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WEAK-STRONG UNIQUENESS FOR THREE DIMENSIONAL INCOMPRESSIBLE ACTIVE LIQUID CRYSTALS

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Abstract The hydrodynamics of active liquid crystal models has attracted much attention in recent years due to many applications of these models. In this paper, we study the weak-strong uniqueness for the Leray-Hopf type weak solutions to the incompressible active liquid crystals in $\mathbb{R}^3$. Our results yield that if there exists a strong solution, then it is unique among the Leray-Hopf type weak solutions associated with the same initial data.

Key words analysis of parabolic and elliptic types; weak-strong uniqueness; active liquid crystals; weak solution; energy equality

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1 Introduction

In this paper, we study the so-called three dimensional incompressible active nematic system [16, 17] which describes the hydrodynamic motion of nematic liquid crystals with active constituent particles. Roughly speaking, this model is a system that couples a forced incompressible Navier-Stokes equation for the fluid velocity field u with a parabolic system for the Q-tensor order parameter. In the Landau-de Gennes theory, the Q-tensor order parameter describes the primary and secondary directions of nematic alignment along with variations in the degree of the nematic order [32]. Since the existence of global weak solutions for this system was recently established in [11], we are interested in investigating the uniqueness properties for such weak solutions of the incompressible active nematic system. In particular, this system turns into the incompressible Navier-Stokes equation if the Q-tensor is isotropic ($Q = 0$). From this point of view, the weak solutions of incompressible active nematic system are expected to have the property of weak-strong uniqueness, analogous to the celebrated works by Prodi [35] and Serrin [39] on the incompressible Navier-Stokes equations. Moreover, there are several relevant works on nematic liquid crystals in the literature, such as for the simplified Ericksen-Leslie system [24] and the generic Q-tensor model [20].

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We begin with a short review of the relevant studies on the incompressible Navier-Stokes equations, especially regarding the uniqueness of weak solutions. Although one may find many different definitions of weak solutions in the literature, the first one is the Leray-Hopf weak solution, which was established by the classical results of Leray (1934) [28] and Hopf (1950) [22]. Up until now, the problem about the uniqueness of Leray-Hopf weak solutions has been open. However, the well-known Ladyzhenskaya-Prodi-Serrin condition showed that if a Leray-Hopf weak solution u satisfies $u \in L_t^q L_x^p$ for some $q < \infty$ and $p > d$ with $\frac{2}{q} + \frac{d}{p} \leq 1$, then all Leray-Hopf weak solutions with the same initial data must coincide (see also [29, 35, 39]). For the endpoint case $(p, q) = (d, \infty)$, that was later established by Kozono and Sohr in [26]. Such types of results are often referred to as weak-strong uniqueness. On the other hand, the Ladyzhenskaya-Prodi-Serrin condition also implies the regularity properties of Leray-Hopf weak solutions. For more details on this, we refer interested readers to [12, 14, 38, 41] and the references therein.

As an alternative, a more general weak solution was considered by Fabes, Jones and Rivière [15] in 1972; this was also referred to as the very weak solution. Their key observation proclaimed the equivalence of the property of weak solutions of class $L_t^q L_x^p$ and solutions of the integral form of the Navier-Stokes equations. By virtue of this fact, they proved that weak solutions of class $L_t^q L_x^p$ are unique if $\frac{2}{q} + \frac{d}{p} \leq 1$ and $d < p < +\infty$. However, for the limit case $(p, q) = (d, \infty)$, we have to work with a smaller class of functional space $C_t L_x^d$ rather than $L_t^\infty L_x^d$, where the iteration scheme is convergent. The corresponding uniqueness results can be found in [27]; see also [18, 31, 33] for further discussions. We also remark that the uniqueness of the weak solutions of class $L_t^\infty L_x^d$ was later studied by Lions and Masmoudi [31] for the dimension $d \geq 4$.

