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THE STABILITY OF BOUSSINESQ EQUATIONS WITH PARTIAL DISSIPATION AROUND THE HYDROSTATIC BALANCE

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Abstract This paper is devoted to understanding the stability of perturbations around the hydrostatic equilibrium of the Boussinesq system in order to gain insight into certain atmospheric and oceanographic phenomena. The Boussinesq system focused on here is anisotropic, and involves only horizontal dissipation and thermal damping. In the 2D case $\mathbb{R}^2$, due to the lack of vertical dissipation, the stability and large-time behavior problems have remained open in a Sobolev setting. For the spatial domain $\mathbb{T} \times \mathbb{R}$, this paper solves the stability problem and gives the precise large-time behavior of the perturbation. By decomposing the velocity u and temperature θ into the horizontal average $(\bar{u}, \bar{\theta})$ and the corresponding oscillation $(\tilde{u}, \tilde{\theta})$, we can derive the global stability in H^2 and the exponential decay of $(\tilde{u}, \tilde{\theta})$ to zero in H^1. Moreover, we also obtain that $(\bar{u}_2, \bar{\theta})$ decays exponentially to zero in H^1, and that $\bar{u}_1$ decays exponentially to $\bar{u}_1(\infty)$ in H^1 as well; this reflects a strongly stratified phenomenon of buoyancy-driven fluids. In addition, we establish the global stability in H^3 for the 3D case $\mathbb{R}^3$.

Key words Boussinesq equations; partial dissipation; stability; decay

2020 MR Subject Classification 35Q35; 35Q86; 76D03; 76D50

1 Introduction

The main purpose of this paper is to study the stability and large-time behavior of the following two-dimensional Boussinesq equations:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \mu \partial_{11} u + \theta e_2, \\ \theta_t + u \cdot \nabla \theta + u_2 + \eta \theta = 0, \\ \mathrm{div} u = 0, \qquad (u, \theta)|_{t=0} = (u_0, \theta_0). \end{cases} \tag{1.1}$$

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Here $u(x,t)$, $p(x,t)$, $\theta(x,t)$ denote the velocity field, pressure, scalar temperature, respectively. The constant μ represents the viscosity, η stands for the thermal damping coefficient, and $e_2 = (0,1)^{\mathrm{T}}$. System (1.1) comes from the 2D Boussinesq equations without thermal diffusion,

$$\begin{cases} v_t + v\cdot\nabla v + \nabla P = \mu\partial_{11}v + \Theta e_2, \\ \Theta_t + v\cdot\nabla\Theta = 0, \\ \mathrm{div} u = 0, \\ (u,\Theta)|_{t=0} = (u_0,\Theta_0). \end{cases} \tag{1.2}$$

The perturbation around the hydrostatic equilibrium of (1.2) is

$$u = v - v_{he}, \quad \theta = \Theta - \Theta_{he}, \quad p = P - P_{he},$$

where

$$v_{he} = (0,0)^T, \quad \Theta_{he} = x_2, \quad P_{he} = \frac{1}{2}x_2^2. \tag{1.3}$$

Furthermore, the perturbed temperature equation adds a damping term $\eta\theta$.

The Boussinesq system models buoyancy-driven fluids such as atmospheric and oceanographic flows, where rotation and stratification play an important role. Enormous effort has been made in this area by many researchers, and they have observed that the 2D Boussinesq equations share an analogous feature with the 3D incompressible Euler or Navier-Stokes equations for axisymmetric swirling flow, and that they have a similar vortex stretching effect to that in three dimensions (see [32, 33, 36], for instance).

There has been substantial progress made on fundamental mathematical issues such as the global existence and regularity of various 2D Boussinesq systems, particularly those with only partial dissipation or fractional dissipation, or indeed no dissipation at all (see for example [2–5, 8–14, 20, 22–28, 31, 35, 41–43]). Due to the physical applications, for instance, in atmospherics and astrophysics, recent investigations on the Boussinesq equations have focused on the stability problem of perturbations around several physically relevant steady states, such as the hydrostatic equilibrium (1.3) and shear flow. The work of Doering *et al* [16] initiated the rigorous study of the stability near the hydrostatic equilibrium of the 2D Boussinesq equations with only velocity dissipation. Later, Tao *et al* [38] established the large-time behavior and the eventual temperature profile. Dong *et al* [19] studied the stability and large time behavior of the 2D Boussinesq system without thermal diffusion under a different boundary condition. In addition, Castro *et al* [7] proved the stability and large-time behavior with only velocity damping instead of dissipation in 2D, and Wan considered the same case with velocity damping in [40]. Other results on perturbations near the shear flow can be found in [15, 37, 44].

Here we list some results on the stability problem very relevant to the Boussinesq equations (1.1). Lai *et al* [29] studied the stability and large-time behavior of the Boussinesq equations with only vertical velocity dissipation and thermal damping in $\mathbb{R}^2$. Later, a followup work by Lai *et al* [30] gave the optimal decay of the stability problem. Ben Said *et al* investigated the stability and decay of the 2D Boussinesq equation with only vertical dissipation and horizontal thermal diffusion [6] in $\mathbb{R}^2$, and with horizontal dissipation and vertical thermal diffusion [1] in $\mathbb{T}\times\mathbb{R}$. The work of Dong *et al* [18] established the stability and exponential decay with horizontal dissipation in $\mathbb{T}\times\mathbb{R}$. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1]$ denotes a 1D periodic box. Motivated by these works, we consider the 2D Boussinesq equations with only horizontal dissipation and thermal damping in $\mathbb{T}\times\mathbb{R}$. Now, we will state our main ideas regarding this problem.

If the temperature is not a concern, then the Boussinesq equations (1.1) can be reduced to anisotropic Navier-Stokes equations as follows:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \mu\partial_{11}u, \\ \mathrm{div} u = 0. \end{cases} \tag{1.4}$$

Taking the differential operator $\nabla\times$ on (1.4) and denoting that $\omega = \nabla\times u$, we have the following vorticity equation:

$$\omega_t + u\cdot\nabla\omega = \mu\partial_{11}\omega. \tag{1.5}$$

The L^2-estimation on $\nabla\omega$ gives that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2 + \mu\|\partial_1\nabla\omega\|_{L^2}^2 &= -\int \nabla u\cdot\nabla\omega\cdot\nabla\omega \\ &= -\int \partial_1 u_1\partial_1\omega\partial_1\omega - \int \partial_1 u_2\partial_2\omega\partial_1\omega \\ &\quad - \int \partial_2 u_1\partial_1\omega\partial_2\omega - \int \partial_2 u_2\partial_2\omega\partial_2\omega. \end{aligned} \tag{1.6}$$

It is easy to observe that the last two terms are difficult to control, due to the lack of vertical dissipation. Therefore, the global well-posedness of anisotropic Navier-Stokes equations above still remains open in the Sobolev setting in $\mathbb{R}^2$. To our knowledge, the difficulty in $\mathbb{R}^2$ can be overcome when the spatial domain is replaced by $\mathbb{T}\times\mathbb{R}$, just as in [17]. The key point of [17] is very simple; it introduces the horizontal average $\bar{f}$ and the corresponding oscillation $\tilde{f}$ of f as

$$\bar{f} = \int_{\mathbb{T}} f(x)\mathrm{d}x_1, \quad \tilde{f} = f - \bar{f}. \tag{1.7}$$

Then we have that

$$\begin{aligned} \int \partial_2 u_1\partial_1\omega\partial_2\omega &= \int \partial_2(\tilde{u}_1+\bar{u}_1)\partial_1\tilde{\omega}\partial_2(\tilde{\omega}+\bar{\omega}) \\ &= \int \partial_2\tilde{u}_1\partial_1\tilde{\omega}\partial_2\tilde{\omega} + \int \partial_2\tilde{u}_1\partial_1\tilde{\omega}\partial_2\bar{\omega} + \int \partial_2\bar{u}_1\partial_1\tilde{\omega}\partial_2\tilde{\omega}, \end{aligned} \tag{1.8}$$

where we have used the orthogonality $\int \tilde{f}\bar{f} = 0$ and $\partial_1\bar{\omega} = 0$ to eliminate the bad term $\int \partial_2\bar{u}_1\partial_1\tilde{\omega}\partial_2\bar{\omega}$.

By applying the anisotropic inequality and Poincaré's inequality,

$$\|\tilde{f}\|_{H^s} \lesssim \|\partial_1\tilde{f}\|_{H^s}, \tag{1.9}$$

the third term of (1.6) can be controlled by the horizontal dissipation. The last term is the same.

When considering the temperature, we note that u_2 in $(1.1)_2$ can offer us a damping term, very analogous to θ. However, $\nabla u_2 = (\partial_1 u_2, -\partial_1 u_1)$ means that the damping of u_2 cannot provide more information on the dissipation than in the case of horizontal dissipation; this makes the situation very different to [29]. This indicates that in $\mathbb{R}^2$, the global well-posedness of (1.1) is also open. All of these things impel us to consider the Boussinesq equations (1.1) in $\mathbb{T}\times\mathbb{R}$.

Changing (1.1) into the first-order derivative formulation

$$\begin{cases}\omega_t + u\cdot\nabla\omega = \mu\partial_{11}\omega + \partial_1\theta,\\ \partial_t(\nabla\theta) + \nabla(u\cdot\nabla\theta) + \nabla u_2 + \eta\nabla\theta = 0,\\ \mathrm{div}u = 0,\end{cases} \tag{1.10}$$

the L^2-inner product gives that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\omega,\nabla\theta)\|_{L^2}^2 + \mu\|\partial_1\omega\|_{L^2}^2 + \eta\|\nabla\theta\|_{L^2}^2 = -\int \nabla u\cdot\nabla\theta\cdot\nabla\theta. \tag{1.11}$$

Since θ has no higher-order dissipation, the right hand side of (1.11) can be only bounded by

$$\int \nabla u\cdot\nabla\theta\cdot\nabla\theta \le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2. \tag{1.12}$$

If the domain is $\mathbb{R}^2$, the anisotropic inequality

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^2}^{\frac{1}{4}}\|\partial_1 f\|_{L^2}^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 f\|_{L^2}^{\frac{1}{4}} \tag{1.13}$$

gives that

$$\int \nabla u\cdot\nabla\theta\cdot\nabla\theta \lesssim \|\nabla u\|_{H^1}^{\frac{1}{2}}\|\partial_1\nabla u\|_{H^1}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^2 \lesssim \|u\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^2, \tag{1.14}$$

which means that the H^2 Sobolev setting is sufficient to provide the dissipation $\|(\partial_1 u,\theta)\|_{H^2}^2$. However, in $\mathbb{T}\times\mathbb{R}$, the anisotropic inequality

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^2}^{\frac{1}{4}}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}(\|\partial_2 f\|_{L^2} + \|\partial_1\partial_2 f\|_{L^2})^{\frac{1}{4}} \tag{1.15}$$

is very different, and the horizontal dissipation $\|\partial_1\nabla u\|_{L^2}$ cannot be separated from $\|\nabla u\|_{L^\infty}$. We note that the thermal damping may provide a good dissipation for θ, which makes the separation of horizontal velocity dissipation unnecessary. This means that we can directly by (1.15), get that

$$\|\nabla u\|_{L^\infty} \lesssim \|u\|_{H^2} + \|\partial_1 u\|_{H^2}. \tag{1.16}$$

Thus the Sobolev setting H^2 suffices to close the energy estimates.

