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A LOW-REGULARITY FOURIER INTEGRATOR FOR THE DAVEY-STEWARTSON II SYSTEM WITH ALMOST MASS CONSERVATION

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Abstract In this work, we propose a low-regularity Fourier integrator with almost mass conservation to solve the Davey-Stewartson II system (hyperbolic-elliptic case). Arbitrary order mass convergence could be achieved by the suitable addition of correction terms, while keeping the first order accuracy in $H^\gamma \times H^{\gamma+1}$ for initial data in $H^{\gamma+1} \times H^{\gamma+1}$ with $\gamma > 1$. The main theorem is that, up to some fixed time T, there exist constants τ_0 and C depending only on T and $\|u\|_{L^\infty\left((0,T);H^{\gamma+1}\right)}$ such that, for any $0 < \tau \leq \tau_0$, we have that

$$\|u(t_n,\cdot) - u^n\|_{H^\gamma} \leq C\tau, \quad \|v(t_n,\cdot) - v^n\|_{H^{\gamma+1}} \leq C\tau,$$

where u^n and v^n denote the numerical solutions at $t_n = n\tau$. Moreover, the mass of the numerical solution $M(u^n)$ satisfies that

$$|M(u^n) - M(u_0)| \leq C\tau^5.$$

Key words Davey-Stewartson II system; low-regularity; exponential integrator; first order accuracy; mass conservation

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1 Introduction

Recently, Ning, Kou and Wang [5] constructed a first-order low-regularity integrator for the Davey-Stewartson II (DS-II) system, which showed the first order accuracy in H^γ for initial data in $H^{\gamma+1}$. However, it is difficult to maintain the geometric structure of underlying PDEs for low-regularity integrators. The geometric structure is not only an important property, but

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also becomes the standard for judging the effectiveness of the numerical methods. In general, the conservative schemes perform much better than the nonconservative schemes.

Ismail and Taha [2] proposed a linearly implicit scheme with mass conservation for solving the coupled nonlinear Schrödinger equation, and the proposed scheme conserved the mass, exactly ruling out any possibility of the blowing up of the numerical solution. Wu and Yao [9] proposed a first-order Fourier integrator with almost mass conservation for solving the cubic nonlinear Schrödinger equation in one dimension. To the best of our knowledge, this is the first attempt to consider the conservation laws of the numerical solution for the exponential-type integrators. For the Korteweg-de Vries equation, Maierhofer and Schratz [4] proposed a symplectic resonance-based method for low regularity initials while preserving the underlying geometric structure. Recently, a new class of embedded low-regularity integrators was proposed for the classical KdV equation; these obtain the first order and the second order accuracy in H^γ for initial data in $H^{\gamma+1}$ [11] and $H^{\gamma+3}$ [10], respectively. Moreover, Ning, Wu and Zhao [6] proposed an embedded low-regularity integrator that achieved first-order accuracy by requiring the boundedness of one additional spatial derivative of the solution for the modified KdV equation. Wang and Zhao [8] achieved the second-order accuracy in time without loss of regularity of the solution by introducing a symmetric exponential-type low-regularity integrator for the Klein-Gordon equation. Schartz, Wang and Zhao [7] proposed an ultra low-regularity integrator for solving the nonlinear Dirac equation, which enabled optimal first-order temporal convergence without requiring any additional regularity on the solution. For the Davey-Stewartson systems, Frauendiener and Klein [1] presented a detailed numerical study of the Davey-Stewartson I system and obtained the relative conservation of the mass.

In this study, inspired by the works of Wu, Yao [9] and Ning, Kou, Wang [5], we develop a new scheme to achieve almost mass conservation while also requiring as low regularity as possible while maintaining first-order convergence for the DS-II system with rough initial data on a torus. Due to the non-elliptic nature of the principal operator in equation (1.1), the resonant set becomes larger, making it impossible to directly apply the method proposed in reference [9] to the DS-II system. Hence, we shall fully exploit the structure of the DS-II system and employ delicate Fourier analysis to overcome the complexity of the phase function.

The DS-II system with the rough initial data on a torus that we study in this work is

$$\begin{cases} \mathrm{i}\partial_t u(t,\boldsymbol{x}) + \partial_{x_1}^2 u(t,\boldsymbol{x}) - \partial_{x_2}^2 u(t,\boldsymbol{x}) = \mu_1 |u(t,\boldsymbol{x})|^2 u(t,\boldsymbol{x}) + \mu_2 u(t,\boldsymbol{x}) \partial_{x_1} v(t,\boldsymbol{x}), \\ \partial_{x_1}^2 v(t,\boldsymbol{x}) + \partial_{x_2}^2 v(t,\boldsymbol{x}) = \partial_{x_1}\left(|u(t,\boldsymbol{x})|^2\right), \quad t>0,\ \boldsymbol{x}=(x_1,x_2)\in\mathbb{T}^2, \end{cases} \tag{1.1}$$

where $\mu_1,\ \mu_2 \in \mathbb{R}$, $\mathbb{T}=(0,2\pi)$, $u=u(t,\boldsymbol{x}) : \mathbb{R}^+\times\mathbb{T}^2\to\mathbb{C}$, $v=v(t,\boldsymbol{x}) : \mathbb{R}^+\times\mathbb{T}^2\to\mathbb{R}$, and $u_0=u(0,\boldsymbol{x})\in H^\gamma(\mathbb{T}^2)\,(\,\gamma\ge 0\,)$ is unknown.

A variable substitution is introduced, namely,

$$\begin{cases} \xi_1 = \dfrac{1}{2}(x_1+x_2), \\ \xi_2 = \dfrac{1}{2}(x_1-x_2); \end{cases}$$

that is,

$$\begin{cases} \phi(t,\boldsymbol{\xi}) = u(t,\boldsymbol{x}), \\ \psi(t,\boldsymbol{\xi}) = v(t,\boldsymbol{x}), \end{cases} \tag{1.2}$$

where $\boldsymbol{\xi}=(\xi_1,\xi_2)$ and $\boldsymbol{x}=(x_1,x_2)$.