In particular, by the standard embeddings, we see that the weak solutions of the Leray-Hopf class satisfy the condition $u \in L_t^q L_x^p$ for $\frac{2}{q} + \frac{d}{p} = \frac{d}{2}$ and $d \geq 2$. This implies that there is a substantial gap between the conditions regarding the existence and uniqueness. Thanks to the Ladyzhenskaya-Prodi-Serrin condition, the uniqueness holds for weak solutions in $L_t^q L_x^p$ such that $\frac{2}{q} + \frac{d}{p} \leq 1$ with $p \geq d$. However, the question regarding uniqueness of weak solutions of class $L_t^q L_x^p$ with $\frac{2}{q} + \frac{d}{p} > 1$ is quite involved. This has remained completely open until a very recent work by Buckmaster and Vicol [5]. More precisely, the first non-uniqueness was shown in class $C_t L_x^{2+}$ for dimension three. Later, in [34], Luo extended a similar non-uniqueness result to the dimension $d \geq 4$. More recently, Cheskidov and Luo [7] established the non-uniqueness of weak solutions of class $L_t^q L_x^\infty$ for $1 \leq q < 2$ and $d \geq 2$. Interestingly, although the weak-strong uniqueness implies the uniqueness of weak solutions of class $C_t L_x^2$ in dimension two, the authors of [8] obtained the non-uniqueness in class $C_t L_x^p$ for $1 \leq p < 2$. In light of these studies, a conjecture is that non-uniqueness holds for weak solutions of class $L_t^q L_x^p$ such that $\frac{2}{q} + \frac{d}{p} > 1$ with $d \geq 2$. Nevertheless, only partial criteria are known for this conjecture. We also refer readers to [1, 3, 6, 25], as well as the references therein for more results on non-uniqueness.

In view of the above results on Navier-Stokes equations, an important step towards understanding the uniqueness properties of weak solutions is via establishing some suitable criteria. Otherwise, one may result in the non-uniqueness of weak solutions. As was shown in [11], there exist global weak solutions to the three dimensional incompressible active nematic system, but their uniqueness is unknown. This motivates us to investigate the uniqueness criteria for the solutions. However, our aim is to get a result similar to that of Serrin for the Navier-Stokes equation. More precisely, we study the following initial value problem of the incompressible active nematic system in $\mathbb{R}^3$:

$$\begin{cases} Q_t + (u\cdot\nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H[Q], \\ u_t + (u\cdot\nabla)u + \nabla p - \mu\Delta u = -\nabla\cdot(\nabla Q\odot\nabla Q) + \nabla\cdot(Q\Delta Q - \Delta Q Q) \\ \qquad\qquad\qquad\qquad\qquad\qquad -\lambda\nabla\cdot(|Q|H[Q]) + \kappa\nabla\cdot Q, \\ \operatorname{div}u = 0, \\ (Q,u)\mid_{t=0} = (Q_0,u_0). \end{cases} \tag{1.1}$$

Here $u\in\mathbb{R}^3$ is the flow velocity, Q is a traceless and symmetric 3×3 matrix which stands for the nematic tensor order parameter, p denotes the usual pressure, $\mu>0$ represents the viscosity coefficient, $\Gamma^{-1}>0$ is the rotational viscosity, $\lambda\in\mathbb{R}$ refers to the nematic alignment parameter, and $D := \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ and $\Omega := \frac{1}{2}\left(\nabla u - \nabla u^T\right)$ are the symmetric and antisymmetric part of the strain tensor with $(\nabla u)_{\alpha\beta} = \partial_\beta u_\alpha$. Moreover, $\nabla Q\odot\nabla Q$ is a symmetric tensor with its component $(\nabla Q\odot\nabla Q)_{\alpha\beta}$ given by $\partial_\beta Q_{\gamma\delta}\partial_\alpha Q_{\gamma\delta}$. In what follows, we use a partial Einstein summation convention; that is, we sum over the repeated indices. The molecular tensor H is defined by

$$H[Q] := \Delta Q - aQ + b\left[Q^2 - \frac{\operatorname{tr}(Q^2)}{3}\mathcal{I}_3\right] - cQ\operatorname{tr}(Q^2),$$

which describes the relaxational dynamics of the nematic phase and can be derived from the Landau-de Gennes free energy (see also [21]). $c>0$ refers to the concentration of active particles, and a, $\kappa\in\mathbb{R}$ are constants related to the interaction of active particles. In addition, from a physical point of view, the material-dependent constant b is usually taken to be a positive number. For active system (1.1), we need to conquer some new difficulties, due to the appearance of the active terms $\lambda|Q|D$, $\kappa\nabla\cdot Q$ and $\lambda\nabla\cdot(|Q|H[Q])$. Additionally, we also mention that the active nematic systems are distinguished from their well-studied passive counterparts (excluding active terms), because they describe the fluids with active constituent particles. In fact, such systems are energy consumed and dissipated by the active particles that drive the system out of equilibrium. For more background and discussion on active liquid crystals, we refer interested readers to [11, 16, 17, 21] and the references therein.

