this inequality with Proposition 2.1, we have

$$|I_{n+1}(a_1,\cdots,a_{n+1})|=\frac{1}{q_{n+1}(q_{n+1}+q_n)}\le\frac{1}{q_{n+1}^2}\le 20\left(\frac{1+\sqrt5}{2}\right)^{-2(n+1)}.\tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we get that

$$\mathcal{H}^s(E_{\inf}(\Delta,\alpha)) \leq \liminf_{j\to\infty}\sum_{n=j}^{\infty} n^{\alpha+s}\mathrm{e}^{2n^{\alpha+s}(1+\log n)}\cdot 20^s\left(\frac{1+\sqrt{5}}{2}\right)^{-2(n+1)s} = 0,$$

and hence that $\mathcal{H}^s(E_{\inf}(\Delta,\alpha)) = 0$.

3.2.2 Case $\alpha \geq 1$

In order to obtain the upper bound of $\dim_{\mathrm{H}} E_{\inf}(\Delta,\alpha)$, our method is to choose a suitable positive real number s such that $\mathcal{H}^s(E_{\inf}(\Delta,\alpha)) \leq 0$. Meanwhile, for the lower bound of $\dim_{\mathrm{H}} E_{\inf}(\Delta,\alpha)$, we shall construct a Cantor-like subset of $E_{\inf}(\Delta,\alpha)$ and then estimate its Hausdorff dimension by the mass distribution principle.

Upper bound Let $0 < \varepsilon < \alpha$ be small. By the definition of lim inf, if $x \in E_{\inf}(\Delta,\alpha)$, then $x \in \Delta$, and there exists $j \geq 1$ such that $a_k(x)a_{k+1}(x) \geq k^{\alpha-\varepsilon}$ for all $k \geq j$, and also that $a_n(x)a_{n+1}(x) \leq n^{\alpha+\varepsilon}$ for infinitely many n's. Thus we are able to prove that

$$E_{\inf}(\Delta,\alpha) \subseteq \bigcup_{j\geq 1}^{\infty} B_j(\alpha,\varepsilon), \tag{3.4}$$

where $B_j(\alpha,\varepsilon)$ is defined by

$$B_j(\alpha,\varepsilon) = \bigcap_{i=j}^{\infty}\bigcup_{n=i}^{\infty}\Big\{x\in\Delta : a_n(x)a_{n+1}(x) \leq n^{\alpha+\varepsilon}, a_k(x)a_{k+1}(x) \geq k^{\alpha-\varepsilon},\ j \leq \forall k \leq n\Big\}. \tag{3.5}$$

By (3.4), the monotonicity and countable stability properties of the Hausdorff dimension indicate that

$$\dim_{\mathrm{H}} E_{\inf}(\Delta,\alpha) \leq \sup_{j\geq 1}\big\{\dim_{\mathrm{H}} B_j(\alpha,\varepsilon)\big\}. \tag{3.6}$$

Next, we give only the upper bound of $\dim_{\mathrm{H}} B_1(\alpha,\varepsilon)$. The other cases are similar. Let

$$\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon) := \Big\{(a_1,\cdots,a_{n+1})\in\mathbb{N}^{n+1} : 1 \leq a_1a_2 \leq \cdots \leq a_na_{n+1} \leq n^{\alpha+\varepsilon},\ a_ka_{k+1} \geq k^{\alpha-\varepsilon},\ 1 \leq \forall k \leq n\Big\}.$$

Then, by (3.5), we have

$$B_1(\alpha,\varepsilon) = \bigcap_{i=1}^{\infty}\bigcup_{n=i}^{\infty}\ \bigcup_{(a_1,\cdots,a_{n+1})\in\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon)} I_{n+1}(a_1,\cdots,a_{n+1}). \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned}\mathcal{H}^s(B_1(\alpha,\varepsilon)) &\leq \liminf_{i\to\infty}\sum_{n=i}^{\infty}\ \sum_{(a_1,\cdots,a_{n+1})\in\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon)} |I_{n+1}(a_1,\cdots,a_{n+1})|^s\\ &= \liminf_{i\to\infty}\sum_{n=i}^{\infty}\big(\sharp\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon)\cdot|I_{n+1}(a_1,\cdots,a_{n+1})|^s\big).\end{aligned} \tag{3.8}$$