Then, the equations (1.1) can be rewritten as

$$\begin{cases} \mathrm{i}\phi_t + \phi_{\xi_1\xi_2} = \mu_1|\phi|^2\phi + \dfrac{1}{2}\mu_2\phi(\psi_{\xi_1} + \psi_{\xi_2}), \\ \psi_{\xi_1\xi_1} + \psi_{\xi_2\xi_2} = (\partial_{\xi_1} + \partial_{\xi_2})(|\phi|^2). \end{cases} \tag{1.3}$$

To avoid confusion of subsequent symbols, the equations (1.3) will be rewritten as equations with respect to x_1, x_2 to obtain that

$$\begin{cases} \mathrm{i}\phi_t + \partial_{x_1x_2}\phi - \phi E(|\phi|^2) = 0, \\ \psi = -(-\Delta)^{-1}(\partial_{x_1} + \partial_{x_2})(|\phi|^2), \end{cases} \tag{1.4}$$

where $E(f) = (\tilde{\mu}_1 + \mu_2\frac{\partial_{x_1x_2}}{\Delta})f, \tilde{\mu}_1 = \mu_1 + \frac{1}{2}\mu_2, \phi_0 = u\big(0, x_1 + x_2, x_1 - x_2\big)$. The derivation process can be found in [5].

Now, we only need to analyze the system (1.4) based on the above variable substitution. The DS-II system is completely integrable and thus has an infinite number of formally conserved quantities. The solution ϕ of system (1.4) in L^2 should satisfy the law of mass conservation,

$$M(\phi(t)) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} |\phi(t,\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = M(\phi_0) = M_0. \tag{1.5}$$

Ning, Kou and Wang [5] constructed a numerical solution with only first order convergence for the mass of the system (1.4):

$$\Psi(f) = \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau} f - \mathrm{i}\tau\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau}\Big[f \cdot E\big(\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\bar{f} \cdot f\big)\Big]. \tag{1.6}$$

Here

$$\omega(z) = \begin{cases} \dfrac{\mathrm{e}^z - 1}{z}, & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases} \tag{1.7}$$

Inspired by the idea of [9], we propose additional correction terms to improve the mass conservation, for example,

$$\Psi(f) = \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau} f + I(f).$$

Then we have that

$$\big\|\Psi(f)\big\|_{L^2}^2 = \|f\|_{L^2}^2 + \langle 2\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau} f + I(f), I(f)\rangle.$$

In order to improve the mass conservation, we add a correction term $J(f)$ and consider the mass

$$\begin{aligned} \big\|\Psi(f) + J(f)\big\|_{L^2}^2 = \|f\|_{L^2}^2 &+ \langle 2\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau} f + I(f), I(f)\rangle \\ &+ 2\langle \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau} f, J(f)\rangle + \langle 2I(f) + J(f), J(f)\rangle. \end{aligned}$$

We set that

$$J(f) = H(f) \cdot \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau} f, \text{ with } H(f) = -\langle 2\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau} f + I(f), I(f)\rangle \|f\|_{L^2}^{-2},$$

so that

$$\big\|\Psi(f) + J(f)\big\|_{L^2}^2 = \|f\|_{L^2}^2 + O(\tau^3),$$

which gives the third order convergence of the mass.

In this work, we focus on the fifth order mass conservation; two additional correction terms should be added. The solution of the modified DS system (1.4) can be written as

$$\phi^n = \Psi(\phi^{n-1}) + J_1(\phi^{n-1}) + J_2(\phi^{n-1}), \tag{1.8}$$

$$\psi^n = -(-\Delta)^{-1}(\partial_{x_1} + \partial_{x_2})(|\phi^n|^2), \tag{1.9}$$

where $n = 1, 2, \cdots, \frac{T}{\tau}$; $\phi^0 = \phi_0$, and the three functionals I, J_1, J_2 are defined as

$$I(U) = \Psi(U) - \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau}U, \tag{1.10}$$

$$J_1(U) = H(U)\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau}U, \tag{1.11}$$

$$J_2(U) = -\frac{1}{2}\big(H(U)\big)^2\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau}U - \Pi_0(|U|^2)^{-1}H(U)\mathrm{Re}\big(\Pi_0(I(U)\mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}\tau}\bar{U})\big)\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau}U, \tag{1.12}$$

and

$$H(U) = -\Pi_0(|U|^2)^{-1}\Big[\mathrm{Re}\big(\Pi_0(I(U)\mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}\tau}\bar{U})\big) + \frac{1}{2}\Pi_0(|I(U)|^2)\Big], \tag{1.13}$$

where $\Pi_0(f)$ is set to be the zero mode of the function f:

$$\Pi_0(f) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Now, we have the main theorem of this work.

Theorem 1.1 Let ϕ^n and ψ^n be the numerical solution of the DS-II system (1.4), obtained from the LRI schemes (1.8) and (1.9) up to some fixed time $T > 0$. Under the assumption that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$, for some $\gamma > 1$, there exist constants τ_0 and $C > 0$ such that, for any $0 < \tau \le \tau_0$, we have that

$$\|\phi(t_n, \cdot) - \phi^n\|_{H^\gamma} \le C\tau, \quad \|\psi(t_n, \cdot) - \psi^n\|_{H^{\gamma+1}} \le C\tau, \quad n = 0, 1, \cdots, \frac{T}{\tau}. \tag{1.14}$$

Moreover,

$$|M(\phi^n) - M(\phi_0)| \le C\tau^5, \tag{1.15}$$

where the constants τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

From equation (1.2), we can obtain the scheme of u and v for $n = 1, 2, \cdots, \frac{T}{\tau}, u^0 = u_0$ as

$$u^n = \tilde{\Psi}(u^{n-1}) + \tilde{J}_1(u^{n-1}) + \tilde{J}_2(u^{n-1}) \tag{1.16}$$

and

$$v^n = -(-\Delta)^{-1}\partial_{x_1}(|u^n|^2), \tag{1.17}$$

where

$$\tilde{\Psi}(f) = \mathrm{e}^{\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}f - \mathrm{i}\tau\mathrm{e}^{\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}\Big[f \cdot E\big(\omega(-2\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau)\bar{f} \cdot f\big)\Big],$$

$$\tilde{I}(U) = \tilde{\Psi}(U) - \mathrm{e}^{\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}U, \qquad \tilde{J}_1(U) = \tilde{H}(U)\mathrm{e}^{\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}U,$$

$$\begin{aligned}\tilde{J}_2(U) = &-\frac{1}{2}\big(\tilde{H}(U)\big)^2\mathrm{e}^{\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}U \\ &- \Pi_0(|U|^2)^{-1}\tilde{H}(U)\mathrm{Re}\big(\Pi_0(\tilde{I}(U)\mathrm{e}^{-\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}\bar{U})\big)\mathrm{e}^{\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}U\end{aligned}$$

and

$$\tilde{H}(U) = -\Pi_0(|U|^2)^{-1}\Big[\mathrm{Re}\big(\Pi_0(I(U)\mathrm{e}^{-\mathrm{i}(\partial^2_{x_1} - \partial^2_{x_2})\tau}\bar{U})\big) + \frac{1}{2}\Pi_0(|\tilde{I}(U)|^2)\Big].$$

Then we get

Corollary 1.2 Let u^n and v^n be the numerical solution of the DS-II system (1.1) obtained from the LRI schemes (1.16) and (1.17) up to some fixed time $T>0$. Under the assumption that $u_0 \in H^{\gamma+1}(\mathbb{T}^2)$, for some $\gamma>1$, there exist constants τ_0 and $C>0$ such that, for any $0<\tau \leq \tau_0$, we have that

$$\left\|u\left(t_n, \cdot\right)-u^n\right\|_{H^\gamma} \leq C \tau, \quad\left\|v\left(t_n, \cdot\right)-v^n\right\|_{H^{\gamma+1}} \leq C \tau, \quad n=0,1, \cdots, \frac{T}{\tau}. \tag{1.18}$$

Moreover,

$$\left|M\left(u^n\right)-M\left(u_0\right)\right| \leq C \tau^5, \tag{1.19}$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^{\infty}((0, T) ; H^{\gamma+1})}$.