When the spatial domain is

$$\Omega = \mathbb{T}\times\mathbb{R}, \tag{1.17}$$

with $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1]$ being a 1D periodic box, the desired stability problem on (1.1) is solvable. For simplicity, we set that $\mu = \eta = 1$. Then we have the following result:

Theorem 1.1 Let $\Omega = \mathbb{T}\times\mathbb{R}$. Assume that $(u_0,\theta_0)\in H^2(\Omega)$ and $\mathrm{div}u_0 = 0$. Then there exists a sufficiently small $\varepsilon > 0$ such that, if

$$\|(u_0,\theta_0)\|_{H^2} \le \varepsilon, \tag{1.18}$$

then (1.1) has a unique small global solution satisfying that

$$\|(u,\theta)(t)\|_{H^2}^2 + \int_0^t (\|u_2(\tau)\|_{L^2}^2 + \|(\partial_1 u,\theta)(\tau)\|_{H^2}^2)\mathrm{d}\tau \le C\varepsilon^2 \tag{1.19}$$

for some generic constant $C > 0$ and all $t > 0$.

The key point in this paper is that Ω allows us to separate the horizontal average from the corresponding oscillation. These two elements have different physical behaviors, as can be seen in the numerical results of [16]. In fact, we obtain that the oscillation part $(\tilde{u}, \tilde{\theta})$ in (1.1) could decay exponentially. By analyzing the structure of the horizontal average in (1.1) and combining the exponential decay of the oscillation part, we also get that the horizontal average part $(\bar{u}_2, \bar{\theta})$ has analogous exponential decay, and that $\bar{u}_1$ decays exponentially to $\bar{u}_1(\infty)$.

Theorem 1.2 Let $(u_0, \theta_0) \in H^2(\Omega)$ and $\mathrm{div} u_0 = 0$. Assume that $\|(u_0, \theta_0)\|_{H^2} \le \varepsilon$ for sufficiently small $\varepsilon > 0$. Let (u, θ) be the corresponding solution to (1.1). Then there exist constants $C_2 < C_1$ such that

$$\|(\tilde{u}, \tilde{\theta})(t)\|_{H^1} \le \|(u_0, \theta_0)\|_{H^1} \mathrm{e}^{-C_1 t}, \tag{1.20}$$

and

$$\|(\bar{u}_1 - \bar{u}_1(\infty), \bar{u}_2)(t)\|_{H^1} \lesssim \|(u_0, \theta_0)\|_{H^1} \mathrm{e}^{-C_1 t}, \ \|\bar{\theta}(t)\|_{H^1} \lesssim \|(u_0, \theta_0)\|_{H^1} \mathrm{e}^{-C_2 t}, \tag{1.21}$$

where $\bar{u}_1(\infty) = \lim\limits_{t \to \infty} \bar{u}_1(t)$. In fact, here we have that $\bar{u}_2 = 0$.

To prove the exponential decay, we need to decompose the Boussinesq equations (1.1) into the oscillation part $(\tilde{u}, \tilde{\theta})$,

$$\begin{cases} \partial_t \tilde{u} + \widetilde{u \cdot \nabla \tilde{u}} + u_2 \partial_2 \bar{u} - \partial_{11} \tilde{u} + \nabla \tilde{p} = \tilde{\theta} e_2, \\ \partial_t \tilde{\theta} + \widetilde{u \cdot \nabla \tilde{\theta}} + u_2 \partial_2 \bar{\theta} + \tilde{\theta} + \tilde{u}_2 = 0, \end{cases} \tag{1.22}$$

and the average part $(\bar{u}, \bar{\theta})$,

$$\begin{cases} \partial_t \bar{u} + \overline{u \cdot \nabla \tilde{u}} + (0, \partial_2 \bar{p})^T = \bar{\theta} e_2, \\ \partial_t \bar{\theta} + \overline{u \cdot \nabla \tilde{\theta}} + \bar{\theta} + \bar{u}_2 = 0. \end{cases} \tag{1.23}$$

By taking energy estimates on (1.22) for $\|(\tilde{u}, \tilde{\theta})\|_{L^2}^2$, $\|(\nabla \tilde{u}, \nabla \tilde{\theta})\|_{L^2}^2$, and carefully evaluating the nonlinear terms with Poincaré's inequality and other anisotropic inequalities, we can establish the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|(\tilde{u}, \tilde{\theta})\|_{H^1}^2 + (2 - C\|(u, \theta)\|_{H^2}) \|(\partial_1 \tilde{u}, \tilde{\theta})\|_{H^1}^2 \le 0. \tag{1.24}$$

When the initial data (u_0, θ_0) is sufficiently small, i.e.,

$$\|(u_0, \theta_0)\|_{H^2} \le \varepsilon \tag{1.25}$$

with $\varepsilon > 0$ being sufficiently small, then $\|(u, \theta)\|_{H^2} \le C\varepsilon$ and

$$2 - C\|(u, \theta)\|_{H^2} \ge 1. \tag{1.26}$$

An application of Poincaré's inequality gives the desired exponential decay.

Similarly, it is easy to observe that $(\bar{u}_2, \bar{\theta})$ satisfies that

$$\begin{cases} \partial_t \bar{u}_2 + \overline{u \cdot \nabla \tilde{u}_2} + \partial_2 \bar{p} = \bar{\theta}, \\ \partial_t \bar{\theta} + \overline{u \cdot \nabla \tilde{\theta}} + \bar{\theta} + \bar{u}_2 = 0, \end{cases} \tag{1.27}$$

and that $(\bar{u}_2, \bar{\theta})$ has the damping dissipation in $(1.27)_2$. This provides the possibility of exponential decay for $(\bar{u}_2, \bar{\theta})$ as long as the nonlinear terms in (1.27) are controlled by the exponential decay estimates on $(\tilde{u}, \tilde{\theta})$ and the structure of pressure p. In fact, by noting that $\bar{u}_2 = 0$, we only need to give the exponential decay of $\bar{\theta}$.

The estimate on $\bar{u}_1$ is very simple. Note that $\bar{u}_1$ satisfies the equation

$$\partial_t \bar{u}_1 + \overline{u \cdot \nabla \tilde{u}_1} = 0. \tag{1.28}$$

This means that, by the exponential decay of $(\tilde{u},\tilde{\theta})$, $\bar{u}_1(t)$ has the limit $\bar{u}_1(\infty)$, and decays exponentially to $\bar{u}_1(\infty)$ in H^1.

Next we also show the stability of the 3D case in $\mathbb{R}^3$. Consider the 3D Boussinesq system with partial dissipation

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \Delta_h u + \theta e_3, \\ \theta_t + u\cdot\nabla\theta + \theta + u_3 = 0, \\ \mathrm{div} u = 0, \\ (u,\theta)|_{t=0} = (u_0,\theta_0). \end{cases} \tag{1.29}$$

Here Δ_h denotes the horizontal Laplacian operator.

The reason for choosing the domain $\mathbb{R}^3$ instead of $\mathbb{T}^2\times\mathbb{R}$ is that the anisotropic inequality

$$\int_{\mathbb{R}^3} fgh\mathrm{d}x \lesssim \|f\|_{L^2}^{\frac{1}{2}}\|\partial_1 f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_3 f\|_{L^2}^{\frac{1}{2}} \tag{1.30}$$

suffices to recover the desired horizontal dissipation. Now we give the following stability result:

Theorem 1.3 Let $\Omega=\mathbb{R}^3$. Assume that $(u_0,\theta_0)\in H^3(\Omega)$ and $\mathrm{div}u_0=0$. Then there exists a sufficiently small $\varepsilon>0$ such that, when

$$\|(u_0,\theta_0)\|_{H^3}\le\varepsilon, \tag{1.31}$$

(1.29) has a unique small global solution so that

$$\|(u,\theta)(t)\|_{H^3}^2 + \int_0^t (\|u_3(\tau)\|_{L^2}^2 + \|(\nabla_h u,\theta)(\tau)\|_{H^3}^2)\mathrm{d}\tau \le C\varepsilon^2 \tag{1.32}$$

for some generic constant $C>0$ and all $t>0$.

The rest of this paper is divided into four sections. Section 2 presents several anisotropic inequalities and some properties on the orthogonal decomposition. Sections 3 and 4 prove Theorem 1.1 and Theorem 1.2, respectively while Section 5 is devoted to establishing Theorem 1.3.

Notation $a\lesssim b$ denotes that $a\le Cb$ for some generic constant. For simplicity, $\int f := \int_\Omega f\mathrm{d}x$ and $\int_0^t f = \int_0^t f(\tau)\mathrm{d}\tau$. $H^k(\Omega)$ denotes the classical Sobolev spaces, and $L^p(\Omega)$ stands for the classical Lebesgue space with the L^p norm.

2 Preliminaries

This section states several anisotropic inequalities to be used extensively in the proof of Theorems 1.1 and 1.2. Moreover, some key properties of the horizontal average and the corresponding oscillation are also listed here.

Since the domain is $\Omega=\mathbb{T}\times\mathbb{R}$, we define the horizontal average of $f=f(x_1,x_2)$ as

$$\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1,x_2)\mathrm{d}x_1. \tag{2.1}$$

Then we decompose f into $\bar{f}$ and the corresponding oscillation part $\tilde{f}$:

$$f = \bar{f} + \tilde{f}. \tag{2.2}$$

The next lemma presents a few properties of $(\bar{f},\tilde{f})$.

Lemma 2.1 Let $\Omega = \mathbb{T} \times \mathbb{R}$. Then

(a) $(\bar{f}, \tilde{f})$ obeys the properties

$$\overline{\partial_1 f} = \partial_1 \bar{f} = 0, \quad \overline{\partial_2 f} = \partial_2 \bar{f}, \quad \bar{\tilde{f}} = 0, \quad \widetilde{\partial_2 f} = \partial_2 \tilde{f}, \quad \widetilde{\partial_1 f} = \partial_1 f = \partial_1 \tilde{f}; \tag{2.3}$$

(b) if f is divergence-free, namely, $\mathrm{div} f = 0$, then $\bar{f}$ and $\tilde{f}$ are both divergence-free, i.e.,

$$\mathrm{div}\bar{f} = \mathrm{div}\tilde{f} = 0. \tag{2.4}$$

(c) $\bar{f}$ and $\tilde{f}$ are orthogonal in L^2, namely,

$$(\bar{f}, \tilde{f}) := \int_\Omega \bar{f}\tilde{f}\mathrm{d}x = 0, \quad \|f\|_{L^2}^2 = \|\bar{f}\|_{L^2}^2 + \|\tilde{f}\|_{L^2}^2. \tag{2.5}$$

In addition, $\|\bar{f}\|_{L^2} \le \|f\|_{L^2}$ and $\|\tilde{f}\|_{L^2} \le \|f\|_{L^2}$.