Before proceeding, we state a version of the Stirling formula that will be used repeatedly in the sequel:

$$\sqrt{2\pi}n^{n+\frac{1}{2}}\mathrm{e}^{-n} \leq n! \leq en^{n+\frac{1}{2}}\mathrm{e}^{-n}.$$

Now let us estimate the length of the basic interval $I_{n+1}(a_1,\cdots,a_{n+1})$ for $(a_1,\cdots,a_{n+1})\in\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon)$. In combination with (2.2), the Stirling formula implies that

$$|I_{n+1}(a_1,\cdots,a_{n+1})|\le\frac{1}{(a_1a_2\cdots a_{n+1})^2}\le\frac{1}{(a_1a_2)\cdot(a_2a_3)\cdots(a_na_{n+1})}$$
$$\le\frac{1}{(n!)^{\alpha-\varepsilon}}\le\frac{1}{\mathrm{e}^{(1-\varepsilon)(\alpha-\varepsilon)n\log n}}. \tag{3.9}$$

Next, an upper bound estimation of $\sharp\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon)$ is given as follows:

Lemma 3.1 For any $\alpha\ge1$ and $\varepsilon>0$, we have

$$\sharp\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon)\le\mathrm{e}^{\frac{\alpha+2\varepsilon-1}{2}n\log n}.$$

Proof We only consider the case in which n is an even integer; a similar proof would go through for the case in which n is odd. For any $0<\varepsilon<2-\alpha$, Lemma 2.6 implies that

$$\sharp\widetilde{\mathcal{C}}_{n+1}(\alpha,\varepsilon)\le\sum_{u=1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\binom{\frac n2-2+u}{\frac n2-1}\cdot\binom{\frac n2+\lfloor\frac{\lfloor n^{\alpha+\varepsilon}\rfloor}{u}\rfloor}{\frac n2+1}$$
$$\le\sum_{u=1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\binom{\frac n2-2+u}{\frac n2-1}\cdot\binom{\frac n2+\lfloor\frac{n^{\alpha+\varepsilon}}{u}\rfloor}{\frac n2+1}. \tag{3.10}$$

To obtain an upper bound, we shall distinguish between two cases: $1\le\alpha<2$ and $\alpha\ge2$.

Case (I): $1\le\alpha<2$**.** For any $0<\varepsilon<2-\alpha$, the right-hand side of (3.10) can be estimated as follows:

$$\begin{aligned}
&\sum_{u=1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\binom{\frac n2-2+u}{\frac n2-1}\cdot\binom{\frac n2+\lfloor\frac{n^{\alpha+\varepsilon}}{u}\rfloor}{\frac n2+1}\\
&=\sum_{u=1}^{\frac n2}\binom{\frac n2-2+u}{\frac n2-1}\cdot\binom{\frac n2+\lfloor\frac{n^{\alpha+\varepsilon}}{u}\rfloor}{\frac n2+1}+\sum_{u=\frac n2+1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\binom{\frac n2-2+u}{\frac n2-1}\cdot\binom{\frac n2+\lfloor\frac{n^{\alpha+\varepsilon}}{u}\rfloor}{\frac n2+1}\\
&\le\frac n2\cdot\binom{\frac n2-2+\frac n2}{\frac n2-1}\cdot\binom{\frac n2+\lfloor n^{\alpha+\varepsilon}\rfloor}{\frac n2+1}+n^{\alpha+\varepsilon}\cdot\binom{\frac n2-2+\lfloor n^{\alpha+\varepsilon}\rfloor}{\frac n2-1}\cdot\binom{\frac n2+\lfloor 2n^{\alpha+\varepsilon-1}\rfloor}{\frac n2+1}\\
&\le\frac n2\cdot\frac{(n/2)^{n/2}\cdot2^{n/2}}{(n/2)!}\cdot\frac{(n^{\alpha+\varepsilon})^{(n/2+1)}\cdot2^{n/2}}{(n/2)!}+n^{\alpha+\varepsilon}\cdot\frac{(n^{\alpha+\varepsilon})^{n/2}\cdot2^{n/2}}{(n/2)!}\cdot n^{2n^{\alpha+\varepsilon-1}}\\
&\le\frac{n^3}{2}\cdot\frac{(2e)^n}{2\pi(n/2)}\cdot(2n^{\alpha+\varepsilon-1})^{n/2}+n^{\alpha+\varepsilon}\cdot\frac{(2e)^{n/2}}{\sqrt{2\pi}(n/2)^{1/2}}\cdot(2n^{\alpha+\varepsilon-1})^{n/2}\cdot n^{2n^{\alpha+\varepsilon-1}}\\
&\le\frac13\mathrm{e}^{4n+\frac{\alpha+\varepsilon-1}{2}n\log n}+\frac23\mathrm{e}^{(\alpha+\varepsilon+1)\log n+2n+2n^{\alpha+\varepsilon-1}\log n+\frac{\alpha+\varepsilon-1}{2}n\log n}\\
&\le\mathrm{e}^{\frac{\alpha+2\varepsilon-1}{2}n\log n}.
\end{aligned}\tag{3.11}$$