Remark 1.3 The almost mass convergence scheme proposed in this work, together with the first-order scheme proposed in [5], has the lowest regularity requirement among all schemes for the DS-II system so far. For example, the Strang splitting method requires the loss of two derivatives.

Remark 1.4 With respect to the Fourier integrator, our scheme also achieves first-order convergence in $H^\gamma \times H^{\gamma+1}$, compared to the first-order scheme of [5], and maintaines fifth-order accuracy for the mass. Thus, the physical properties of the solution to the equation can be well preserved.

Remark 1.5 By repeating the same argument, and adding a additional correction term

$$J(f)=H(f) \cdot \mathrm{e}^{\mathrm{i} \partial_{x_1 x_2} \tau} f,$$

with

$$H(f)=-\langle 2 \mathrm{e}^{\mathrm{i} \partial_{x_1 x_2} \tau} f+I I(f), I I(f)\rangle\|f\|_{L^2}^{-2},$$

and

$$I I=\Psi+J_1+J_2-\mathrm{e}^{\mathrm{i} \partial_{x_1 x_2} \tau},$$

we obtaind 7-order convergent scheme in mass.

2 Preliminaries

2.1 Some Notations

First, we present some notations and tools for future derivation and analysis. We use $A \lesssim B$ or $B \lesssim A$ to denote the statement that $A \leq C B$ for some absolute constant $C>0$, which may vary from line to line but is independent of τ or n. We denote that $A \sim B$ for $A \lesssim B \lesssim A$. We use $O(Y)$ to denote any quantity X such that $X \lesssim Y$.

For $\boldsymbol{k}:=(k_1, k_2) \in \mathbb{Z}^2$, $\boldsymbol{x}:=(x_1, x_2) \in \mathbb{T}^2$, we denote that

$$\boldsymbol{k} \cdot \boldsymbol{x}=k_1 x_1+k_2 x_2, \quad|\boldsymbol{k}|^2=\left|k_1\right|^2+\left|k_2\right|^2 .$$

We denote $\langle\cdot, \cdot\rangle$ to be the real L^2-inner product, that is

$$\langle f, g\rangle=\operatorname{Re} \int_{\mathbb{T}^2} f(\boldsymbol{x}) \overline{g(\boldsymbol{x})} \mathrm{d} \boldsymbol{x} .$$

The discrete Fourier transform of a function f on $\mathbb{T}^2$ is defined by

$$\hat{f}_{\boldsymbol{k}}=\frac{1}{(2 \pi)^2} \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} f_{\boldsymbol{k}},$$

and its inverse is

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \hat{f}_{\boldsymbol{k}}.$$

Then the L^2 norm and the $H^\gamma(\mathbb{T}^2)$ for $\gamma \geq 0$ norm are defined as

$$\|f\|_{L^2}^2 = (2\pi)^2 \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\hat{f}_{\boldsymbol{k}}|^2 = (2\pi)^2 \|\hat{f}_{\boldsymbol{k}}\|_{L^2(\mathbb{Z}^2)}^2, \tag{2.1}$$

$$\|f\|_{H^\gamma(\mathbb{T}^2)}^2 = \|J^\gamma f\|_{L^2(\mathbb{T}^2)}^2 = (2\pi)^2 \sum_{\boldsymbol{k}\in\mathbb{Z}^2} (1+|\boldsymbol{k}|^2)^\gamma |\hat{f}_{\boldsymbol{k}}|^2, \tag{2.2}$$

where the operator is

$$J^s = (1-\Delta)^{\frac{s}{2}}, \quad \forall s \in \mathbb{R}. \tag{2.3}$$

Moreover, the discrete Fourier transform of $(-\Delta)^{-1}$ and $|\nabla|^{-1}$ can be represented by

$$\widehat{(-\Delta)^{-1}} f(k) = \begin{cases} |\boldsymbol{k}|^{-2} \hat{f}_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \neq 0, \\ 0, & \text{if } \boldsymbol{k} = 0. \end{cases} \tag{2.4}$$

$$\widehat{|\nabla|^{-1}} f(\boldsymbol{k}) = \begin{cases} |\boldsymbol{k}|^{-1} \hat{f}_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \neq 0, \\ 0, & \text{if } \boldsymbol{k} = 0. \end{cases} \tag{2.5}$$

$T_m(M;\varphi)$ is set to be a class of qualities which is defined in the Fourier space by

$$\mathcal{F} T_m(M;\varphi)(\boldsymbol{k}) = O\Big(\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\cdots+\boldsymbol{k}_m} |M(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_m)|\, |\hat{\varphi}_{\boldsymbol{k}_1}(t)| \cdots |\hat{\varphi}_{\boldsymbol{k}_m}(t)| \Big), \tag{2.6}$$

where $\boldsymbol{k}_j = (k_{j1}, k_{j2}) \in \mathbb{Z}^2$, $j = \{1,\cdots,m\}$, and M is a function regarding $\boldsymbol{k}_1,\cdots,\boldsymbol{k}_m$.

Furthermore, the isometric property of the operator $\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t}$ gives that

$$\|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t} f\|_{H^\gamma} = \|f\|_{H^\gamma} \tag{2.7}$$

for all $f \in H^\gamma$, $\gamma > 1$ and $t \in \mathbb{R}$.