Proof It is easy to verify (a). If $\mathrm{div} f = 0$, then

$$\mathrm{div}\bar{f} = \overline{\mathrm{div} f} = 0, \quad \mathrm{div}\tilde{f} = \widetilde{\mathrm{div}} f = 0. \tag{2.6}$$

For (c), by the definitions of $\bar{f}$ and $\tilde{f}$,

$$(\bar{f}, \tilde{f}) = \int_\Omega \bar{f}\tilde{f}\mathrm{d}x = \int_{\mathbb{R}} \bar{f}(\int_{\mathbb{T}} \tilde{f}(x_1, x_2)\mathrm{d}x_1)\mathrm{d}x_2 = 0. \tag{2.7}$$

This completes the proof of Lemma 2.1. □

Next we introduce some important lemmas related to the anisotropic inequalities.

Lemma 2.2 For any 1D function $f(x)$,

(1) supposing that $f(x) \in H^1(\mathbb{R})$, then

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|\partial_x f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}; \tag{2.8}$$

(2) supposing that $f(x) \in H^1(\mathbb{T})$, then

$$\|f\|_{L^\infty(\mathbb{T})} \le \sqrt{2}\|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x f\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L^2(\mathbb{T})}. \tag{2.9}$$

In particular,

$$\|\tilde{f}\|_{L^\infty(\mathbb{T})} \le \sqrt{2}\|\tilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x \tilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}}. \tag{2.10}$$

Proof This Lemma is actually a special case of the Gagliardo-Nirenberg inequality [34]. For (1), since

$$|f(x)|^2 = \int_\infty^x 2f(y)\partial_x f(y)\mathrm{d}y, \quad \text{for any } x \in \mathbb{R}, \tag{2.11}$$

we have that

$$\|f\|_{L^\infty(\mathbb{R})}^2 \le \int_{\mathbb{R}} 2|f(y)\partial_x f(y)|\mathrm{d}y \le 2\|f\|_{L^2(\mathbb{R})}\|\partial_x f\|_{L^2(\mathbb{R})}. \tag{2.12}$$

Similarly, for (2), by the mean-value theorem, there exists a $s \in \mathbb{T}$ such that

$$|f(x)|^2 - |\bar{f}|^2 = |f(x)|^2 - |f(s)|^2 = \int_s^x 2f(y)\partial_x f(y)\mathrm{d}y, \quad \text{for any } x \in \mathbb{T}, \tag{2.13}$$

which gives that

$$\|f\|_{L^\infty(\mathbb{T})}^2 \le \int_{\mathbb{T}} 2|f(y)\partial_x f(y)|\mathrm{d}y + |\bar{f}|^2 \le 2\|f\|_{L^2(\mathbb{T})}\|\partial_x f\|_{L^2(\mathbb{T})} + \|f\|_{L^2(\mathbb{T})}^2. \tag{2.14}$$

Thus,

$$\|f\|_{L^\infty(\mathbb{T})} \le (2\|f\|_{L^2(\mathbb{T})}\|\partial_x f\|_{L^2(\mathbb{T})} + \|f\|_{L^2(\mathbb{T})}^2)^{\frac{1}{2}} \le \sqrt{2}\|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x f\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L^2(\mathbb{T})}. \tag{2.15}$$

This completes the proof of Lemma 2.2. □

Lemma 2.2 can be immediately applied to the 2D case, just like Poincaré's inequality and the anisotropic inequalities.

Lemma 2.3 Let $\Omega = \mathbb{T}\times\mathbb{R}$. Then, for any integer $k\geq 0$,

$$\|\tilde{f}\|_{H^k} \lesssim \|\partial_1\tilde{f}\|_{H^k} \lesssim \|\partial_1 f\|_{H^k}. \tag{2.16}$$

In addition,

$$\|\tilde{f}\|_{L^\infty(\Omega)} \lesssim \|\partial_1\tilde{f}\|_{H^1(\Omega)} \lesssim \|\partial_1 f\|_{H^1(\Omega)}. \tag{2.17}$$

Proof Since, by the mean-value theorem, there exists $s\in\mathbb{T}$ such that $\tilde{f}(s,x_2)=0$, we immediately, get by Hölder's inequality, that

$$|\partial^\alpha\tilde{f}(x_1,x_2)| = \left|\int_s^{x_1}\partial^\alpha\partial_1\tilde{f}(y_1,x_2)\mathrm{d}y_1\right| \leq \left(\int_{\mathbb{T}}|\partial^\alpha\partial_1\tilde{f}(y_1,x_2)|^2\mathrm{d}y_1\right)^{\frac{1}{2}}, \tag{2.18}$$

with a multi-index α satisfying that $|\alpha|\leq k$. This implies that

$$\|\partial^\alpha\tilde{f}\|_{L^2} \leq \|\partial_1\partial^\alpha\tilde{f}\|_{L^2}, \tag{2.19}$$

and hence,

$$\|\tilde{f}\|_{H^k} \lesssim \|\partial_1\tilde{f}\|_{H^k}. \tag{2.20}$$

By Lemma 2.2 and (2.16), we have that

$$\begin{aligned}\|\tilde{f}\|_{L^\infty(\Omega)} &\lesssim \|\tilde{f}\|^{\frac{1}{2}}_{L^\infty_{x_2}L^2_{x_1}}\|\partial_1\tilde{f}\|^{\frac{1}{2}}_{L^\infty_{x_2}L^2_{x_1}} \lesssim \Big\|\|\tilde{f}\|_{L^\infty_{x_2}}\Big\|^{\frac{1}{2}}_{L^2_{x_1}}\Big\|\|\partial_1\tilde{f}\|_{L^\infty_{x_2}}\Big\|^{\frac{1}{2}}_{L^2_{x_1}}\\ &\lesssim \Big\|\|\tilde{f}\|^{\frac{1}{2}}_{L^2_{x_2}}\|\partial_2\tilde{f}\|^{\frac{1}{2}}_{L^2_{x_2}}\Big\|^{\frac{1}{2}}_{L^2_{x_1}}\Big\|\|\partial_1\tilde{f}\|^{\frac{1}{2}}_{L^2_{x_2}}\|\partial_1\partial_2\tilde{f}\|^{\frac{1}{2}}_{L^2_{x_2}}\Big\|^{\frac{1}{2}}_{L^2_{x_1}}\\ &\lesssim \|\tilde{f}\|^{\frac{1}{4}}_{L^2}\|\partial_1\tilde{f}\|^{\frac{1}{4}}_{L^2}\|\partial_2\tilde{f}\|^{\frac{1}{4}}_{L^2}\|\partial_1\partial_2\tilde{f}\|^{\frac{1}{4}}_{L^2} \lesssim \|\partial_1\tilde{f}\|_{H^1}.\end{aligned} \tag{2.21}$$

□

Lemma 2.4 Let $\Omega=\mathbb{T}\times\mathbb{R}$. Then, for any $f,g,h\in L^2(\Omega)$ with $\partial_1 f\in L^2(\Omega)$ and $\partial_2 g\in L^2(\Omega)$, it holds that

$$\left|\int_\Omega fgh\mathrm{d}x\right| \lesssim \|f\|^{\frac{1}{2}}_{L^2}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|g\|^{\frac{1}{2}}_{L^2}\|\partial_2 g\|^{\frac{1}{2}}_{L^2}\|h\|_{L^2}. \tag{2.22}$$

In particular,

$$\left|\int_\Omega \tilde{f}gh\mathrm{d}x\right| \lesssim \|\tilde{f}\|^{\frac{1}{2}}_{L^2}\|\partial_1\tilde{f}\|^{\frac{1}{2}}_{L^2}\|g\|^{\frac{1}{2}}_{L^2}\|\partial_2 g\|^{\frac{1}{2}}_{L^2}\|h\|_{L^2}. \tag{2.23}$$

For any $f\in H^2(\Omega)$,

$$\|f\|_{L^\infty(\Omega)} \lesssim \|f\|^{\frac{1}{4}}_{L^2}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac{1}{4}}\|\partial_2 f\|^{\frac{1}{4}}_{L^2}(\|\partial_2 f\|_{L^2}+\|\partial_1\partial_2 f\|_{L^2})^{\frac{1}{4}} \tag{2.24}$$

and

$$\|\tilde{f}\|_{L^\infty(\Omega)} \lesssim \|\tilde{f}\|^{\frac{1}{4}}_{L^2}\|\partial_1\tilde{f}\|^{\frac{1}{4}}_{L^2}\|\partial_2\tilde{f}\|^{\frac{1}{4}}_{L^2}\|\partial_1\partial_2\tilde{f}\|^{\frac{1}{4}}_{L^2}. \tag{2.25}$$

Proof Directly, by Hölder's inequality and Lemma 2.2, we have that

$$\begin{aligned}\left|\int_\Omega fgh\mathrm{d}x\right| &\leq \|f\|_{L^\infty_{x_1}L^2_{x_2}}\|g\|_{L^2_{x_1}L^\infty_{x_2}}\|h\|_{L^2}\\ &\lesssim \Big\|\|f\|^{\frac{1}{2}}_{L^2_{x_1}}(\|f\|_{L^2_{x_1}}+\|\partial_1 f\|_{L^2_{x_1}})^{\frac{1}{2}}\Big\|_{L^2_{x_2}}\Big\|\|g\|^{\frac{1}{2}}_{L^2_{x_2}}\|\partial_2 g\|^{\frac{1}{2}}_{L^2_{x_2}}\Big\|_{L^2_{x_1}}\|h\|_{L^2}\end{aligned}$$

$$\lesssim \|f\|_{L^2}^{\frac{1}{2}}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}. \tag{2.26}$$

Similarly, by Lemma 2.2,

$$\begin{aligned}\|f\|_{L^\infty(\Omega)} &\lesssim \left\| \|f\|_{L^2_{x_2}}^{\frac{1}{2}} \|\partial_2 f\|_{L^2_{x_2}}^{\frac{1}{2}} \right\|_{L^\infty_{x_1}} \lesssim \left\| \|f\|_{L^\infty_{x_1}} \right\|_{L^2_{x_2}}^{\frac{1}{2}} \left\| \|\partial_2 f\|_{L^\infty_{x_1}} \right\|_{L^2_{x_2}}^{\frac{1}{2}} \\ &\lesssim \left\| \|f\|_{L^2_{x_1}}^{\frac{1}{2}} (\|f\|_{L^2_{x_1}} + \|\partial_1 f\|_{L^2_{x_1}})^{\frac{1}{2}} \right\|_{L^2_{x_2}}^{\frac{1}{2}} \left\| \|\partial_2 f\|_{L^2_{x_1}}^{\frac{1}{2}} (\|\partial_2 f\|_{L^2_{x_1}} + \|\partial_1\partial_2 f\|_{L^2_{x_1}})^{\frac{1}{2}} \right\|_{L^2_{x_2}}^{\frac{1}{2}} \\ &\lesssim \|f\|_{L^2}^{\frac{1}{4}}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}(\|\partial_2 f\|_{L^2}+\|\partial_1\partial_2 f\|_{L^2})^{\frac{1}{4}}. \end{aligned} \tag{2.27}$$