Case (II): $\alpha\ge2$**.** We split the right-hand side of (3.10) into three parts:

$$\sum_{u=1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\binom{\frac n2-2+u}{\frac n2-1}\cdot\binom{\frac n2+\lfloor\frac{n^{\alpha+\varepsilon}}{u}\rfloor}{\frac n2+1}=\sum_{u=1}^{\frac n2}+\sum_{u=\frac n2+1}^{\lfloor 2n^{\alpha+\varepsilon-1}\rfloor}+\sum_{u=\lfloor 2n^{\alpha+\varepsilon-1}\rfloor+1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}. \tag{3.12}$$

From the upper bound in (3.11), we infer that

$$\sum_{u=1}^{\frac n2}\le\frac13\mathrm{e}^{4n+\frac{\alpha+\varepsilon-1}{2}n\log n}\le\frac13\mathrm{e}^{\frac{\alpha+2\varepsilon-1}{2}n\log n}.$$

By the Stirling formula, we have

$$\begin{aligned}\sum_{u=\frac{n}{2}+1}^{\lfloor 2n^{\alpha+\varepsilon-1}\rfloor} &\le \sum_{u=\frac{n}{2}+1}^{\lfloor 2n^{\alpha+\varepsilon-1}\rfloor} \frac{u^{n/2}\cdot 2^{n/2}}{(n/2)!}\cdot\frac{(n^{\alpha+\varepsilon}/u)^{(n/2+1)}\cdot 2^{n/2}}{(n/2)!}\\ &\le (2n^{\alpha+\varepsilon-1})^2\cdot (n^{\alpha+\varepsilon})^{n/2}\cdot\frac{(2e)^n}{2\pi(n/2)^{n+1}}\\ &\le \mathrm{e}^{2(\alpha+\varepsilon-1)\log n+3n+\frac{\alpha+\varepsilon-2}{2}n\log n}\\ &\le \frac{1}{3}\mathrm{e}^{\frac{\alpha+2\varepsilon-1}{2}n\log n}.\end{aligned}$$

Let

$$f(u)=\frac{n}{2}\log u+\frac{n^{\alpha+\varepsilon}}{u}\log n,\ \ u\in[\lfloor 2n^{\alpha+\varepsilon-1}\rfloor+1,\lfloor n^{\alpha+\varepsilon}\rfloor].$$

Then it is clear that the function $f(u)$ is decreasing on interval $[\lfloor 2n^{\alpha+\varepsilon-1}\rfloor+1,\lfloor 2n^{\alpha+\varepsilon-1}\rfloor\log n]$, and increasing on interval $[\lfloor 2n^{\alpha+\varepsilon-1}\rfloor\log n,\lfloor n^{\alpha+\varepsilon}\rfloor]$. Thus we have

$$\begin{aligned}f(u)&\le\max\left\{f(\lfloor 2n^{\alpha+\varepsilon-1}\rfloor+1),\ f(\lfloor n^{\alpha+\varepsilon}\rfloor)\right\}\\ &\le\max\left\{n+\frac{\alpha+\varepsilon}{2}n\log n,\ 2\log n+\frac{\alpha+\varepsilon}{2}n\log n\right\}=n+\frac{\alpha+\varepsilon}{2}n\log n.\end{aligned}\tag{3.13}$$