2.2 Some Preliminary Estimates

Lemma 2.1 (Kato-Ponce inequality [3]) For any $\gamma > 1$, $f, g \in H^\gamma(\mathbb{T}^d)$, the following inequality holds:

$$\|fg\|_{H^\gamma} \lesssim \|f\|_{H^\gamma} \|g\|_{H^\gamma}. \tag{2.8}$$

Lemma 2.2 Let $\gamma > 1$ and $\varphi \in H^\gamma$. Then the following inequality holds:

$$\|T_3(k_{11}(k_{22}+k_{32}) + k_{21}(k_{12}+k_{32}) + k_{31}(k_{12}+k_{22}); \varphi)\|_{H^{-\gamma}} \lesssim \|\varphi\|_{L^\infty((0,T);H^\gamma)}^3.$$

Proof We assume that $\hat{\varphi}_{\boldsymbol{k}_j} > 0$, $j = \{1,2,3\}$ for any $\boldsymbol{k}_j$, otherwise one may replace these by $|\hat{\varphi}_{\boldsymbol{k}_j}|$.

Using the definition in (2.6) and Sobolev's embedding theorem, we have that

$$\begin{aligned} &\big\|T_3(k_{11}(k_{22}+k_{32}) + k_{21}(k_{12}+k_{32}) + k_{31}(k_{12}+k_{22}); \varphi)\big\|_{H^{-\gamma}} \\ \lesssim\ & \Big\| \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} (|\boldsymbol{k}_1||\boldsymbol{k}_2| + |\boldsymbol{k}_1||\boldsymbol{k}_3| + |\boldsymbol{k}_2||\boldsymbol{k}_3|)\, \hat{\varphi}_{\boldsymbol{k}_1} \hat{\varphi}_{\boldsymbol{k}_2} \hat{\varphi}_{\boldsymbol{k}_3} \Big\|_{H^{-\gamma}} \\ \lesssim\ & \|(|\nabla|\varphi)^2 \varphi\|_{H^{-\gamma}} \lesssim \|(|\nabla|\varphi)^2 \varphi\|_{L^1}. \end{aligned}$$

From Lemma 2.1, we get that

$$\begin{aligned} &\|T_3(k_{11}(k_{22}+k_{32}) + k_{21}(k_{12}+k_{32}) + k_{31}(k_{12}+k_{22}); \varphi)\|_{H^{-\gamma}} \\ \lesssim\ & \|\nabla\varphi\|_{L^2}^2 \|\varphi\|_{L^\infty} \lesssim \|\varphi\|_{L^\infty((0,T);H^\gamma)}^3. \end{aligned}$$

Therefore, Lemma 2.2 is proven. □

2.3 Review of the First-Order Numerical Scheme Construction

Ning, Kou and Wang [5] solved the DS-II system (1.4) using Duhamel's formula

$$\phi(t)=\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t}\phi(t_0)-\mathrm{i}\int_{t_0}^{t}\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t-s)}\big[\phi(s)\cdot E(|\phi(s)|^2)\big]\mathrm{d}s. \tag{2.9}$$

By the twisted variable $\varphi(t)=\mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}t}\phi(t)$, we get that

$$\varphi(t)=\varphi(t_0)-\mathrm{i}\int_{t_0}^{t}\mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}s}\Big[\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}s}\varphi(s)\cdot E\big(|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}s}\varphi(s)|^2\big)\Big]\mathrm{d}s.$$

By taking the discrete Fourier transformation, we get that

$$\begin{aligned}\hat{\varphi}_{\boldsymbol{k}}(t_{n+1})=\hat{\varphi}_{\boldsymbol{k}}(t_n)-\mathrm{i}\int_0^{\tau}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3}\mathrm{e}^{\mathrm{i}\alpha(t_n+s)}\big[\tilde{\mu}_1+\mu_2\frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big]\\ \cdot\,\hat{\bar{\varphi}}_{\boldsymbol{k}_1}(t_n+s)\hat{\varphi}_{\boldsymbol{k}_2}(t_n+s)\hat{\varphi}_{\boldsymbol{k}_3}(t_n+s)\mathrm{d}s,\end{aligned} \tag{2.10}$$

where $\alpha=k_1k_2+k_{11}k_{12}-k_{21}k_{22}-k_{31}k_{32}$, and we let $\beta=k_{11}(k_{22}+k_{32})+k_{21}(k_{12}+k_{32})+k_{31}(k_{12}+k_{22})$. Hence, we have that $\alpha=2k_{11}k_{12}+\beta$.

Only the dominant quadratic term $2\mathrm{i}sk_{11}k_{12}$ is chosen for the integration, so the integration can be carried out fully in Fourier space as

$$\int_0^{\tau}\mathrm{e}^{2\mathrm{i}sk_{11}k_{12}}\mathrm{d}s=\tau\omega(2\mathrm{i}\tau k_{11}k_{12}).$$

Hence,

$$\begin{aligned}\varphi(t_{n+1})&=\varphi(t_n)-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}t_n}\Big[\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi(t_n)E\big(\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\overline{\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi(t_n)}\cdot\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi(t_n)\big)\Big]\\&\quad+\mathcal{R}_1^n+\mathcal{R}_2^n\\&:=\Phi^n(\varphi(t_n))+\mathcal{R}_1^n+\mathcal{R}_2^n,\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}\mathcal{R}_1^n=-\mathrm{i}\int_0^{\tau}\mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}(t_n+s)}\Big[\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi(t_n+s)\cdot E\big(|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi(t_n+s)|^2\big)\\-\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi(t_n)\cdot E\big(|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi(t_n)|^2\big)\Big]\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_2^n=-\mathrm{i}\sum_{\boldsymbol{k}\in\mathbb{Z}^2}\sum_{\substack{\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3}\in\mathbb{Z}^2\\ \boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3}}\mathrm{e}^{\mathrm{i}t_n\alpha}\big[\tilde{\mu}_1+\mu_2\frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big]\\ \cdot\,\hat{\bar{\varphi}}_{\boldsymbol{k}_1}\hat{\varphi}_{\boldsymbol{k}_2}\hat{\varphi}_{\boldsymbol{k}_3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\int_0^{\tau}\mathrm{e}^{2\mathrm{i}sk_{11}k_{12}}(\mathrm{e}^{\mathrm{i}s\beta}-1)\mathrm{d}s.\end{aligned}$$

Then we get the scheme of the first order low-regularity integrator (LRI) for solving the DS-II system (1.4):

$$\phi^n=\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau}\phi^{n-1}-\mathrm{i}\tau\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}\tau}\Big[\phi^{n-1}\cdot E\big(\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\overline{\phi^{n-1}}\cdot\phi^{n-1}\big)\Big], \tag{2.12}$$

$$\psi^n=-(-\Delta)^{-1}(\partial_{x_1}+\partial_{x_2})|\phi^n|^2. \tag{2.13}$$

Here the numerical solution of $\phi^n=\phi^n(\boldsymbol{x})$ and $\psi^n=\psi^n(\boldsymbol{x})$ is the numerical solution, for $n=1,2,3,\cdots$, and the scheme reaches first order accuracy.