In order to assess the progress on active liquid crystals, we now review several related studies about the passive nematic models of the Q-tensor. In [36, 37], Paicu and Zarnescu proved the existence of global weak solutions to the coupled incompressible Navier-Stokes equation and the Q-tensor system in $\mathbb{R}^d$ $(d=2,3)$. For dimension two, the existence of regular solutions and the weak-strong uniqueness of weak solutions were further studied. Later, the existence and regularity of weak solutions on the d-dimensional torus over a certain singular potential was established by Wilkinson in [42]. For the coupled compressible Navier-Stokes equation and the Q-tensor system, Wang, Xu and Yu [43] derived the existence and long-time dynamics of globally defined weak solutions. More recently, Xiao [44] investigated the existence of global strong solutions to three dimensional incompressible Q-tensor model, as well as the corresponding weak-strong uniqueness. For more results and discussions see [2, 13, 23]. Nevertheless, all of these results are about the passive nematic liquid crystals without active terms and the concentration equation.

For the active liquid crystals, the existence of global weak solutions to the incompressible active nematic system in dimensions two and three was established by Chen, Majumdar, Wang and Zhang [11]. In the same way as for the passive system [37], the weak-strong uniqueness has only been studied in dimensional two. For the inhomogeneous incompressible active liquid crystals, Lian and Zhang [30] derived the global weak solutions in a three-dimensional bounded domain. Moreover, the compressible flow of the active nematic system was studied in [9]; see the survey paper [10] and references therein for a more detailed discussion.

As mentioned above, the weak-strong uniqueness of system (1.1) was only established in $\mathbb{R}^2$ [11]. It is easy to see that system (1.1) contains several strongly nonlinear terms and generates considerable analytical difficulties. In fact, the highest order terms will not completely cancel out in the higher regularity estimates for dimension three. Conversely, in dimension two, one can obtain the estimates on $\|Q(t,\cdot)\|_{W^{1,\infty}}$ and $\|u(t,\cdot)\|_{L^\infty}$ to close the estimates. Thus the existence of global strong solutions follows. On the other hand, inspired by the work by Xiao [44], it is also possible to obtain the existence of strong solutions in some suitable function spaces in $\mathbb{R}^3$. Nevertheless, we are more interested in the weak-strong uniqueness for weak solutions of system (1.1). To this end, we introduce the notion of Leray-Hopf type weak solutions to the incompressible active nematic system (1.1) as follows:

Definition 1.1 (Leray-Hopf type weak solution) A pair (Q,u) is called a Leray-Hopf type weak solution to (1.1) with the initial data $(Q_0,u_0)\in H^1(\mathbb{R}^3,S_0^3)\times L^2_\sigma(\mathbb{R}^3)$ if the following conditions hold:

(i) $Q\in L^\infty_{\rm loc}(\mathbb{R}_+;H^1(\mathbb{R}^3))\cap L^2_{\rm loc}(\mathbb{R}_+;H^2(\mathbb{R}^3))$ and $u\in L^\infty_{\rm loc}(\mathbb{R}^+;L^2(\mathbb{R}^3))\cap L^2_{\rm loc}(\mathbb{R}_+;H^1(\mathbb{R}^3))$;