With the same method, we deduce from the Stirling formula and (3.13) that

$$\begin{aligned}\sum_{u=\lfloor 2n^{\alpha+\varepsilon-1}\rfloor+1}^{\lfloor n^{\alpha+\varepsilon}\rfloor} &\le \sum_{u=\lfloor 2n^{\alpha+\varepsilon-1}\rfloor+1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\frac{u^{n/2}\cdot 2^{n/2}}{(n/2)!}\cdot n^{\frac{n^{\alpha+\varepsilon}}{u}}\\ &\le \sum_{u=\lfloor 2n^{\alpha+\varepsilon-1}\rfloor+1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\mathrm{e}^{\frac{n}{2}\log u+\frac{n^{\alpha+\varepsilon}}{u}\log n-\frac{1}{2}n\log n+3n}\\ &\le n^{\alpha+\varepsilon}\cdot\mathrm{e}^{4n+\frac{\alpha+\varepsilon-1}{2}n\log n}\le\frac{1}{3}\mathrm{e}^{\frac{\alpha+2\varepsilon-1}{2}n\log n}.\end{aligned}$$

Putting these three estimates into (3.12) completes the proof. □

Now we are in a position to obtain the upper bound of $\dim_{\mathrm{H}} E_{\mathrm{inf}}(\Delta,\alpha)$. Taking

$$(1-\varepsilon)(\alpha-\varepsilon)s=\frac{\alpha+2\varepsilon-1}{2}+\varepsilon,$$

by Lemma 3.1, we conclude from (3.8) and (3.9) that

$$\mathcal{H}^s(B_1(\alpha,\varepsilon))\le\liminf_{i\to\infty}\sum_{n=i}^{\infty}\mathrm{e}^{\frac{\alpha+2\varepsilon-1}{2}n\log n}\cdot\frac{1}{\mathrm{e}^{(\frac{\alpha+2\varepsilon-1}{2}+\varepsilon)n\log n}}=0.$$

This shows that

$$\dim_{\mathrm{H}} B_1(\alpha,\varepsilon)\le s=\frac{\alpha+4\varepsilon-1}{2(1-\varepsilon)(\alpha-\varepsilon)}.$$

Then the desired upper bound follows by (3.6), i.e.,

$$\dim_{\mathrm{H}} E_{\mathrm{inf}}(\Delta,\alpha)\le\frac{\alpha-1}{2\alpha}.$$

Lower bound By the upper bound estimate, we have $\dim_{\mathrm{H}} E_{\mathrm{inf}}(\Delta,1)=0$. In what follows, we assume that $\alpha>1$. Let

$$G(\alpha)=\left\{x\in(0,1):\ a_{2n-1}(x)=1,\ 2n\lfloor(2n)^{\alpha-1}\rfloor+1\le a_{2n}(x)\le(2n+1)\lfloor(2n)^{\alpha-1}\rfloor,\forall n\ge 1\right\}.$$

Then it is obvious that

$$G(\alpha) \subseteq E_{\inf}(\Delta, \alpha). \tag{3.14}$$

For any $n \geq 1$, let

$$C_{2n}(\alpha) = \Big\{(a_1, \cdots, a_{2n}) \in \mathbb{N}^{2n} : 2k\lfloor (2k)^{\alpha-1} \rfloor + 1 \leq a_{2k}(x) \leq (2k+1)\lfloor (2k)^{\alpha-1} \rfloor, \\ a_{2k-1}(x) = 1,\ 1 \leq \forall k \leq n\Big\}.$$

Then it is easy to see that

$$G(\alpha) = \bigcap_{n=1}^{\infty} \bigcup_{(a_1,\cdots,a_{2n})\in C_{2n}(\alpha)} I_{2n}(a_1, \cdots, a_{2n}).$$

Suppose that μ is the probability measure supported on $G(\alpha)$ such that

$$\mu(I_{2n}(a_1, \cdots, a_{2n})) = \frac{1}{\sharp C_{2n}(\alpha)} = \prod_{k=1}^{n} \frac{1}{\lfloor (2k)^{\alpha-1} \rfloor} \tag{3.15}$$

for any $n \geq 1$ and $(a_1, \cdots, a_{2n}) \in C_{2n}(\alpha)$.