Theorem 2.3 ([5]) Let ϕ^n and ψ^n be the numerical solution of the DS-II system (1.4) obtained from the schemes (2.12) and (2.13) up to some fixed time $T > 0$. Assume that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$ for some $\gamma > 1$. Then there exist constants $\tau_0 > 0$ and $C > 0$ such that, for any $0 < \tau \leq \tau_0$, we have that

$$\|\phi(t_n) - \phi^n\|_{H^\gamma} \leq C\tau, \quad \|\psi(t_n) - \psi^n\|_{H^{\gamma+1}} \leq C\tau, \quad n = 0, 1, \cdots, \frac{T}{t},$$

where the constants τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

We list some estimates from Ning, Kou and Wang [5].

The estimates for $\mathcal{R}_1^n$ and $\mathcal{R}_2^n$ are as follows:

Lemma 2.4 ([5]) Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$. Then there exist constants $\tau_0 > 0$ and $C > 0$ such that, for any $0 < \tau \leq \tau_0$,

$$\|\mathcal{R}_1^n\|_{H^\gamma} \leq C\tau^2,$$

where τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^\gamma)}$.

Lemma 2.5 ([5]) Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$. Then there exist constants $\tau_0 > 0$ and $C > 0$ such that, for any $0 < \tau \leq \tau_0$,

$$\|\mathcal{R}_2^n\|_{H^\gamma} \leq C\tau^2,$$

where τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

By combining Lemmas 2.4 and 2.5, the local error estimate of the numerical propagator is obtained.

Lemma 2.6 (Local error [5]) Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$. Then there exist constants $\tau_0 > 0$ and $C > 0$ such that, for any $0 < \tau \leq \tau_0$,

$$\|\varphi(t_{n+1}) - \Phi^n(\varphi(t_n))\|_{H^\gamma} \leq C\tau^2,$$

where τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

The stability of the solution is as follows:

Lemma 2.7 (Stability [5]) Let $f, g \in H^\gamma$. Then, for $\gamma > 1$, the following estimate holds:

$$\|\Phi^n(f) - \Phi^n(g)\|_{H^\gamma} \leq (1 + C\tau)\,\|f - g\|_{H^\gamma} + C\tau\,\|f - g\|_{H^\gamma}^3.$$

Here C depends only on $\|f\|_{H^\gamma}$.

3 The Almost Mass-Conserved Scheme

3.1 Construction of the Numerical Integrator

Let $\varphi^n = \mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}t_n}\phi^n$. Accordingly, from (1.10)–(1.13), we have that

$$\varphi^{n+1} = \varphi^n + I^n(\varphi^n) + J_1^n(\varphi^n) + J_2^n(\varphi^n), \tag{3.1}$$

where Φ^n is defined as in (2.11), that is,

$$I^n(\varphi^n) = \Phi^n(\varphi^n) - \varphi^n, \tag{3.2}$$

and the functionals J_1^n, J_2^n are given by

$$J_1^n(\varphi^n) = H^n(\varphi^n)\varphi^n, \tag{3.3}$$

$$J_2^n(\varphi^n) = -\frac{1}{2}\big(H^n(\varphi^n)\big)^2\varphi^n - \big(\|\varphi^n\|_{L^2}^2\big)^{-1}H^n(\varphi^n)\big\langle I^n(\varphi^n),\varphi^n\big\rangle\varphi^n, \tag{3.4}$$

and

$$H^n(\varphi^n) = -\big(\|\varphi^n\|_{L^2}^2\big)^{-1}\Big(\langle I^n(\varphi^n),\varphi^n\rangle + \frac{1}{2}\|I^n(\varphi^n)\|_{L_2}^2\Big). \tag{3.5}$$

The proof of Theorem 1.1 depends on some key lemmas. First, the convergence order of $I^n(\varphi)$ is given.

Lemma 3.1 Let $\gamma > 1$. Assume that $\varphi \in H^\gamma$. Then there exists a constant $C > 0$ such that

$$\|I^n(\varphi)\|_{L^2} \lesssim C\tau, \tag{3.6}$$

where C depends on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

Proof By (2.11) and (3.2), we have that

$$I^n(\varphi) = -\mathrm{i}\tau \mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}t_n}\left[\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi E\big(\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\overline{\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi}\cdot \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi\big)\right].$$

Hence, we have that

$$\begin{aligned}\|I^n(\varphi)\|_{L^2} &= \big\| - \mathrm{i}\tau \mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}t_n}\big[\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi E\big(\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\overline{\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi}\cdot \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi\big)\big]\big\|_{L^2}\\ &\le \tau\big\|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi E\big(\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\overline{\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi}\cdot \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi\big)\big\|_{L^2}\\ &\le \tau\big\|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi\big\|_{L^\infty}\big\|E\big(\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\overline{\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi}\cdot \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi\big)\big\|_{L^2}.\end{aligned}$$

By $\|Ef\|_{L^2} \le C\|f\|_{L^2}$, we have that

$$\|I^n(\varphi)\|_{L^2} \lesssim \tau\big\|\varphi\big\|_{L^\infty}\big\|\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)\overline{\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi}\cdot \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi\big\|_{L^2},$$

Together with Lemma 2.1 and $\|\omega(-2\mathrm{i}\partial_{x_1x_2}\tau)f\|_{H^\gamma} \le C\|f\|_{H^\gamma}$, we obtain that

$$\|I^n(\varphi)\|_{L^2} \lesssim \tau\big\|\varphi\big\|_{H^\gamma}\big\|\overline{\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}t_n}\varphi}\big\|_{H^\gamma}\big\|\varphi\big\|_{H^\gamma} \lesssim \tau\|\varphi\|^3_{L^\infty((0,T);H^\gamma)}.$$

This proves the lemma. □

Next, we give the convergence order of $\langle I^n(\varphi),\varphi\rangle$.