(ii) For any compactly supported $\phi\in C^\infty([0,\infty)\times\mathbb{R}^3;\mathcal{S}_0^3)$ and $\psi\in C^\infty([0,\infty)\times\mathbb{R}^3;\mathbb{R}^3)$ with $\nabla\cdot\psi=0$, we have

$$\begin{aligned}&\int_0^\infty (Q:\partial_t\phi)+\Gamma(\Delta Q:\phi)+(Q:u\cdot\nabla\phi)-(Q\Omega-\Omega Q-\lambda|Q|D:\phi){\rm d}t\\&=\Gamma\int_0^\infty\left(aQ-b\left[Q^2-\frac{{\rm tr}(Q^2)}{3}\mathcal{I}_3\right]+cQ{\rm tr}(Q^2):\phi\right){\rm d}t-(Q_0(x):\phi(0,x))\end{aligned}\tag{1.2}$$

and

$$\begin{aligned}&\int_0^\infty (u,\partial_t\psi)+(u,u\cdot\nabla\psi)-\mu(\nabla u,\nabla\psi){\rm d}t+\int_0^\infty\left(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla\psi^T\right){\rm d}t\\&=\int_0^\infty\left(\kappa Q-\lambda|Q|H[Q]:\nabla\psi^T\right){\rm d}t-(u_0(x),\psi(0,x)).\end{aligned}\tag{1.3}$$

(iii) Letting $T\in(0,+\infty)$, there holds for all $t\in[0,T]$,

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|Q(t)\|_{H^1}^2+2\mu\int_0^t\|\nabla u(s)\|_{L^2}^2{\rm d}s\\&\quad+2a\Gamma\int_0^t\|Q(s)\|_{L^2}^2{\rm d}s+2(a+1)\Gamma\int_0^t\|\nabla Q(s)\|_{L^2}^2{\rm d}s+2\Gamma\int_0^t\|\Delta Q(s)\|_{L^2}^2{\rm d}s\\&\le\|u_0\|_{L^2}^2+\|Q_0\|_{H^1}^2-2\int_0^t(a\lambda|Q|Q+\kappa Q:\nabla u)\,{\rm d}s+2\int_0^t(\lambda|Q|D:Q)\,{\rm d}s\\&\quad+2\Gamma\int_0^t\left(b\left[Q^2-\frac{{\rm tr}(Q^2)}{3}\mathcal{I}_3\right]-cQ{\rm tr}(Q^2):Q-\Delta Q\right){\rm d}s\\&\quad+2\lambda\int_0^t\left(|Q|\left\{b\left[Q^2-\frac{{\rm tr}(Q^2)}{3}\mathcal{I}_3\right]-cQ{\rm tr}(Q^2)\right\}:\nabla u\right){\rm d}s.\end{aligned}\tag{1.4}$$

By Chen-Majumdar-Wang-Zhang [11], it is known that, for initial data $(Q_0,u_0)\in H^1(\mathbb{R}^3, S_0^3)\times L^2_\sigma(\mathbb{R}^3)$, there exists a weak solution (Q,u) to system (1.1) such that the above conditions (i) and (ii) hold. However, it is unclear whether their weak solutions are indeed the Leray-Hopf type weak solutions. In fact, this energy inequality automatically holds for sufficiently regular solutions of (1.1); see Theorem 1.2 and Lemma 2.1 below for the details. Therefore, it is reasonable to consider such Leray-Hopf type weak solutions for (1.1).

Since the Ladyzhenskaya-Prodi-Serrin condition guarantees that Leray-Hopf weak solution of the Navier-Stokes equation satisfies the energy equality, one may expect a very similar property for the incompressible active nematic system. Moreover, as in [39], the energy equality is essential to study the weak-strong uniqueness. Hence, one of the main results in this paper is to prove that Leray-Hopf type weak solution of (1.1) verifies an energy equality under some additional regularity property. More precisely, we have

Theorem 1.2 Let (Q,u) and (R,v) be two Leray-Hopf type weak solutions to (1.1) with the same initial data $(Q_0,u_0)\in H^1(\mathbb{R}^3,S_0^3)\times L^2_\sigma(\mathbb{R}^3)$. Assume that for some $2\le p\le 3$,

$$\Delta Q\in L^q(0,T;L^p(\mathbb{R}^3))\quad\text{and}\quad \nabla u\in L^q(0,T;L^p(\mathbb{R}^3)),\quad \frac{2}{q}+\frac{3}{p}=2. \tag{1.5}$$