Let us estimate the μ-measure of each ball $B(x, \rho)$ with $x \in G(\alpha)$ and $\rho > 0$. By the construction of $G(\alpha)$, for each $x \in G(\alpha)$, there exists an $(2n)$-tuple $(a_1, \cdots, a_{2n}) \in C_{2n}(\alpha)$ such that $x \in I_{2n}(a_1, \cdots, a_{2n})$ for any $n \geq 1$. For any $\rho > 0$, let $2n$ be the integer such that

$$|I_{2n+2}(a_1, \cdots, a_{2n+2})| \leq \rho < |I_{2n}(a_1, \cdots, a_{2n})|. \tag{3.16}$$

Then, by Lemma 2.2, we have

$$B(x, \rho) \subseteq I_{2n-2}(a_1, \cdots, a_{2n-2}).$$

Combining (3.15) with (3.16), we deduce from (2.2) and Lemma 2.4 that

$$\begin{aligned}
\liminf_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} &\geq \liminf_{n\to\infty} \frac{\log \mu(I_{2n-2}(a_1, \cdots, a_{2n-2}))}{\log |I_{2n+2}(a_1, \cdots, a_{2n+2})|} \\
&\geq \liminf_{n\to\infty} \frac{-\sum\limits_{k=1}^{n-1} \log(\lfloor (2k)^{\alpha-1} \rfloor)}{-2(2n+2)\log 2 - 2\alpha \sum\limits_{k=1}^{n+1} \log(2k+1)} \\
&= \frac{\alpha-1}{2\alpha}.
\end{aligned}$$

This shows that

$$\dim_{\mathrm{H}} G(\alpha) \geq \frac{\alpha-1}{2\alpha}. \tag{3.17}$$

In view of (3.14) and (3.17), the proof of lower bound is complete.

4 On the Products of More than Two Consecutive Partial Quotients

The following is a multifractal analysis of the convergence exponent of the products of more than two consecutive partial quotients: let $m \geq 2$ be a positive integer and denote by

$$\hat{\tau}(x) := \inf\Big\{s \geq 0 : \sum_{n=1}^{\infty} \big(a_n(x) \cdots a_{n+m}(x)\big)^{-s} < \infty\Big\}$$

the convergence exponent of the sequence $\{a_n(x)\cdots a_{n+m}(x)\}_{n\geq 1}$. For any $0\leq\alpha<\infty$, let $\hat{\Gamma}(\alpha)=\big\{x\in(0,1):\hat{\tau}(x)=\alpha\big\}$ be the level sets of the convergence exponent $\hat{\tau}(x)$. In the sequel, we shall present the multifractal spectrum of $\hat{\tau}(x)$. Applying a result of Bakhtawar and Feng [1, Lemma 3.1], we have

$$\dim_{\rm H}\big\{x\in(0,1):\ a_n(x)\cdots a_{n+m}(x)\to\infty \text{ as } n\to\infty\big\}=\frac{1}{2}. \tag{4.1}$$

Using (4.1) and similar to those subsets in the proof of Theorem 1.1, we have

Theorem 4.1 For any $0\leq\alpha<\infty$, we have

$$\dim_{\rm H}\hat{\Gamma}(\alpha)=\frac{1}{2}.$$

Note that if the sequence $\{a_n(x)\cdots a_{n+m}(x)\}_{n\geq 1}$ is non-decreasing, then the multifractal spectrum of its convergence exponent is equal to $\dim_{\rm H}(\hat{\Gamma}(\alpha)\cap\hat{\Delta})$, where

$$\hat{\Delta}=\Big\{x\in(0,1):\ a_n(x)\leq a_{n+m+1}(x),\ \forall\ n\geq 1\Big\}.$$

By (1.3), we write

$$\hat{\Gamma}(\alpha)\cap\hat{\Delta}=\left\{x\in\hat{\Delta}:\liminf_{n\to\infty}\frac{\log(a_n(x)\cdots a_{n+m}(x))}{\log n}=\frac{1}{\alpha}\right\}:=E_{\inf}(\hat{\Delta},\frac{1}{\alpha}).$$

Theorem 4.2 For any $0\leq\alpha<\infty$, we have

$$\dim_{\rm H}E_{\inf}(\hat{\Delta},\alpha)=\begin{cases}0, & 0\leq\alpha<1;\\ \dfrac{\alpha-1}{2\alpha}, & \alpha\geq 1.\end{cases}$$

In what follows, we will modify the calculations in the previous section and sketch the proof of Theorem 4.2. We also consider the cases of $0\leq\alpha<1$ and $\alpha\geq 1$ separately, as before.