The rest of Lemma 2.4 can be obtained by applying Poincaré's inequality (2.16). □

3 Proof of Theorem 1.1

Since the local well-posedness can be shown by a standard method in [32], here we focus on the global *a priori* estimates on the solution in $H^2(\Omega)$. If we define the energy functional as

$$E(t) := \sup_{0\le\tau\le t}\|(u,\theta)(\tau)\|_{H^2}^2 + \int_0^t \|(\partial_1 u,\theta)(\tau)\|_{H^2}^2 \mathrm{d}\tau, \tag{3.1}$$

then our main efforts can be devoted to establishing the energy inequality

$$E(t) \le CE(0) + CE(t)^{\frac{3}{2}} \tag{3.2}$$

for some generic constant C and all $t > 0$. Once (3.2) holds, a standard bootstrapping argument from [39] can reveal the global bounded energy. If we make the ansatz

$$E(t) \le 2CE(0) \le 2C\varepsilon^2, \tag{3.3}$$

then (3.2) will indicate a smaller bound for $E(t)$ when $\varepsilon > 0$ is sufficiently small. In fact, if (3.3) holds, then

$$E(t) \le \frac{1}{1-\sqrt{2C}C\varepsilon}CE(0) \le \frac{3}{4}\times 2CE(0), \tag{3.4}$$

with a small ε satisfying that

$$\frac{1}{1-\sqrt{2C}C\varepsilon} \le \frac{3}{2}, \quad \text{i.e.,} \quad \varepsilon \le \frac{1}{3\sqrt{2}}C^{-\frac{3}{2}}. \tag{3.5}$$

Therefore, the bootstrapping argument asserts that $E(t)$ is bounded uniformly for all $t > 0$, namely, that

$$E(t) \le C\varepsilon^2. \tag{3.6}$$

Now we show that (3.2) holds. Since the equivalence of the Sobolev norms holds, namely,

$$\|f\|_{H^2}^2 \sim \|f\|_{L^2}^2 + \|f\|_{\dot H^2}^2, \tag{3.7}$$

we only need to give the L^2 estimates of (u,θ) and $(\nabla\omega, \Delta\theta)$. Here, $\omega = \nabla\times u$ and $\|f\|_{\dot H^2}^2 := \sum\limits_{i=1}^{2}\|\partial_i^2 f\|_{L^2}^2$. A L^2-estimate on (1.1) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,\theta)\|_{L^2}^2 + 2\|(\partial_1 u,\theta)\|_{L^2}^2 = 0. \tag{3.8}$$

To give the L^2 estimate of $(\nabla\omega, \Delta\theta)$, we make use of the second-order derivative equation of (1.1) as follows:

$$\begin{cases} \partial_t \nabla\omega + \nabla(u\cdot\nabla\omega) = \partial_{11}\nabla\omega + \nabla\partial_1\theta, \\ \partial_t(\Delta\theta) + \Delta(u\cdot\nabla\theta) + \Delta u_2 + \Delta\theta = 0. \end{cases} \tag{3.9}$$

By taking L^2-inner product with $(\nabla\omega, \Delta\theta)$, we have that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\omega, \Delta\theta)\|_{L^2}^2 + \|(\partial_1\nabla\omega, \Delta\theta)\|_{L^2}^2 \\ &= (\nabla\partial_1\theta, \nabla\omega) - (\Delta u_2, \Delta\theta) - (\nabla(u\cdot\nabla\omega), \nabla\omega) - (\Delta(u\cdot\nabla\theta), \Delta\theta) \\ &= A_1 + A_2 + A_3. \end{aligned} \tag{3.10}$$

Here it is easy to verify that

$$A_1 = (\nabla\partial_1\theta, \nabla\omega) - (\Delta u_2, \Delta\theta) = (\Delta\theta, \partial_1\omega) - (\Delta u_2, \Delta\theta) = 0, \tag{3.11}$$

where we have used the fact that $\partial_1\omega = \Delta u_2$, by the free-divergence of velocity.

Also, by $\mathrm{div} u = 0$,

$$\begin{aligned} A_2 &= -\int \nabla u\cdot\nabla\omega\cdot\nabla\omega \\ &= -\int \partial_1 u_1 \partial_1\omega\partial_1\omega - \int \partial_1 u_2 \partial_2\omega\partial_1\omega - \int \partial_2 u_1\partial_1\omega\partial_2\omega - \int \partial_2 u_2\partial_2\omega\partial_2\omega \\ &= A_{21} + A_{22} + A_{23} + A_{24}. \end{aligned} \tag{3.12}$$

Obviously, by Sobolev imbedding inequality,

$$\begin{aligned} |A_{21}| + |A_{22}| &\lesssim \|\partial_1 u_1\|_{L^\infty}\|\partial_1\omega\|_{L^2}^2 + \|\partial_1 u_2\|_{L^\infty}\|\partial_2\omega\|_{L^2}\|\partial_1\omega\|_{L^2} \\ &\lesssim \|\partial_1 u\|_{H^2}\|\nabla\omega\|_{L^2}\|\partial_1\omega\|_{L^2} \lesssim \|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.13}$$

For A_{23}, by Lemmas 2.1, 2.3, 2.4 and the Sobolev imbedding inequality, we have that

$$\begin{aligned} A_{23} &= -\int (\partial_2\bar{u}_1 + \partial_2\tilde{u}_1)\partial_1\tilde{\omega}(\partial_2\tilde{\omega} + \partial_2\bar{\omega}) \\ &= -\int \partial_2\bar{u}_1\partial_1\tilde{\omega}\partial_2\tilde{\omega} - \int \partial_2\tilde{u}_1\partial_1\tilde{\omega}\partial_2\tilde{\omega} - \int \partial_2\tilde{u}_1\partial_1\tilde{\omega}\partial_2\bar{\omega} \\ &\lesssim \|\partial_2\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\tilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1^2\tilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\tilde{\omega}\|_{L^2} + \|\partial_2\tilde{u}_1\|_{L^4}\|\partial_1\tilde{\omega}\|_{L^4}\|\partial_2\tilde{\omega}\|_{L^2} \\ &\quad + \|\partial_2\tilde{u}_1\|_{L^4}\|\partial_1\tilde{\omega}\|_{L^4}\|\partial_2\bar{\omega}\|_{L^2} \\ &\lesssim \|u\|_{H^2}\|\partial_1\tilde{\omega}\|_{H^1}\|\partial_2\tilde{\omega}\|_{L^2} + \|\partial_2\tilde{u}_1\|_{H^1}\|\partial_1\tilde{\omega}\|_{H^1}\|\partial_2\omega\|_{L^2} \\ &\lesssim \|u\|_{H^2}\|\partial_1\tilde{\omega}\|_{H^1} + \|\partial_1\tilde{u}_1\|_{H^2}\|\partial_1\tilde{\omega}\|_{H^1}\|\partial_2\omega\|_{L^2} \\ &\lesssim \|u\|_{H^2}\|\partial_1 u\|_{H^2}. \end{aligned} \tag{3.14}$$

Here, $\int \partial_2\bar{u}_1\partial_1\tilde{\omega}\partial_2\bar{\omega}$ vanishes, due to the orthogonality in Lemma 2.1.

Similar to A_{23},

$$\begin{aligned} A_{24} &= \int \partial_1\tilde{u}_1\partial_2(\tilde{\omega}+\bar{\omega})\partial_2(\bar{\omega}+\tilde{\omega}) = \int \partial_1\tilde{u}_1\partial_2\tilde{\omega}\partial_2\tilde{\omega} + \int \partial_1\tilde{u}_1\partial_2\tilde{\omega}\partial_2\bar{\omega} + \int \partial_1\tilde{u}_1\partial_2\bar{\omega}\partial_2\tilde{\omega} \\ &\lesssim \|\partial_1\tilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\tilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\tilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\tilde{\omega}\|_{L^2}^{\frac{1}{2}}(\|\partial_2\tilde{\omega}\|_{L^2} + \|\partial_2\bar{\omega}\|_{L^2}) \\ &\lesssim \|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.15}$$

For A_3, we directly have, by $\mathrm{div} u=0$ and the anisotropic inequalities in Lemma 2.4, that

$$\begin{aligned}A_3 &= -\int \Delta u\cdot\nabla\theta\Delta\theta - 2\int \nabla u\cdot\nabla^2\theta\Delta\theta\\ &\lesssim \|\Delta u\|_{L^2}^{\frac12}(\|\Delta u\|_{L^2}+\|\partial_1\Delta u\|_{L^2})^{\frac12}\|\nabla\theta\|_{L^2}^{\frac12}\|\partial_2\nabla\theta\|_{L^2}^{\frac12}\|\Delta\theta\|_{L^2}\\ &\quad+\|\nabla u\|_{L^2}^{\frac14}(\|\nabla u\|_{L^2}+\|\partial_1\nabla u\|_{L^2})^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}(\|\partial_2\nabla u\|_{L^2}+\|\partial_1\partial_2\nabla u\|_{L^2})^{\frac14}\|\nabla^2\theta\|_{L^2}\|\Delta\theta\|_{L^2}\\ &\lesssim (\|u\|_{H^2}+\|\partial_1 u\|_{H^2})\|\theta\|_{H^2}^2. \qquad (3.16)\end{aligned}$$

Summarizing the estimates of A_1, A_2 and A_3, and combining (3.8) and (3.10) and then integrating over the time, we have that

$$\begin{aligned}\|(u,\theta)(t)\|_{H^2}^2+2\int_0^t\|(\partial_1 u,\theta)\|_{H^2}^2 &\le \|(u_0,\theta_0)\|_{H^2}^2 + C\int_0^t \|(u,\theta)\|_{H^2}\|(\partial_1 u,\theta)\|_{H^2}^2\\ &\le \|(u_0,\theta_0)\|_{H^2}^2+CE(t)^{\frac32}, \qquad (3.17)\end{aligned}$$

which implies that (3.2) holds. Hence,

$$\|(u,\theta)(t)\|_{H^2}^2+\int_0^t\|(\partial_1 u,\theta)(\tau)\|_{H^2}^2\mathrm{d}\tau \le C\varepsilon^2. \qquad (3.18)$$

Next we will estimate the dissipation $\int_0^t\|u_2\|_{L^2}^2$. We recall that (u_2,θ) satisfies that

$$\begin{cases}\partial_t u_2 + u\cdot\nabla u_2+\partial_2 p = \partial_{11}u_2+\theta,\\ \partial_t\theta+u\cdot\nabla\theta+\theta+u_2=0.\end{cases} \qquad (3.19)$$

Taking the L^2-inner product with (θ,u_2), we immediately get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u_2,\theta)+\|u_2\|_{L^2}^2 &= -(u\cdot\nabla u_2,\theta)-(u\cdot\nabla\theta,u_2)-(\partial_{11}u_2,\theta)-(\partial_2 p,\theta)+\|\theta\|_{L^2}-(\theta,u_2)\\ &= -(\partial_{11}u_2,\theta)-(\partial_2 p,\theta)+\|\theta\|_{L^2}-(\theta,u_2)\\ &\le \frac12\|u_2\|_{L^2}^2+2\|\theta\|_{L^2}^2+\frac12\|\partial_{11}u_2\|_{L^2}^2-(\partial_2 p,\theta). \qquad (3.20)\end{aligned}$$

We need to estimate $(\partial_2 p,\theta)$. Note that p satisfies that

$$\begin{aligned}\Delta p &= -2(\partial_1 u_2\partial_2 u_1+(\partial_2 u_2)^2)+\partial_2\theta = -2(\partial_1\tilde u_2\partial_2 u_1+\partial_2\tilde u_2\partial_2 u_2)+\partial_2\theta\\ &= -2[\partial_1(\tilde u_2\partial_2 u_1)-\tilde u_2\partial_2\partial_1 u_1+\partial_2(\tilde u_2\partial_2 u_2)-\tilde u_2\partial_2\partial_2 u_2]+\partial_2\theta\\ &= -2\partial_1(\tilde u_2\partial_2 u_1)-2\partial_2(\tilde u_2\partial_1 u_1)+\partial_2\theta, \qquad (3.21)\end{aligned}$$

where we have used that

$$\partial_2 u_2 = -\partial_1 u_1 = -\partial_1\tilde u_1 = \partial_2\tilde u_2, \qquad (3.22)$$

by $\mathrm{div} u=0$ and Lemma 2.1.