Lemma 3.2 Let $\gamma > 1$. Assume that $\varphi \in H^\gamma$. Then there exists a constant $C > 0$ such that

$$|\langle I^n(\varphi),\varphi\rangle| \lesssim C\tau^2, \tag{3.7}$$

where C depends on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

Proof We perform a Fourier transformation on $I^n(\varphi)$ to obtain that

$$\widehat{I^n}(\varphi) = -\mathrm{i}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3}\int_0^\tau \mathrm{e}^{\mathrm{i}\alpha t_n}\mathrm{e}^{2\mathrm{i}k_{11}k_{12}s}\mathrm{d}s\big[\tilde{\mu}_1 + \mu_2\frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big]\hat{\bar{\varphi}}_{\boldsymbol{k}_1}\hat{\varphi}_{\boldsymbol{k}_2}\hat{\varphi}_{\boldsymbol{k}_3}.$$

Together with $\alpha = 2k_{11}k_{12} + \beta$, we get that

$$\begin{aligned}\widehat{I^n}(\varphi) = -\mathrm{i}&\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3}\int_0^\tau \mathrm{e}^{\mathrm{i}\alpha t_n}\left[\mathrm{e}^{\mathrm{i}\alpha s} - \mathrm{e}^{2\mathrm{i}k_{11}k_{12}s}(\mathrm{e}^{2\mathrm{i}\beta s}-1)\right]\mathrm{d}s\\ &\cdot\big[\tilde{\mu}_1 + \mu_2\frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big]\hat{\bar{\varphi}}_{\boldsymbol{k}_1}\hat{\varphi}_{\boldsymbol{k}_2}\hat{\varphi}_{\boldsymbol{k}_3}.\end{aligned} \tag{3.8}$$

Therefore, we can write $\widehat{I^n}(\varphi)$ as

$$\widehat{I^n}(\varphi) = -\mathrm{i}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3}\int_0^\tau \mathrm{e}^{\mathrm{i}\alpha(t_n+s)}\mathrm{d}sW\hat{\bar{\varphi}}_{\boldsymbol{k}_1}\hat{\varphi}_{\boldsymbol{k}_2}\hat{\varphi}_{\boldsymbol{k}_3} + (\widehat{\mathcal{R}_2^n})_{\boldsymbol{k}}, \tag{3.9}$$

where $\widehat{(\mathcal{R}_2^n)}_{\boldsymbol{k}}$ is defined as

$$\begin{aligned}\widehat{(\mathcal{R}_2^n)}_{\boldsymbol{k}} = \mathrm{i} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} &\int_0^\tau \mathrm{e}^{\mathrm{i}\alpha t_n} \mathrm{e}^{2\mathrm{i}k_{11}k_{12}s} (\mathrm{e}^{2\mathrm{i}\beta s} - 1)\mathrm{d}s \\ &\cdot \big[\tilde{\mu}_1 + \mu_2 \frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big] \hat{\bar{\varphi}}_{\boldsymbol{k}_1} \hat{\varphi}_{\boldsymbol{k}_2} \hat{\varphi}_{\boldsymbol{k}_3}.\end{aligned}$$

By establishing a Fourier inversion of the (3.8) equation, we obtain that

$$I^n(\varphi) = -\mathrm{i} \int_0^\tau \mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}(t_n+s)} \Big(\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)} \varphi \cdot E\big(|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi|^2\big) \Big) \mathrm{d}s + \mathcal{R}_2^n. \tag{3.10}$$

By taking the inner product of I^n and substituting (3.10) into the equation, we get that

$$\begin{aligned}\langle I^n(\varphi), \varphi\rangle &= \Big\langle -\mathrm{i} \int_0^\tau \mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}(t_n+s)} \big(\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)} \varphi \cdot E\big(|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi|^2\big)\big) \mathrm{d}s, \varphi \Big\rangle + \langle \mathcal{R}_2^n, \varphi\rangle \\ &= \int_0^\tau \Big\langle -\mathrm{i}\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)} \varphi \cdot E\big(|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi|^2\big), \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi \Big\rangle \mathrm{d}s + \langle \mathcal{R}_2^n, \varphi\rangle.\end{aligned}$$

Since

$$\Big\langle -\mathrm{i}f \cdot E\big(|f|^2\big), f \Big\rangle = \mathrm{Re} \int_{\mathbb{T}^d} -\mathrm{i}f \cdot E\big(|f|^2\big) \cdot \bar{f} \mathrm{d}x = 0,$$

we get that

$$\int_0^\tau \Big\langle -\mathrm{i}\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)} \varphi \cdot E\big(|\mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi|^2\big), \mathrm{e}^{\mathrm{i}\partial_{x_1x_2}(t_n+s)}\varphi \Big\rangle \mathrm{d}s = 0.$$

Hence, we have that

$$\langle I^n(\varphi), \varphi\rangle = \langle \mathcal{R}_2^n, \varphi\rangle.$$

According to Lemma 2.2, we get that

$$|\langle I^n(\varphi), \varphi\rangle| = |\langle \mathcal{R}_2^n, \varphi\rangle| \le \|\mathcal{R}_2^n\|_{H^{-\gamma}} \|\varphi\|_{H^\gamma} \lesssim C\tau^2,$$

where C depends on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

This proves Lemma 3.2. □

3.2 The Proof of Theorem 1.1

Since $\varphi^n = \mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}t_n}\phi^n$, $\varphi(t_n) = \mathrm{e}^{-\mathrm{i}\partial_{x_1x_2}t_n}\phi(t_n)$, we only need to prove that the conclusion of Theorem 1.1 holds for φ^n and $\varphi(t_n)$.

From (3.1), we have that

$$\varphi^{n+1} = \varphi^n + I^n(\varphi^n) + J_1^n(\varphi^n) + J_2^n(\varphi^n).$$

Then, we get that

$$\varphi^{n+1} - \varphi(t_{n+1}) = \Phi^n(\varphi^n) - \Phi^n(\varphi(t_n)) + \Phi^n(\varphi(t_n)) - \varphi(t_{n+1}) + J_1^n(\varphi^n) + J_2^n(\varphi^n).$$

By Lemmas 2.6 and 2.7, we find that

$$\|\varphi(t_{n+1}) - \Phi^n(\varphi(t_n))\|_{H^\gamma} \le C\tau^2,$$

and

$$\|\Phi^n(\varphi^n) - \Phi^n(\varphi(t_n))\|_{H^\gamma} \le (1+C\tau)\|\varphi^n - \varphi(t_n)\|_{H^\gamma} + C\tau\|\varphi^n - \varphi(t_n)\|_{H^\gamma}^3.$$

From (3.5), Lemma 3.1 and Lemma 3.2, we have that

$$\begin{aligned}|H^n(\varphi^n)| &\le \left(\|\varphi^n\|_{L^2}^2\right)^{-1}\Big(|\langle I^n(\varphi^n),\varphi^n\rangle| + \frac{1}{2}\|I^n(\varphi^n)\|_{L^2}^2\Big)\\ &\le C\tau^2\left(\|\varphi^n\|_{H^\gamma}^2 + \|\varphi^n\|_{H^\gamma}^4\right) \le C\tau^2(1+\|\varphi^n-\varphi(t_n)\|_{H^\gamma}^4),\end{aligned} \tag{3.11}$$

which yields that

$$\|J_1^n(\varphi^n)\|_{H^\gamma} = |H^n(\varphi^n)|\,\|\varphi^n\|_{H^\gamma} \le C\tau^2(1+\|\varphi^n-\varphi(t_n)\|_{H^\gamma}^5). \tag{3.12}$$