4.1 Case $0\leq\alpha<1$

Lemma 4.3 Let $0<s<1-\alpha$ be small and let

$$\mathcal{C}_{n+m}(\alpha,s):=\big\{(a_1,\cdots,a_{n+m})\in\mathbb{N}^{n+m}:1\leq a_1\cdots a_{m+1}\leq\cdots\leq a_n\cdots a_{n+m}\leq n^{\alpha+s}\big\}.$$

Then

$$\sharp\mathcal{C}_{n+m}(\alpha,s)\leq{\rm e}^{(m+1)n^{\alpha+s}(1+2\log n)}.$$

Proof We only provide a proof in the case of $n=(m+1)k$, where $k\geq 1$ is an integer. Similar proofs would go through for the other cases. Using the fact that if $1\leq n_1\leq n_2$ are two positive integers, then $\binom{n_2}{n_1}$ increases in n_2 for a fixed n_1, we obtain that

$$\sharp\mathcal{C}_{n+1}(\alpha,s)\leq n^{\alpha+s}\binom{k-2+\lfloor n^{\alpha+s}\rfloor}{k-1}\cdot\binom{k+\lfloor n^{\alpha+s}\rfloor}{k+1}$$

(see (3.2)), where n is even. Thus, for the case where $m=2,\ n=3k$ and $\ell=\lfloor n^{\alpha+s}\rfloor$, we apply Lemma 2.6 and the same method as that of (3.2) to obtain that

$$\begin{aligned}\sharp\mathcal{C}_{n+2}(\alpha,s)&=\sum_{u=1}^{\ell}\left\{\binom{k-2+u}{k-1}\cdot\sum_{v=1}^{\lfloor\frac{\ell}{u}\rfloor}\left\{\binom{k-1+v}{k}\binom{k+\lfloor\frac{\frac{\ell}{u}}{v}\rfloor}{k+1}\right\}\right\}\\&\leq n^{2(\alpha+s)}\binom{k-2+\lfloor n^{\alpha+s}\rfloor}{k-1}\cdot\binom{k-1+\lfloor n^{\alpha+s}\rfloor}{k}\cdot\binom{k+\lfloor n^{\alpha+s}\rfloor}{k+1}.\end{aligned}$$

Compared with the estimate of $\sharp\mathcal{C}_{n+1}(\alpha,s)$ and $\sharp\mathcal{C}_{n+2}(\alpha,s)$, we have

$$\begin{aligned}\sharp\mathcal{C}_{n+m}(\alpha,s) \le n^{m(\alpha+s)}&\binom{k-2+\lfloor n^{\alpha+s}\rfloor}{k-1}\\ &\cdot\underbrace{\binom{k-1+\lfloor n^{\alpha+s}\rfloor}{k}\cdots\binom{k-1+\lfloor n^{\alpha+s}\rfloor}{k}}_{m-1}\cdot\binom{k+\lfloor n^{\alpha+s}\rfloor}{k+1}\\ &\le n^{m(\alpha+s)}\left\{\binom{k-1+\lfloor n^{\alpha+s}\rfloor}{k}\right\}^{m+1}\\ &\le n^{m(\alpha+s)}\Big((\frac{n}{m+1}+1)\cdots(\frac{n}{m+1}+n^{\alpha+s})\Big)^{m+1}\\ &\le n^{m(\alpha+s)}\big(\frac{n}{m+1}+n^{\alpha+s}\big)^{(m+1)n^{\alpha+s}}\le \mathrm{e}^{(m+1)n^{\alpha+s}(1+2\log n)}.\end{aligned}$$

□

Putting (3.1), (3.3) and Lemma 4.3 together, we conclude that $\mathcal{H}^s(E_{\inf}(\hat{\Delta},\alpha))=0$, and thus complete the proof.