Then we deduce, by Lemma 2.4 and Lemma 2.3, that

$$\begin{aligned}-(\partial_2 p,\theta) &= 2(\partial_2\partial_1\Delta^{-1}(\tilde u_2\partial_2 u_1),\theta)+2(\partial_2^2\Delta^{-1}(\tilde u_2\partial_1 u_1),\theta)-(\partial_2^2\Delta^{-1}\theta,\theta)\\ &= 2(\tilde u_2\partial_2 u_1,\partial_2\partial_1\Delta^{-1}\theta)+2(\tilde u_2\partial_1 u_1,\partial_2^2\Delta^{-1}\theta)-(\partial_2^2\Delta^{-1}\theta,\theta)\\ &\le C\|\tilde u_2\|_{L^2}^{\frac12}\|\partial_1\tilde u_2\|_{L^2}^{\frac12}\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2\partial_1\Delta^{-1}\theta\|_{L^2}\\ &\quad+C\|\tilde u_2\|_{L^\infty}\|\partial_1 u_1\|_{L^2}\|\partial_2^2\Delta^{-1}\theta\|_{L^2}+\|\partial_2^2\Delta^{-1}\theta\|_{L^2}\|\theta\|_{L^2}\\ &\le C\|u\|_{H^2}\|\partial_1 u\|_{L^2}\|\theta\|_{L^2}+\|\theta\|_{L^2}^2. \qquad (3.23)\end{aligned}$$

Here we have used the Calderón-Zygmund singular integral theory (see [21]). In particular, $\|\partial_2^2\Delta^{-1}\theta\|_{L^2} \le \|\theta\|_{L^2}$, by Parseval's theorem.

Returning to (3.20), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u_2,\theta)+\frac{1}{2}\|u_2\|_{L^2}^2 &\le 3\|\theta\|_{L^2}^2+\frac{1}{2}\|\partial_{11}u_2\|_{L^2}^2+C\|u\|_{H^2}\|\partial_1 u\|_{L^2}\|\theta\|_{L^2}\\ &\le 3\|\theta\|_{L^2}^2+\frac{1}{2}\|\partial_{11}u_2\|_{L^2}^2+C\|u\|_{H^2}\|(\partial_1 u,\theta)\|_{L^2}^2\\ &\le C\|(\partial_1 u,\theta)\|_{H^2}^2,\end{aligned}\tag{3.24}$$

where we have used (3.18). Integrating over the time, we have that

$$\begin{aligned}(u_2,\theta)+\frac{1}{2}\int_0^t\|u_2\|_{L^2}^2 &\le (u_2,\theta)(0)+C\int_0^t\|(\partial_1 u,\theta)\|_{H^2}^2\\ &\le \|(u_0,\theta_0)\|_{H^2}^2+C\int_0^t\|(\partial_1 u,\theta)\|_{H^2}^2.\end{aligned}\tag{3.25}$$

Combining this with (3.18) and $(u_2,\theta)\le\frac{1}{2}\|(u,\theta)\|_{L^2}^2$, we can get

$$\|(u,\theta)(t)\|_{H^2}^2+\int_0^t(\|u_2(\tau)\|_{L^2}^2+\|(\partial_1 u,\theta)(\tau)\|_{H^2}^2)\mathrm{d}\tau\le C\varepsilon^2.\tag{3.26}$$

Finally, we will show the uniqueness. Let (u^1,θ^1) and (u^2,θ^2) be the solution to (1.1) in Theorem 1.1. Denote that $(u^0,\theta^0,p^0)=(u^1-u^2,\theta^1-\theta^2,p^1-p^2)$. Then we get the equations of (u^0,θ^0,p^0) as

$$\begin{cases}\partial_t u^0+u^1\cdot\nabla u^0+u^0\cdot\nabla u^2+\nabla p^0=\partial_{11}u^0+\theta^0 e_2,\\ \partial_t\theta^0+u^1\cdot\nabla\theta^0+u^0\cdot\nabla\theta^2+\theta^0+u_2^0=0,\\ \mathrm{div}u^0=0,\\ (u^0,\theta^0)|_{t=0}=(0,0).\end{cases}\tag{3.27}$$

The L^2-inner product with (u^0,θ^0) gives that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u^0,\theta^0)\|_{L^2}^2+\|(\partial_1 u^0,\theta^0)\|_{L^2}^2 &= -(u^0\cdot\nabla u^2,u^0)-(u^0\cdot\nabla\theta^2,\theta^0)\\ &\lesssim \|u^0\|_{L^2}^{\frac{1}{2}}(\|u^0\|_{L^2}+\|\partial_1 u^0\|_{L^2})^{\frac{1}{2}}\|\nabla u^2\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u^2\|_{L^2}^{\frac{1}{2}}\|u^0\|_{L^2}\\ &\quad+\|u^0\|_{L^2}^{\frac{1}{2}}(\|u^0\|_{L^2}+\|\partial_1 u^0\|_{L^2})^{\frac{1}{2}}\|\nabla\theta^2\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\theta^2\|_{L^2}^{\frac{1}{2}}\|\theta^0\|_{L^2}\\ &\le\frac{1}{2}\|\partial_1 u^0\|_{L^2}^2+C\|(u^0,\theta^0)\|_{L^2}^2.\end{aligned}\tag{3.28}$$

Here we have used Lemma 2.4 and the bound of $\|(u^2,\theta^2)\|_{H^2}$. Then Grönwall's inequality yields that

$$\|(u^0,\theta^0)(t)\|_{L^2}^2\le\|(u^0,\theta^0)(0)\|_{L^2}^2\mathrm{e}^{Ct},\tag{3.29}$$

which means that $(u^0,\theta^0)(t)=0$ for any $t>0$, due to the zero initial data.

4 Proof of Theorem 1.2

This section proves Theorem 1.2. We deal with the equations for $(\tilde{u},\tilde{\theta})$ and $(\bar{u},\bar{\theta})$ and make use of the properties of the orthogonal decomposition and anisotropic inequalities.

By Lemma 2.1, we have that $\partial_1 \bar{u} = 0$ and

$$\overline{u \cdot \nabla \bar{u}} = \overline{u_1 \partial_1 \bar{u}} + \overline{u_2 \partial_2 \bar{u}} = \bar{u}_2 \partial_2 \bar{u}. \tag{4.1}$$

Since $\mathrm{div} u = 0$, we can introduce a stream function ψ such that

$$u = \nabla^{\perp} \psi = (-\partial_2 \psi, \partial_1 \psi). \tag{4.2}$$

Then

$$\bar{u}_2 = \overline{\partial_1 \psi} = 0, \quad u_2 = \tilde{u}_2, \tag{4.3}$$

and

$$\overline{u \cdot \nabla \bar{u}} = 0. \tag{4.4}$$

Thus we can decompose the equations of (1.1) into a system of $(\bar{u}, \bar{\theta})$,

$$\begin{cases} \partial_t \bar{u} + \overline{u \cdot \nabla \tilde{u}} + (0, \partial_2 \bar{p})^T = \bar{\theta} e_2, \\ \partial_t \bar{\theta} + \overline{u \cdot \nabla \tilde{\theta}} + \bar{\theta} = 0, \end{cases} \tag{4.5}$$

and a system of $(\tilde{u}, \tilde{\theta})$,

$$\begin{cases} \partial_t \tilde{u} + \widetilde{u \cdot \nabla \tilde{u}} + u_2 \partial_2 \bar{u} - \partial_{11} \tilde{u} + \nabla \tilde{p} = \tilde{\theta} e_2, \\ \partial_t \tilde{\theta} + \widetilde{u \cdot \nabla \tilde{\theta}} + u_2 \partial_2 \bar{\theta} + \tilde{\theta} + \tilde{u}_2 = 0. \end{cases} \tag{4.6}$$

Proof of (1.20) First we show the decay of $(\tilde{u}, \tilde{\theta})$ in H^1. The L^2-estimate on (4.6) yields that

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|(\tilde{u}, \tilde{\theta})\|_{L^2}^2 + \|(\partial_1 \tilde{u}, \tilde{\theta})\|_{L^2}^2 &= -\int \widetilde{u \cdot \nabla \tilde{u}} \cdot \tilde{u} - \int u_2 \partial_2 \bar{u} \cdot \tilde{u} - \int \widetilde{u \cdot \nabla \tilde{\theta}} \cdot \tilde{\theta} - \int u_2 \partial_2 \bar{\theta} \cdot \tilde{\theta} \\ &= B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{4.7}$$

By the orthogonality in Lemma 2.1 and $\mathrm{div} u = 0$,

$$B_1 = -\int \widetilde{u \cdot \nabla \tilde{u}} \cdot \tilde{u} = -\int u \cdot \nabla \tilde{u} \cdot \tilde{u} = 0. \tag{4.8}$$

Similarly,

$$B_3 = -\int \widetilde{u \cdot \nabla \tilde{\theta}} \cdot \tilde{\theta} = -\int u \cdot \nabla \tilde{\theta} \cdot \tilde{\theta} = 0. \tag{4.9}$$

Directly from the orthogonality in Lemma 2.1, Lemma 2.4 and Lemma 2.3, we have that

$$\begin{aligned} B_2 &= -\int u_2 \partial_2 \bar{u} \cdot \tilde{u} = -\int \tilde{u}_2 \partial_2 \bar{u} \cdot \tilde{u} \\ &\lesssim \|\partial_2 \bar{u}\|_{L^2} \|\tilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \tilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\tilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \tilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|u\|_{H^1} \|\partial_1 \tilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \tilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \tilde{u}\|_{L^2} \\ &\lesssim \|u\|_{H^1} \|\partial_1 \tilde{u}\|_{L^2}^2. \end{aligned} \tag{4.10}$$

Analogously to A_2,

$$\begin{aligned} B_4 &= -\int u_2 \partial_2 \bar{\theta} \cdot \tilde{\theta} = -\int \tilde{u}_2 \partial_2 \bar{\theta} \cdot \tilde{\theta} \\ &\lesssim \|\tilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \tilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \bar{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \bar{\theta}\|_{L^2}^{\frac{1}{2}} \|\tilde{\theta}\|_{L^2} \\ &\lesssim \|\theta\|_{H^2} \|\partial_1 \tilde{u}_2\|_{L^2} \|\tilde{\theta}\|_{L^2} \end{aligned}$$

$$\lesssim \|\theta\|_{H^2}\|(\partial_1\tilde{u},\tilde{\theta})\|_{L^2}^2. \tag{4.11}$$

Collecting all of the estimates for $B_1 - B_4$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\tilde{u},\tilde{\theta})\|_{L^2}^2 + (2 - C\|(u,\theta)\|_{H^2})\|(\partial_1\tilde{u},\tilde{\theta})\|_{L^2}^2 \le 0. \tag{4.12}$$

From (3.18), $\|(u,\theta)\|_{H^2} \le C\varepsilon$, and then

$$2 - C\|(u,\theta)\|_{H^2} \ge 1, \tag{4.13}$$

with $\varepsilon > 0$ being sufficiently small. Poincaré's inequality in Lemma 2.3 leads to the desired exponential decay of $\|(\tilde{u},\tilde{\theta})\|_{L^2}$:

$$\|(\tilde{u},\tilde{\theta})\|_{L^2} \le \|(u_0,\theta_0)\|_{L^2}\mathrm{e}^{-C_1 t}. \tag{4.14}$$

In fact, the constant here C_1 satisfies that $C_1 > 1$.