Similarly, from (3.4), (3.11), Lemma 3.1 and Lemma 3.2, we have that

$$\begin{aligned}\|J_2^n(\varphi^n)\|_{H^\gamma} &\le \frac{1}{2}|H^n(\varphi^n)|^2\|\varphi^n\|_\gamma + \left(\|\varphi^n\|_{L^2}^2\right)^{-1}|H^n(\varphi^n)||\langle I^n(\varphi^n),\varphi^n\rangle|\|\varphi^n\|_\gamma\\ &\le C\tau^4(1+\|\varphi^n-\varphi(t_n)\|_{H^\gamma}^9).\end{aligned}$$

Putting this together with the above estimates, we conclude that, for any $\tau \le 1$,

$$\|\varphi^{n+1}-\varphi(t_{n+1})\|_{H^\gamma} \le C\tau^2 + (1+C\tau)\|\varphi^n-\varphi(t_n)\|_{H^\gamma} + C\tau\|\varphi^n-\varphi(t_n)\|_{H^\gamma}^9, \tag{3.13}$$

where the constant C depends only on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

By iteration and the Grönwall's inequalities, we get that

$$\left\|\varphi(t_n)-\varphi^n\right\|_{H^\gamma} \le C\tau^2\sum_{j=0}^{n}(1+C\tau)^j \le C\tau, \quad n=0,1,\cdots,\frac{T}{\tau}, \tag{3.14}$$

which implies the first-order convergence and the following estimate:

$$\|\varphi^n\|_{H^\gamma} \le C, \quad n=0,1,\cdots,\frac{T}{\tau}. \tag{3.15}$$

Here the positive constant C depends only on T and $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

From the DS-II system (1.4), we know that $\psi^n = -(-\Delta)^{-1}(\partial_x+\partial_y)|\phi^n|^2$. Meanwhile, using the first estimate in (1.14), we have that

$$\begin{aligned}\|\psi(t_n)-\psi^n\|_{H^{\gamma+1}} &\le \left\|-(-\Delta)^{-1}(\partial_{x_1}+\partial_{x_2})\left(|\phi(t_n)|^2-|\phi^n|^2\right)\right\|_{H^{\gamma+1}}\\ &\le C\left\||\phi(t_n)|^2-|\phi^n|^2\right\|_{H^\gamma}\\ &\le C\|\phi(t_n)-\phi^n\|_{H^\gamma}(\|\phi(t_n)-\phi^n\|_{H^\gamma}+\|\phi(t_n\|_{H^\gamma})\\ &\le C\tau,\end{aligned}$$

where the constant C depends only on $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$. This proves (1.14).

Now, we prove the almost mass conservation law. From (3.1), we have that

$$\begin{aligned}\|\varphi^{n+1}\|_{L^2}^2 &= \left\langle\varphi^{n+1},\varphi^{n+1}\right\rangle\\ &= \|\varphi^n\|_{L^2}^2 + 2\langle I^n(\varphi^n),\varphi^n\rangle + 2\langle J_1^n(\varphi^n),\varphi^n\rangle + \|I^n(\varphi^n)\|_{L^2}^2\\ &\quad + 2\langle J_2^n(\varphi^n),\varphi^n\rangle + 2\langle I^n(\varphi^n),J_1^n(\varphi^n)\rangle + \|J_1^n(\varphi^n)\|_{L^2}^2\\ &\quad + 2\langle I^n(\varphi^n),J_2^n(\varphi^n)\rangle + 2\langle J_1^n(\varphi^n),J_2^n(\varphi^n)\rangle + \|J_2^n(\varphi^n)\|_{L^2}^2\\ &:= \|\varphi^n\|_{L^2}^2 + I + II + III.\end{aligned} \tag{3.16}$$

By combining (3.3), (3.4) and (3.5), we get that

$$I=0, \quad II=0.$$

From Lemma 3.2, (3.3), (3.4), (3.11) and (3.14), we obtain that

$$2\left|\left\langle I^n(\varphi^n), J_2^n(\varphi^n)\right\rangle\right| \le \left|2\big(\|\varphi^n\|_{L^2}^2\big)^{-1} H^n(\varphi^n)\left\langle I^n(\varphi^n), \varphi^n\right\rangle^2\right| + \left|\big(H^n(\varphi^n)\big)^2\left\langle I^n(\varphi^n), \varphi^n\right\rangle\right| \le C\tau^6,$$

$$2\left|\left\langle J_1^n(\varphi^n), J_2^n(\varphi^n)\right\rangle\right| \le \left|\left[\big(H^n(\varphi^n)\big)^3\|\varphi^n\|_{L^2}^2 + 2\big(H^n(\varphi^n)\big)^2\left\langle I^n(\varphi^n), \varphi^n\right\rangle\right]\right| \le C\tau^6$$

and

$$\|J_2^n(\varphi^n)\|_{L^2}^2 \le \big(\|\varphi^n\|_{L^2}^2\big)^{-1}\big(H^n(\varphi^n)\big)^2\left\langle I^n(\varphi^n), \varphi^n\right\rangle + \big(H^n(\varphi^n)\big)^3\left\langle I^n(\varphi^n), \varphi^n\right\rangle + \frac{1}{4}\big(H^n(\varphi^n)\big)^4\|\varphi^n\|_{L^2}^2 \le C\tau^8.$$

Then, we have that

$$III \le C\tau^6.$$

Therefore, we conclude that

$$\big|\|\varphi^{n+1}\|_{L^2}^2 - \|\varphi^n\|_{L^2}^2\big| \le C\tau^6; \tag{3.17}$$

that is,

$$|M(\varphi^{n+1}) - M(\varphi^n)| \le C\tau^6. \tag{3.18}$$

Then get that

$$|M(\varphi^n) - M(\phi_0)| \le C\tau^5.$$

This finishes the proof of Theorem 1.1.