4.2 Case $\alpha\ge 1$

Lemma 4.4 Let $\varepsilon>0$ be small and let $\alpha\ge 1$, and let

$$\begin{aligned}\widetilde{\mathcal{C}}_{n+m}(\alpha,\varepsilon):=\Big\{&(a_1,\cdots,a_{n+m})\in\mathbb{N}^{n+m}:1\le a_1\cdots a_{m+1}\le\cdots\le a_n\cdots a_{n+m}\le n^{\alpha+\varepsilon},\\ &a_k\cdots a_{k+m}\ge k^{\alpha-\varepsilon},\ 1\le\forall k\le n\Big\}.\end{aligned}$$

Then we have

$$\sharp\widetilde{\mathcal{C}}_{n+m}(\alpha,\varepsilon)\le \mathrm{e}^{\frac{\alpha+(m+1)\varepsilon-1}{m+1}n\log n}.$$

Proof We only provide a proof for the case of $n=(m+1)k$. The remaining cases can be dealt with in similar manner. In the proof of Lemma 3.1, we observe that, for $n=2k$, the upper bound of the sum

$$\sum_{u=1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\binom{k-2+u}{k-1}\binom{k+\lfloor\frac{\lfloor n^{\alpha+\varepsilon}\rfloor}{u}\rfloor}{k+1}$$

is comparable to its first term, $\binom{k+\lfloor n^{\alpha+\varepsilon}\rfloor}{k+1}$. More precisely, there exist real numbers $b_1>1$ and $0<c_1<1$ such that

$$\sum_{u=1}^{\lfloor n^{\alpha+\varepsilon}\rfloor}\binom{k-2+u}{k-1}\binom{k+\lfloor\frac{\lfloor n^{\alpha+\varepsilon}\rfloor}{u}\rfloor}{k+1}\le \mathrm{e}^{b_1n^{c_1}\log n}\cdot\binom{k+\lfloor n^{\alpha+\varepsilon}\rfloor}{k+1}\le \mathrm{e}^{\frac{\alpha+2\varepsilon-1}{2}n\log n}.$$

Then, in the case where $m=2,\ n=3k$ and $\ell=\lfloor n^{\alpha+\varepsilon}\rfloor$, we can use the same method as that of Lemma 3.1 to obtain that

$$\begin{aligned}\sharp\widetilde{\mathcal{C}}_{n+2}(\alpha,\varepsilon)&=\sum_{u=1}^{\ell}\left\{\binom{k-2+u}{k-1}\cdot\sum_{v=1}^{\lfloor\frac{\ell}{u}\rfloor}\left\{\binom{k-1+v}{k}\binom{k+\lfloor\frac{\frac{\ell}{u}}{v}\rfloor}{k+1}\right\}\right\}\\ &\le\sum_{u=1}^{\ell}\left\{\binom{k-2+u}{k-1}\cdot b_2\mathrm{e}^{n^{c_2}\log n}\cdot\binom{k+\lfloor\frac{\ell}{u}\rfloor}{k+1}\right\}\\ &\le b_2b_3\mathrm{e}^{n^{c_2}\log n+n^{c_3}\log n}\cdot\binom{k+\lfloor n^{\alpha+\varepsilon}\rfloor}{k+1},\end{aligned}$$

where the real numbers satisfy $b_2, b_3 > 1$ and $0 < c_2, c_3 < 1$. As for the case of $n = (m+1)k$, there exist m sums in the formula presented in Lemma 2.6. By an induction on the number of sums, we could find real numbers $b > 1$ and $0 < c < 1$ such that

$$\begin{aligned}\sharp\widetilde{\mathcal{C}}_{n+m}(\alpha,\varepsilon) &\le \mathrm{e}^{bn^c\log n}\cdot\binom{k+\lfloor n^{\alpha+\varepsilon}\rfloor}{k+1} = \mathrm{e}^{bn^c\log n}\cdot\binom{\frac{n}{m+1}+\lfloor n^{\alpha+\varepsilon}\rfloor}{\frac{n}{m+1}+1}\\ &\le \mathrm{e}^{bn^c\log n}\cdot\frac{(n^{\alpha+\varepsilon})^{n/(m+1)}}{(n/(m+1))!} \le \mathrm{e}^{\frac{\alpha+(m+1)\varepsilon-1}{m+1}n\log n}.\end{aligned}$$