Then, for any $t\in[0,T]$, we have

$$\begin{aligned}(u(t),v(t)) = {}& \|u_0\|_{L^2}^2-2\mu\int_0^t(\nabla v,\nabla u)\mathrm{d}s+\int_0^t(v,v\cdot\nabla u)+(u,u\cdot\nabla v)\mathrm{d}s\\ &+\int_0^t\left(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla v\right)\mathrm{d}s\\ &+\int_0^t\left(\nabla R\odot\nabla R-R\Delta R+\Delta RR:\nabla u\right)\mathrm{d}s\\ &-\int_0^t\left(\kappa Q-\lambda|Q|H[Q]:\nabla v\right)\mathrm{d}s-\int_0^t\left(\kappa R-\lambda|R|H[R]:\nabla u\right)\mathrm{d}s\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}&(Q(t):R(t))+(\nabla Q(t):\nabla R(t))\\ ={}&\|Q_0\|_{H^1}^2-2a\Gamma\int_0^t(Q:R)\,\mathrm{d}s-2(a+1)\Gamma\int_0^t(\nabla Q:\nabla R)\mathrm{d}s-2\Gamma\int_0^t(\Delta Q:\Delta R)\mathrm{d}s\\ &+\int_0^t(Q:u\cdot\nabla R)\mathrm{d}s+\int_0^t(\Delta R:u\cdot\nabla Q)\mathrm{d}s+\int_0^t(R:v\cdot\nabla Q)\mathrm{d}s+\int_0^t(\Delta Q:v\cdot\nabla R)\mathrm{d}s\\ &-\int_0^t(Q\Omega-\Omega Q-\lambda|Q|D:R-\Delta R)\mathrm{d}s-\int_0^t(R\overline{\Omega}-\overline{\Omega}R-\lambda|R|\overline{D}:Q-\Delta Q)\mathrm{d}s\\ &+\Gamma\int_0^t\left(b\left[Q^2-\frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right]-cQ\mathrm{tr}(Q^2):R-\Delta R\right)\mathrm{d}s\\ &+\Gamma\int_0^t\left(b\left[R^2-\frac{\mathrm{tr}(R^2)}{3}\mathcal{I}_3\right]-cR\mathrm{tr}(R^2):Q-\Delta Q\right)\mathrm{d}s.\end{aligned} \tag{1.7}$$

In particular, (Q,u) satisfies the energy equality

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|Q(t)\|_{H^1}^2+2\mu\int_0^t\|\nabla u(s)\|_{L^2}^2\mathrm{d}s\\ &+2a\Gamma\int_0^t\|Q(s)\|_{L^2}^2\mathrm{d}s+2(a+1)\Gamma\int_0^t\|\nabla Q(s)\|_{L^2}^2\mathrm{d}s+2\Gamma\int_0^t\|\Delta Q(s)\|_{L^2}^2\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \|u_0\|_{L^2}^2 + \|Q_0\|_{H^1}^2 - 2\int_0^t (a\lambda|Q|Q + \kappa Q : \nabla u)\,\mathrm{d}s + 2\int_0^t (\lambda|Q|D : Q)\,\mathrm{d}s \\
&\quad + 2\Gamma \int_0^t \left(b\left[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right] - cQ\mathrm{tr}(Q^2) : Q - \Delta Q \right)\mathrm{d}s \\
&\quad + 2\lambda \int_0^t \left(|Q| \left\{ b\left[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right] - cQ\mathrm{tr}(Q^2) \right\} : \nabla u \right)\mathrm{d}s, \qquad (1.8)
\end{aligned}
$$

for all $t \in [0, T]$.

Remark 1.3 We list a few remarks here concerning Theorem 1.2.

(i) This result implies that if a Leray-Hopf type weak solution of (1.1) satisfies the condition (1.5), then energy inequality (1.4) should hold as a strict equality.

(ii) It is known from classical results by Shinbrot [40] that the energy equality of the Navier-Stokes equation holds under the condition $u \in L_t^q L_x^p$ such that $\frac{2}{q} + \frac{2}{p} \le 1$ with $p \ge 4$. This condition is weaker than the Ladyzhenskaya-Prodi-Serrin condition. Hence, an interesting problem is to find a weaker condition for (1.1) such that energy equality holds.