• Next, we turn to the exponential decay of $\|(\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}$. Applying ∇ to (4.6) and taking the L^2-inner product with $(\nabla\tilde{u},\nabla\tilde{\theta})$, we have that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}^2 + \|(\partial_1\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}^2 \\
&= -\int \nabla(\widetilde{u\cdot\nabla\tilde{u}})\cdot\nabla\tilde{u} - \int \nabla(u_2\partial_2\bar{u})\cdot\nabla\tilde{u} - \int \nabla(\widetilde{u\cdot\nabla\tilde{\theta}})\cdot\nabla\tilde{\theta} - \int \nabla(u_2\partial_2\bar{\theta})\cdot\nabla\tilde{\theta} \\
&= D_1 + D_2 + D_3 + D_4.
\end{aligned} \tag{4.15}$$

By Lemma 2.1, we get that

$$\begin{aligned}
D_1 &= -\int \nabla(\widetilde{u\cdot\nabla\tilde{u}})\cdot\nabla\tilde{u} = -\int \nabla(u\cdot\nabla\tilde{u})\cdot\nabla\tilde{u} = -\int \nabla u\cdot\nabla\tilde{u}\cdot\nabla\tilde{u} \\
&= -\int \partial_1 u_1\partial_1\tilde{u}\cdot\partial_1\tilde{u} - \int \partial_1 u_2\partial_2\tilde{u}\cdot\partial_1\tilde{u} - \int \partial_2 u_1\partial_1\tilde{u}\cdot\partial_2\tilde{u} - \int \partial_2 u_2\partial_2\tilde{u}\cdot\partial_2\tilde{u} \\
&= D_{11} + D_{12} + D_{13} + D_{14}.
\end{aligned} \tag{4.16}$$

By Lemma 2.4 and Lemma 2.3, we have that

$$\begin{aligned}
D_{11} &= -\int \partial_1 u_1\partial_1\tilde{u}\cdot\partial_1\tilde{u} \\
&\lesssim \|\partial_1 u_1\|_{L^2}\|\partial_1\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\tilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u_1\|_{L^2}\|\partial_1\partial_1\tilde{u}\|_{L^2}^{\frac{3}{2}}\|\partial_2\partial_1\tilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^1}\|\partial_1\nabla\tilde{u}\|_{L^2}^2.
\end{aligned} \tag{4.17}$$

Similarly, we also get

$$|D_{12}|, |D_{13}| \lesssim \|u\|_{H^1}\|\partial_1\nabla\tilde{u}\|_{L^2}^2. \tag{4.18}$$

By $\mathrm{div}u = 0$, and in a manner similar to D_{11}, we have that

$$\begin{aligned}
D_{14} &= \int \partial_1 u_1\partial_2\tilde{u}\cdot\partial_2\tilde{u} = \int \partial_1\tilde{u}_1\partial_2\tilde{u}\cdot\partial_2\tilde{u} \\
&\lesssim \|\partial_2\tilde{u}\|_{L^2}\|\partial_1\tilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\tilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\tilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^1}\|\partial_1\nabla\tilde{u}\|_{L^2}^2.
\end{aligned} \tag{4.19}$$

Thus, D_1 is bounded by

$$|D_1| \lesssim \|u\|_{H^1}\|\partial_1\nabla\tilde{u}\|_{L^2}^2. \tag{4.20}$$

For D_3, we obtain from Lemmas 2.1–2.4 that

$$\begin{aligned}
D_3 &= -\int \nabla(\widetilde{u\cdot\nabla\tilde{\theta}})\cdot\nabla\tilde{\theta} = -\int \nabla(u\cdot\nabla\tilde{\theta})\cdot\nabla\tilde{\theta} = -\int \nabla u\cdot\nabla\tilde{\theta}\cdot\nabla\tilde{\theta}\\
&= -\int \nabla\tilde{u}\cdot\nabla\tilde{\theta}\cdot\nabla\tilde{\theta} - \int \nabla\bar{u}\cdot\nabla\tilde{\theta}\cdot\nabla\tilde{\theta}\\
&\lesssim \|\nabla\tilde{u}\|_{L^2}^{\frac12}\|\partial_1\nabla\tilde{u}\|_{L^2}^{\frac12}\|\nabla\tilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\nabla\tilde{\theta}\|_{L^2}^{\frac12}\|\nabla\tilde{\theta}\|_{L^2} + \|\nabla\bar{u}\|_{L^\infty_{x_2}}\|\nabla\tilde{\theta}\|_{L^2}^2\\
&\lesssim \|(u,\theta)\|_{H^2}\|(\partial_1\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}^2 + \|\nabla\bar{u}\|_{H^1_{x_2}}\|\nabla\tilde{\theta}\|_{L^2}^2\\
&\lesssim \|(u,\theta)\|_{H^2}\|(\partial_1\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}^2. \qquad (4.21)
\end{aligned}$$

For D_2, by Lemma 2.1, (4.3) and $\mathrm{div}u=0$, we have that

$$\begin{aligned}
D_2 &= -\int \nabla(u_2\partial_2\bar{u})\cdot\nabla\tilde{u}\\
&= -\int \partial_1 u_2\partial_2\bar{u}\cdot\partial_1\tilde{u} - \int u_2\partial_2\partial_1\bar{u}\cdot\partial_1\tilde{u} - \int \partial_2 u_2\partial_2\bar{u}\cdot\partial_2\tilde{u} - \int u_2\partial_2\partial_2\bar{u}\cdot\partial_2\tilde{u}\\
&= -\int \partial_1\tilde{u}_2\partial_2\bar{u}\cdot\partial_1\tilde{u} + \int \partial_1\tilde{u}_1\partial_2\bar{u}\cdot\partial_2\tilde{u} - \int \tilde{u}_2\partial_2\partial_2\bar{u}\cdot\partial_2\tilde{u}\\
&= D_{21}+D_{22}+D_{23}. \qquad (4.22)
\end{aligned}$$

Then by Lemma 2.3 and Lemma 2.4, we have that

$$\begin{aligned}
D_{21} &= -\int \partial_1\tilde{u}_2\partial_2\bar{u}\cdot\partial_1\tilde{u}\\
&\lesssim \|\partial_2\bar{u}\|_{L^2}\|\partial_1\tilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\partial_1\tilde{u}_2\|_{L^2}^{\frac12}\|\partial_1\tilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1\tilde{u}\|_{L^2}^{\frac12}\\
&\lesssim \|u\|_{H^1}\|\partial_1\nabla\tilde{u}\|_{L^2}^2. \qquad (4.23)
\end{aligned}$$

Similarly to D_{21}, we then have that

$$|D_{22}| \lesssim \|u\|_{H^1}\|\partial_1\nabla\tilde{u}\|_{L^2}^2, \quad \|D_{23}\| \lesssim \|u\|_{H^2}\|\partial_1\nabla\tilde{u}\|_{L^2}^2. \qquad (4.24)$$

Thus,

$$|D_2| \lesssim \|u\|_{H^2}\|\partial_1\nabla\tilde{u}\|_{L^2}^2. \qquad (4.25)$$

For D_4, by (4.3), Lemma 2.4, Lemma 2.2 and Lemma 2.3, we have that

$$\begin{aligned}
D_4 &= -\int \nabla(\tilde{u}_2\partial_2\bar{\theta})\cdot\nabla\tilde{\theta} = -\int \tilde{u}_2\nabla\partial_2\bar{\theta}\cdot\nabla\tilde{\theta} - \int \nabla\tilde{u}_2\partial_2\bar{\theta}\cdot\nabla\tilde{\theta}\\
&\lesssim \|\tilde{u}_2\|_{L^2_{x_1}L^\infty_{x_2}}\|\nabla\partial_2\bar{\theta}\|_{L^2_{x_2}}\|\nabla\tilde{\theta}\|_{L^2} + \|\nabla\tilde{\theta}\|_{L^2}\|\nabla\tilde{u}_2\|_{L^2}^{\frac12}\|\partial_1\nabla\tilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\bar{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_2\bar{\theta}\|_{L^2}^{\frac12}\\
&\lesssim \|\tilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\tilde{u}_2\|_{L^2}^{\frac12}\|\theta\|_{H^2}\|\nabla\tilde{\theta}\|_{L^2} + \|\theta\|_{H^2}\|(\partial_1\nabla\tilde{u}_2,\nabla\tilde{\theta})\|_{L^2}^2\\
&\lesssim \|\theta\|_{H^2}\|(\partial_1\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}^2. \qquad (4.26)
\end{aligned}$$

Collecting all of the estimates from D_1 to D_4, we have the following energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}^2 + (2 - C\|(u,\theta)\|_{H^2})\|(\partial_1\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2}^2 \le 0. \qquad (4.27)$$

Since $\|(u,\theta)\|_{H^2} \le C\varepsilon$, Poincaré's inequality in Lemma 2.3 yields that if we have sufficiently small $\varepsilon>0$, then

$$\|(\nabla\tilde{u},\nabla\tilde{\theta})\|_{L^2} \le \|(\nabla u_0,\nabla\theta_0)\|_{L^2}\mathrm{e}^{-C_1 t}. \qquad (4.28)$$

Combined with the decay estimate (4.14), this completes the proof of (1.20) in Theorem 1.2.