□

Upper bound For any $(a_1,\cdots,a_{n+m}) \in \widetilde{\mathcal{C}}_{n+m}(\alpha,\varepsilon)$, from (2.2), we have

$$\begin{aligned}|I_{n+m}(a_1,\cdots,a_{n+m})| &\le \frac{1}{(a_1a_2\cdots a_{n+m})^2} \le \Big(\frac{1}{(a_1a_2\cdots a_{m+1})\cdots(a_na_{n+1}\cdots a_{n+m})}\Big)^{\frac{2}{m+1}}\\ &\le \Big(\frac{1}{(n!)^{\alpha-\varepsilon}}\Big)^{\frac{2}{m+1}} \le \Big(\frac{1}{\mathrm{e}^{(1-\varepsilon)(\alpha-\varepsilon)n\log n}}\Big)^{\frac{2}{m+1}}.\end{aligned} \tag{4.2}$$

Taking

$$\frac{2}{m+1}(1-\varepsilon)(\alpha-\varepsilon)s = \frac{\alpha+(m+1)\varepsilon-1}{m+1}+\varepsilon,$$

we deduce from (3.6), (3.8), (4.2) and Lemma 4.4 that $\mathcal{H}^s(E_{\inf}(\hat{\Delta},\alpha)) = 0$, and thus

$$\dim_{\mathrm{H}} E_{\inf}(\hat{\Delta},\alpha) \le \frac{\alpha-1}{2\alpha}.$$

Lower bound Using the same method as the proof of Theorem 1.2 for the case $\alpha \ge 1$, we construct a suitable Cantor-like subset $\hat{G}(\alpha)$ of $E_{\inf}(\hat{\Delta},\alpha)$, and then apply Lemma 2.4 to obtain the lower bound of $E_{\inf}(\hat{\Delta},\alpha)$. For the sake of completeness, we give an outline of the proof. Noticing that $\dim_{\mathrm{H}} E_{\inf}(\hat{\Delta},1) = 0$, we consider the case $\alpha > 1$. Let

$$\begin{aligned}\hat{G}(\alpha) = \Big\{x\in(0,1):\ & a_{(m+1)n-1}(x) = \cdots = a_{(m+1)n-m}(x) = 1, (m+1)n\lfloor((m+1)n)^{\alpha-1}\rfloor+1\\ &\le a_{(m+1)n}(x) \le ((m+1)n+1)\lfloor((m+1)n)^{\alpha-1}\rfloor,\ \forall n\ge 1\Big\}.\end{aligned}$$

It is obvious that

$$\hat{G}(\alpha) \subseteq E_{\inf}(\hat{\Delta},\alpha).$$

For any $n \ge 1$, let

$$\begin{aligned}\hat{C}_{(m+1)n}(\alpha) = \Big\{&(a_1,\cdots,a_{(m+1)n}) \in \mathbb{N}^{(m+1)n}:\ a_{(m+1)k-1}(x) = \cdots = a_{(m+1)k-m}(x) = 1,\\ &(m+1)k\lfloor((m+1)k)^{\alpha-1}\rfloor+1 \le a_{(m+1)k}(x)\\ &\le ((m+1)n+1)\lfloor((m+1)k)^{\alpha-1}\rfloor, 1\le\forall k\le n\Big\}.\end{aligned}$$

Let ν be a probability measure supported on $\hat{G}(\alpha)$ such that, for $(a_1,\cdots,a_{(m+1)n}) \in \hat{C}_{(m+1)n}(\alpha)$,

$$\nu(I_{(m+1)n}(a_1,\cdots,a_{(m+1)n})) = \frac{1}{\sharp\hat{C}_{(m+1)n}(\alpha)} = \prod_{k=1}^{n}\frac{1}{\lfloor((m+1)k)^{\alpha-1}\rfloor}. \tag{4.3}$$

For any $\rho > 0$ small enough, let $(m+1)n$ be the integer such that

$$|I_{(m+1)(n+1)}(a_1,\cdots,a_{(m+1)(n+1)})| \le \rho < |I_{(m+1)n}(a_1,\cdots,a_{(m+1)n})|. \tag{4.4}$$

In view of (4.3) and (4.4), we deduce from Lemma 2.4 and (2.2) that

$$\dim_{\mathrm{H}} E_{\inf}(\hat{\Delta},\alpha) \ge \dim_{\mathrm{H}} \hat{G}(\alpha) \ge \frac{\alpha-1}{2\alpha}.$$

Conflict of Interest The authors declare no conflict of interest.

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