4 Numerical Experiments

In this section, we present the numerical experiments of the scheme to justify the main theorem. Since ψ^n can be calculated via equation (1.4), we only need to test ϕ^n in this section. To get the initial data with the desired regularity, we construct $\phi_0(x)$ through the following strategy [23]: we choose $N = 2^9$ as an even integer and discrete the spatial domain $\mathbb{T}^2$ with grid points $x_{j,k} = (\frac{2j\pi}{N}, \frac{2k\pi}{N}),\ j,k = 0,\cdots,N$. Then we take a uniformly distributed random array $\mathrm{rand}(N,N) \in [0,1]^{N\times N}$ and an $N\times N$ vector $\boldsymbol{\Phi}$ whose elements were defined as

$$\boldsymbol{\Phi}_{j,k,l} = \mathrm{rand}(N,N) + i\,\mathrm{rand}(N,N), \quad (j,k = 0,\cdots,N-1).$$

In our numerical experiments, we set that

$$\phi_0(\boldsymbol{x}) := \frac{|\partial_{\boldsymbol{x},N}|^{-\gamma}\boldsymbol{\Phi}}{\||\partial_{\boldsymbol{x},N}|^{-\gamma}\boldsymbol{\Phi}\|_{L^\infty}}, \quad \boldsymbol{x} \in \mathbb{T}^2, \tag{4.1}$$

where the pseudo-differential operator $|\partial_{\boldsymbol{x},N}|^{-\gamma}$ for $\gamma \ge 0$ reads as follows: for Fourier modes $\boldsymbol{k} = (k_1,k_2)$ and $k_j = -N/2,\cdots,N/2-1$, for $j = 1,2$,

$$\big(|\partial_{\boldsymbol{x},N}|^{-\gamma}\big)_{\boldsymbol{k}} = \begin{cases} |\boldsymbol{k}|^{-\gamma}, & \text{if } \boldsymbol{k} \ne 0, \\ 0, & \text{if } \boldsymbol{k} = 0. \end{cases}$$

Since the almost mass conservation scheme given in this work has high accuracy, direct numerical calculation may not capture the convergence order of the mass error. In this experiment, we enlarge the initial value by 10^5 times and calculate the relative errors. The results are shown in Figures 1 and 2. They illustrate that scheme (3.1) achieves first-order convergence in H^γ and fifth-order mass convergence for the initial data in $H^{\gamma+1}, \gamma = 2,3$, and gives higher order mass convergence than the scheme given in [5].

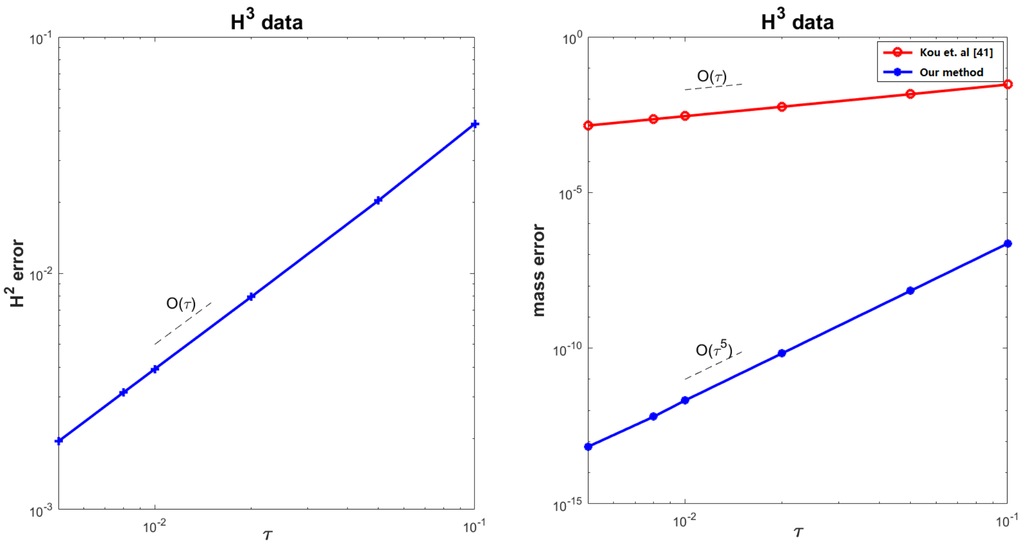

Figure 1 Convergence of (3.1): error $\frac{\|\phi_{ref}-\phi^n\|_{H^\gamma}}{\|\phi_{ref}\|_{H^\gamma}}$ (left) and error $\frac{|M(\phi^n)-M_0|}{|M_0|}$ (right) when $\gamma=2$

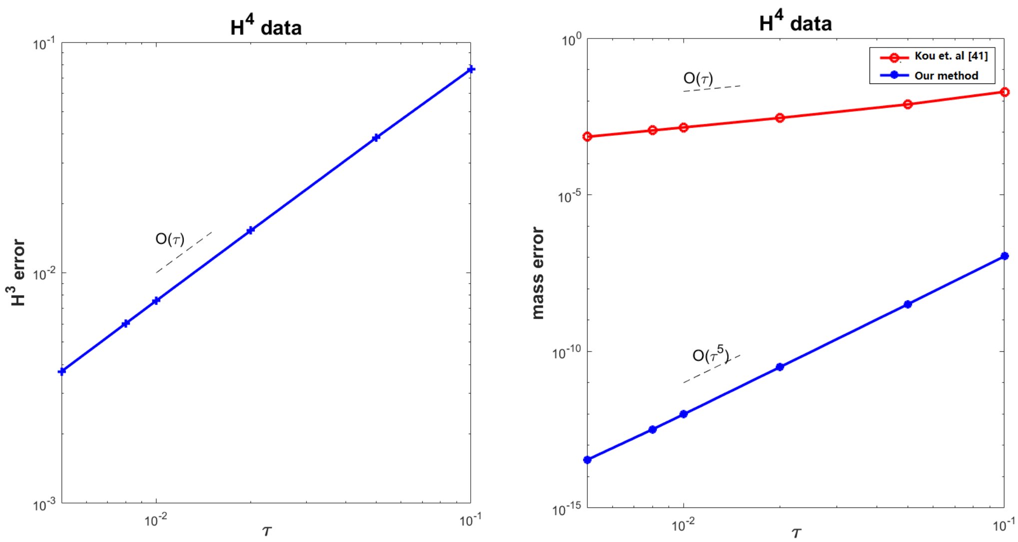

Figure 2 Convergence of (3.1): error $\frac{\|\phi_{ref}-\phi^n\|_{H^\gamma}}{\|\phi_{ref}\|_{H^\gamma}}$ (left) and error $\frac{|M(\phi^n)-M_0|}{|M_0|}$ (right) when $\gamma=3$

5 Conclusion

In this work, we constructed a first-order Fourier integrator with almost mass conservation for solving the DS-II system on a torus under rough initial data. Based on the numerical scheme in [5], we designed a modified numerical scheme to obtain the first-order convergence in $H^\gamma \times H^{\gamma+1}$ with rough initial data in $H^{\gamma+1} \times H^{\gamma+1}$, and the fifth-order mass convergence. In addition, we found that the scheme could be readily extended to construct the arbitrary high-order mass convergence.

Conflict of Interest The authors declare no conflict of interest.

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