Proof of (1.21) Now we aim to show the exponential decay of the average part $(\bar{u},\bar{\theta})$. We recall that $\bar{u}_2=0$ from (4.3), and that $\bar{\theta}$ satisfies the equation

$$\partial_t\bar{\theta}+\overline{u\cdot\nabla\tilde{\theta}}+\bar{\theta}=0. \tag{4.29}$$

Then the inner product of (4.29) in $H^1_{x_2}$ yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\theta}\|^2_{H^1_{x_2}}+\|\bar{\theta}\|^2_{H^1_{x_2}}=-(\overline{u\cdot\nabla\tilde{\theta}},\bar{\theta})_{H^1_{x_2}}. \tag{4.30}$$

By Lemma 2.1, integrating by parts and using Sobolev's imbedding inequality,

$$\begin{aligned}-(\overline{u\cdot\nabla\tilde{\theta}},\bar{\theta})_{H^1_{x_2}}&=-(\overline{\tilde{u}\cdot\nabla\tilde{\theta}},\bar{\theta})_{H^1_{x_2}}=-(\overline{\nabla\cdot(\tilde{u}\tilde{\theta})},\bar{\theta})_{H^1_{x_2}}=(\overline{\tilde{u}_2\tilde{\theta}},\partial_2\bar{\theta})_{H^1_{x_2}}\\&\le\|\overline{\tilde{u}_2\tilde{\theta}}\|_{H^1_{x_2}}\|\partial_2\bar{\theta}\|_{H^1_{x_2}}\lesssim\overline{\|\tilde{u}_2\tilde{\theta}\|_{H^1_{x_2}}}\|\partial_2\bar{\theta}\|_{H^1_{x_2}}\\&\lesssim\|\partial_2\bar{\theta}\|_{H^1_{x_2}}(\overline{\|\tilde{\theta}\|_{L^\infty_{x_2}}\|\tilde{u}_2\|_{H^1_{x_2}}}+\overline{\|\tilde{\theta}\|_{H^1_{x_2}}\|\tilde{u}_2\|_{L^\infty_{x_2}}})\\&\lesssim\|\partial_2\bar{\theta}\|_{H^1_{x_2}}\overline{\|\tilde{\theta}\|_{H^1_{x_2}}\|\tilde{u}_2\|_{H^1_{x_2}}}\\&\lesssim\|\theta\|_{H^2}\|(\tilde{u},\tilde{\theta})\|^2_{H^1}.\end{aligned} \tag{4.31}$$

Combining this with (4.30), (1.20) and the bound for $\|\theta\|_{H^2}$, we can choose a constant $C_2<\min\{1,C_1\}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\theta}\|^2_{H^1_{x_2}}+2C_2\|\bar{\theta}\|^2_{H^1_{x_2}}\le C\varepsilon\|(u_0,\theta_0)\|^2_{H^1}\mathrm{e}^{-2C_1t}. \tag{4.32}$$

Then Grönwall's inequality yields that

$$\|\bar{\theta}\|^2_{H^1_{x_2}}\le\|\bar{\theta}_0\|^2_{H^1_{x_2}}\mathrm{e}^{-2C_2t}+C\varepsilon\|(u_0,\theta_0)\|^2_{H^1}\int_0^t\mathrm{e}^{-2C_2(t-\tau)}\mathrm{e}^{-2C_1\tau}\mathrm{d}\tau\lesssim\|(u_0,\theta_0)\|^2_{H^1}\mathrm{e}^{-2C_2t}. \tag{4.33}$$

This means that

$$\|\bar{\theta}\|_{H^1}\lesssim\|(u_0,\theta_0)\|_{H^1}\mathrm{e}^{-C_2t}. \tag{4.34}$$

- We recall the 1D equation for $\bar{u}_1$:

$$\partial_t\bar{u}_1+\overline{u\cdot\nabla\tilde{u}_1}=0. \tag{4.35}$$

The L^∞-estimate gives that, for any $t_1\le t_2$,

$$\begin{aligned}\|\bar{u}_1(t_1)-\bar{u}_1(t_2)\|_{L^\infty(\mathbb{R})}&\le\int_{t_1}^{t_2}\|\overline{\tilde{u}\cdot\nabla\tilde{u}_1}(\tau)\|_{L^\infty(\mathbb{R})}\mathrm{d}\tau\le\int_{t_1}^{t_2}\overline{\|\tilde{u}\cdot\nabla\tilde{u}_1(\tau)\|_{L^\infty_{x_2}(\mathbb{R})}}\mathrm{d}\tau\\&\lesssim\int_{t_1}^{t_2}\overline{\|\tilde{u}(\tau)\|_{L^\infty_{x_2}}\|\nabla\tilde{u}_1(\tau)\|_{L^\infty_{x_2}}}\mathrm{d}\tau\\&\lesssim\int_{t_1}^{t_2}\overline{\|\tilde{u}(\tau)\|^{\frac{1}{2}}_{L^2_{x_2}}\|\partial_2\tilde{u}(\tau)\|^{\frac{1}{2}}_{L^2_{x_2}}\|\nabla\tilde{u}_1(\tau)\|^{\frac{1}{2}}_{L^2_{x_2}}\|\partial_2\nabla\tilde{u}_1(\tau)\|^{\frac{1}{2}}_{L^2_{x_2}}}\mathrm{d}\tau\\&\lesssim\int_{t_1}^{t_2}\|\tilde{u}\|_{H^1}\|\nabla\tilde{u}_1\|^{\frac{1}{2}}_{L^2}\|\partial_2\nabla\tilde{u}_1\|^{\frac{1}{2}}_{L^2}\mathrm{d}\tau\\&\lesssim\int_{t_1}^{t_2}\|u\|_{H^2}\|\tilde{u}\|_{H^1}\mathrm{d}\tau\lesssim\int_{t_1}^{t_2}\varepsilon\|(u_0,\theta_0)\|_{H^1}\mathrm{e}^{-C_1\tau}\mathrm{d}\tau\\&\lesssim\varepsilon\|(u_0,\theta_0)\|_{H^1}\mathrm{e}^{-C_1t_1},\end{aligned} \tag{4.36}$$

where we have applied Sobolev's imbedding inequality, (1.20) and the bound for $\|u\|_{H^2}$. Since $\|\bar{u}_1\|_{L^\infty}\lesssim\|\bar{u}_1\|_{H^2}\le\|u\|_{H^2}\le C\varepsilon$, we obtain that there exists a limit denoted by $\bar{u}_1(\infty)$ such that

$$\lim_{t\to\infty}\bar{u}_1(t)=\bar{u}_1(\infty). \tag{4.37}$$

Then an H^1-estimate on (4.35) similar to the L^∞-estimate and Lemma 2.1 yield that

$$\begin{aligned}\|\bar{u}_1(t)-\bar{u}_1(\infty)\|_{H^1_{x_2}} &\le \int_t^\infty \|\overline{u\cdot\nabla\tilde{u}_1}\|_{H^1_{x_2}}|\mathrm{d}t \le \int_t^\infty \|\overline{\tilde{u}\cdot\nabla\tilde{u}_1}\|_{H^1_{x_2}}|\mathrm{d}\tau\\ &\le \int_t^\infty \overline{\|\tilde{u}\cdot\nabla\tilde{u}_1\|_{H^1_{x_2}}}\mathrm{d}\tau \lesssim \int_t^\infty \overline{\|\tilde{u}\|_{H^1_{x_2}}\|\nabla\tilde{u}_1\|_{H^1_{x_2}}}\mathrm{d}\tau\\ &\lesssim \int_t^\infty \|u\|_{H^2}\|\tilde{u}\|_{H^1}\mathrm{d}\tau \lesssim \int_t^\infty \varepsilon\|(u_0,\theta_0)\|_{H^1}\mathrm{e}^{-C_1t}\mathrm{d}\tau\\ &\lesssim \|(u_0,\theta_0)\|_{H^1}\mathrm{e}^{-C_1t},\end{aligned} \tag{4.38}$$

which implies that

$$\|\bar{u}_1(t)-\bar{u}_1(\infty)\|_{H^1}\lesssim\|(u_0,\theta_0)\|_{H^1}\mathrm{e}^{-C_1t}. \tag{4.39}$$

Thus we have completed all of the decay estimates for Theorem 1.2. □

5 Proof of Theorem 1.3

The aim of this section is to show the proof of Theorem 1.3. For this purpose, we first introduce two anisotropic inequalities.

Lemma 5.1 Let $\Omega=\mathbb{R}^3$. Then the following estimates hold:

$$\int_\Omega|fgh|\mathrm{d}x\lesssim\|f\|_{L^2}^{\frac12}\|\partial_1 f\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_2 g\|_{L^2}^{\frac12}\|h\|_{L^2}^{\frac12}\|\partial_3 h\|_{L^2}^{\frac12}, \tag{5.1}$$

$$\int_\Omega|fgh|\mathrm{d}x\lesssim\|f\|_{L^2}^{\frac14}\|\partial_1 f\|_{L^2}^{\frac14}\|\partial_2 f\|_{L^2}^{\frac14}\|\partial_1\partial_2 f\|_{L^2}^{\frac14}\|g\|_{L^2}^{\frac12}\|\partial_3 g\|_{L^2}^{\frac12}\|h\|_{L^2}. \tag{5.2}$$

Proof The proof is very trivial, and is attained directly from Lemma 2.2. By Hölder's inequality and Lemma 2.2, we have that

$$\begin{aligned}\int_\Omega|fgh|\mathrm{d}x &\le \|f\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|g\|_{L^2_{x_1}L^\infty_{x_2}L^2_{x_3}}\|h\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\ &\lesssim \Big\|\|f\|_{L^2_{x_1}}^{\frac12}\|\partial_1 f\|_{L^2_{x_1}}^{\frac12}\Big\|_{L^2_{x_2,x_3}}\Big\|\|g\|_{L^2_{x_2}}^{\frac12}\|\partial_2 g\|_{L^2_{x_2}}^{\frac12}\Big\|_{L^2_{x_1,x_3}}\Big\|\|h\|_{L^2_{x_3}}^{\frac12}\|\partial_3 f\|_{L^2_{x_3}}^{\frac12}\Big\|_{L^2_{x_1,x_2}}\\ &\lesssim \|f\|_{L^2}^{\frac12}\|\partial_1 f\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_2 g\|_{L^2}^{\frac12}\|h\|_{L^2}^{\frac12}\|\partial_3 h\|_{L^2}^{\frac12},\end{aligned} \tag{5.3}$$

and that

$$\begin{aligned}\int_\Omega|fgh|\mathrm{d}x &\le \|f\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|g\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|h\|_{L^2}\\ &\lesssim \Big\|\|f\|_{L^2_{x_1}}^{\frac12}\|\partial_1 f\|_{L^2_{x_1}}^{\frac12}\Big\|_{L^\infty_{x_2}L^2_{x_3}}\Big\|\|g\|_{L^2_{x_3}}^{\frac12}\|\partial_3 g\|_{L^2_{x_3}}^{\frac12}\Big\|_{L^2_{x_1,x_2}}\|h\|_{L^2}\\ &\lesssim \Big\|\|f\|_{L^\infty_{x_2}}\Big\|_{L^2_{x_1,x_3}}^{\frac12}\Big\|\|\partial_1 f\|_{L^\infty_{x_2}}\Big\|_{L^2_{x_1,x_3}}^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_3 g\|_{L^2}^{\frac12}\|h\|_{L^2}\\ &\lesssim \Big\|\|f\|_{L^2_{x_2}}^{\frac12}\|\partial_2 f\|_{L^2_{x_2}}^{\frac12}\Big\|_{L^2_{x_1,x_3}}^{\frac12}\Big\|\|\partial_1 f\|_{L^2_{x_2}}^{\frac12}\|\partial_1\partial_2 f\|_{L^2_{x_2}}^{\frac12}\Big\|_{L^2_{x_1,x_3}}^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_3 g\|_{L^2}^{\frac12}\|h\|_{L^2}\\ &\lesssim \|f\|_{L^2}^{\frac14}\|\partial_1 f\|_{L^2}^{\frac14}\|\partial_2 f\|_{L^2}^{\frac14}\|\partial_1\partial_2 f\|_{L^2}^{\frac14}\|g\|_{L^2}^{\frac12}\|\partial_3 g\|_{L^2}^{\frac12}\|h\|_{L^2}.\end{aligned} \tag{5.4}$$

□

Proof of Theorem 1.3 Since the local well-posedness and the global well-posedness from *a priori* estimates and a standard bootstrapping argument are similar to the arguments used in Theorem 1.1, here we only give some *a priori* estimates. Define the energy functional $M(t)$ by

$$M(t)=\sup_{0\le\tau\le t}\|(u,\theta)(\tau)\|_{H^3}^2+\int_0^t\|(\nabla_h u,\theta)(\tau)\|_{H^3}^2\mathrm{d}\tau. \tag{5.5}$$

Our efforts show that

$$M(t)\le CM(0)+CM(t)^{\frac{3}{2}}. \tag{5.6}$$

For the norm equivalence

$$\|f\|_{H^3}^2\sim\|f\|_{L^2}^2+\sum_{i=1}^{3}\|\partial_i^3 f\|_{L^2}^2, \tag{5.7}$$

it suffices to bound $\|(u,\theta)\|_{L^2}$ and $\|(u,\theta)\|_{\dot H^3}^2:=\sum\limits_{i=1}^{3}\|(\partial_i^3 u,\partial_i^3\theta)\|_{L^2}^2$. First, the L^2-inner product of (1.29) yields that

$$\|(u,\theta)\|_{L^2}^2+2\int_0^t\|(\nabla_h u,\theta)\|_{L^2}\mathrm{d}\tau=\|(u_0,\theta_0)\|_{L^2}^2. \tag{5.8}$$

Applying ∂_i^3 to (1.29) and taking the L^2-inner product with $(\partial_i^3 u,\partial_i^3\theta)$, we have that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u,\theta)\|_{\dot H^3}^2+\|(\nabla_h u,\theta)\|_{\dot H^3}^2&=-\sum_{i=1}^{3}\int\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u-\sum_{i=1}^{3}\int\partial_i^3(u\cdot\nabla\theta)\cdot\partial_i^3\theta\\&=H_1+H_2.\end{aligned} \tag{5.9}$$

We can decompose H_1 into

$$\begin{aligned}H_1&=-\sum_{i=1}^{2}\int\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u-\sum_{k=1}^{2}\int\partial_3^3(u_k\partial_k u)\cdot\partial_3^3 u-\int\partial_3^3(u_3\partial_3 u)\cdot\partial_3^3 u\\&=H_{11}+H_{12}+H_{13}.\end{aligned} \tag{5.10}$$

Due to $\mathrm{div}u=0$ and Sobolev's imbedding inequality,

$$\begin{aligned}H_{11}&=-\sum_{i=1}^{2}\sum_{l=1}^{3}C_3^l\int\partial_i^l u\cdot\partial_i^{3-l}\nabla u\cdot\partial_i^3 u\\&\lesssim\|\nabla_h^3 u\|_{L^2}(\|\nabla_h^3 u\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla_h^2 u\|_{L^4}\|\nabla_h\nabla u\|_{L^4}+\|\nabla_h u\|_{L^\infty}\|\nabla_h^2\nabla u\|_{L^2})\\&\lesssim\|u\|_{H^3}\|\nabla_h u\|_{H^3}^2.\end{aligned} \tag{5.11}$$

For H_{12}, by $\mathrm{div}u=0$ and Lemma 5.1, we have that

$$\begin{aligned}H_{12}&=-\sum_{k=1}^{2}\sum_{l=1}^{3}C_3^l\int\partial_3^l u_k\partial_3^{3-l}\partial_k u\cdot\partial_3^3 u\\&\lesssim\sum_{k=1}^{2}\sum_{l=1}^{3}\|\partial_3^l u_k\|_{L^2}^{\frac12}\|\partial_1\partial_3^l u_k\|_{L^2}^{\frac12}\|\partial_3^{3-l}\partial_k u\|_{L^2}^{\frac12}\|\partial_3\partial_3^{3-l}\partial_k u\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 u\|_{L^2}^{\frac12}\\&\lesssim\|u\|_{H^3}\|\nabla_h u\|_{H^3}^2.\end{aligned} \tag{5.12}$$

Since $\mathrm{div}u=0$, $\partial_3 u_3=-\nabla_h\cdot u_h$, by Lemma 5.1 we obtain that

$$H_{13}=-\sum_{l=1}^{3}C_3^l\int\partial_3^l u_3\partial_3^{3-l}\partial_3 u\cdot\partial_3^3 u$$

$$
\begin{aligned}
&= \sum_{l=1}^{3} C_3^l \int \partial_3^{l-1}\nabla_h \cdot u_h \partial_3^{3-l}\partial_3 u \cdot \partial_3^3 u \\
&\lesssim \sum_{l=1}^{3} \|\partial_3^{l-1}\nabla_h\|_{L^2}^{\frac12}\|\partial_3\partial_3^{l-1}\nabla_h\|_{L^2}^{\frac12}\|\partial_3^{3-l}\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3^{3-l}\partial_3 u\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 u\|_{L^2}^{\frac12} \\
&\lesssim \|u\|_{H^3}\|\nabla_h u\|_{H^3}^2. \qquad (5.13)
\end{aligned}
$$

Therefore,

$$
H_1 \lesssim \|u\|_{H^3}\|\nabla_h u\|_{H^3}^2. \qquad (5.14)
$$

Also, by $\mathrm{div}u = 0$ and Sobolev's imbedding inequality, we have that

$$
\begin{aligned}
H_2 &= -\sum_{i=1}^{3}\sum_{l=1}^{3} C_3^l \int \partial_i^l u \cdot \partial_i^{3-l}\nabla\theta \cdot \partial_i^3\theta \\
&\lesssim \|\nabla^3\theta\|_{L^2}(\|\nabla^3 u\|_{L^2}\|\nabla\theta\|_{L^\infty} + \|\nabla^2 u\|_{L^4}\|\nabla\nabla\theta\|_{L^4} + \|\nabla u\|_{L^\infty}\|\nabla^2\nabla\theta\|_{L^2}) \\
&\lesssim \|u\|_{H^3}\|\theta\|_{H^3}^2. \qquad (5.15)
\end{aligned}
$$

Inserting the estimates of H_1 and H_2 into (5.9) and combining with (5.8), then we get that

$$
\begin{aligned}
\|(u,\theta)\|_{H^3}^2 + 2\int_0^t \|(\nabla_h u,\theta)\|_{H^3}^2 \mathrm{d}\tau &\le \|(u_0,\theta_0)\|_{H^3}^2 + C\int_0^t \|u\|_{H^3}\|(\nabla_h u,\theta)\|_{H^3}^2 \mathrm{d}\tau \\
&\le M(0) + CM(t)^{\frac32}. \qquad (5.16)
\end{aligned}
$$

This means that (5.6) holds and that $M(t) \le CM(0)$. Next, we estimate the dissipation $\int_0^t \|u_3\|_{L^2}^2 \mathrm{d}\tau$. We recall that (u_3,θ) satisfies the equations

$$
\begin{cases}
\partial_t u_3 + u\cdot\nabla u_3 + \partial_3 p = \Delta_h u_3 + \theta e_3, \\
\partial_t\theta + u\cdot\nabla\theta + \theta + u_3 = 0.
\end{cases} \qquad (5.17)
$$

Taking the L^2-inner product with (θ,u_3), we obtain that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(u_3,\theta) + \|u_3\|_{L^2}^2 &= -(u\cdot\nabla u_3,\theta) - (u\cdot\nabla\theta,u_3) - (\partial_3 p,\theta) - (\Delta_h u_3,\theta) + \|\theta\|_{L^2}^2 - (\theta,u_3) \\
&= -(\partial_3 p,\theta) - (\Delta_h u_3,\theta) + \|\theta\|_{L^2}^2 - (\theta,u_3) \\
&\le \frac12\|u_3\|_{L^2}^2 + 2\|\theta\|_{L^2}^2 + \frac12\|\Delta_h u_3\|_{L^2}^2 - (\partial_3 p,\theta). \qquad (5.18)
\end{aligned}
$$

Since the pressure p satisfies that

$$
\Delta p = -\mathrm{div}(u\cdot\nabla u) + \partial_3\theta, \qquad (5.19)
$$

we get, by Lemma 5.1, that

$$
\begin{aligned}
-(\partial_3 p,\theta) &= (\partial_3\Delta^{-1}\mathrm{div}(u\cdot\nabla u),\theta) - (\partial_3^2\Delta^{-1}\theta,\theta) \\
&= (u\cdot\nabla u, \partial_3\Delta^{-1}\nabla\theta) - (\partial_3^2\Delta^{-1}\theta,\theta) \\
&\le C\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\partial_3\Delta^{-1}\nabla\theta\|_{L^2}^{\frac12}\|\partial_3\partial_3\Delta^{-1}\nabla\theta\|_{L^2}^{\frac12} \\
&\quad + \|\partial_3^2\Delta^{-1}\theta\|_{L^2}\|\theta\|_{L^2} \\
&\le C\|u\|_{H^1}\|\theta\|_{H^1}\|\nabla_h u\|_{H^1} + \|\theta\|_{L^2}^2 \\
&\le C\|u\|_{H^1}\|(\nabla_h u,\theta)\|_{H^1}^2 + \|\theta\|_{L^2}^2. \qquad (5.20)
\end{aligned}
$$

Plugging this into (5.18), using the bound for $M(t)$ and integrating over the time, we obtain that

$$(u_3,\theta)+\frac{1}{2}\int_0^t\|u_3\|_{L^2}^2\mathrm{d}\tau\le\|(u_0,\theta_0)\|_{L^2}^2+C\int_0^t\|(\nabla_h u,\theta)\|_{H^2}^2\mathrm{d}\tau. \tag{5.21}$$

Coupled with $M(t)\le CM(0)$ and using the fact that $|(u_3,\theta)|\le\frac{1}{2}\|(u,\theta)\|_{L^2}^2$, we have that

$$\int_0^t\|u_3\|_{L^2}^2\mathrm{d}\tau\le CM(0). \tag{5.22}$$

The uniqueness is similar to the 2D case, so we omit it. Thus we have completed the proof of Theorem 1.3. □

Conflict of Interest The authors declare no conflict of interest.

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