(iii) In fact, the energy equality (1.8) holds for all weak solutions studied in [11] under the condition (1.5).

Now we would like to introduce the definition of strong solutions to the incompressible active nematic system (1.1).

Definition 1.4 (Strong solution) Let (Q, u) be a weak solution to system (1.1) with initial data $(Q_0, u_0) \in H^1(\mathbb{R}^3, S_0^3) \times L_\sigma^2(\mathbb{R}^3)$ such that conditions (i) and (ii) hold. If (Q, u) further belongs to the class

$$
\Delta Q \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{and} \quad \nabla u \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} = 2,
$$

then we call it a strong solution of (1.1).

Regular solutions for system (1.1) have been proven to exist in $\mathbb{R}^2$ [11], but their definition of strong solutions is different with the above one. Our notion is a strong solution in the sense that a weak solution has some appropriate extra regularity. Moreover, it is easy to see that the Leray-Hopf type weak solution to (1.1) is a strong solution if it satisfies the additional condition (1.5).

Now, to state our headline theorem, we can obtain the following weak-strong uniqueness for the Leray-Hopf type weak solutions to (1.1):

Theorem 1.5 Let (Q, u) and (R, v) be two Leray-Hopf type weak solutions to system (1.1) with the same initial data $(Q_0, u_0) \in H^1(\mathbb{R}^3, S_0^3) \times L_\sigma^2(\mathbb{R}^3)$. For some $2 \le p \le 3$, if (Q, u) satisfies

$$
\Delta Q \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{and} \quad \nabla u \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} = 2,
$$

then $(Q, u) \equiv (R, v)$ on $[0, T]$.

Remark 1.6 A few remarks are in order:

(i) The advantage of such a result is that $(\Delta Q, \nabla u)$ is only required to be of class $L^q(0, T; L^p)$ with $\frac{2}{q} + \frac{3}{p} = 2$, and (R, v) is in the larger class satisfying the usual energy inequality.

(ii) For the Navier-Stokes equation, it is known that the uniqueness criteria (1.5) related to ∇u also holds for $p > \frac{3}{2}$ (see also [4]). However, the range $2 \le p \le 3$ is essential. In fact, the same as in [20], this more restrictive situation is due to the highest order nonlinear terms (see the estimates (3.6) and (3.19) given in the proof of Theorem 1.2).

The rest of this paper is organized as follows. In Section 2, some notations and propositions are introduced. In Section 3, we establish the existence of the energy equality for the Leray-Hopf type weak solutions to (1.1) with the additional regularity condition (1.5). Thereafter, Section 4 is devoted to proving Theorem 1.5. Finally, Appendix A contains some important preliminary calculations used in this paper.

2 Preliminaries

In this section, we collect the notations that will be used throughout the paper. For $1 \le p \le \infty$, let $\|\cdot\|_{L^p}$ be the standard norm of the Lebesgue spaces $L^p(\mathbb{R}^d)$, while $\|\cdot\|_{H^s}$ stands for the norm on the usual Sobolev spaces $H^s(\mathbb{R}^d)$. In addition, the space $L^p_\sigma(\mathbb{R}^d)$ is defined by

$$L^p_\sigma(\mathbb{R}^d) := \left\{u \in L^p(\mathbb{R}^d)^n : \text{div } u = 0\right\}.$$

We denote the norm of space $L^q_t L^p_x$ by

$$\|u\|_{L^q(0,T;L^p)} := \begin{cases} \left(\displaystyle\int_0^t \|u(t)\|^q_{L^p} \mathrm{d}t\right)^{\frac{1}{q}}, & \text{if } 1 \le p < \infty, \qquad (2.1) \\ \operatorname*{ess\ sup}_{t\in[0,T]} \|u(t)\|_{L^p}, & \text{if } p = \infty. \qquad (2.2) \end{cases}$$

The space of symmetric traceless Q-tensors in d-dimension is defined by

$$S^d_0 := \left\{Q \in \mathbb{M}^{d\times d} : Q_{\alpha\beta} = Q_{\beta\alpha}, \quad \text{tr}(Q) = 0, \quad \alpha, \beta = 1, 2, \cdots, d\right\},$$

where $\mathbb{M}^{d\times d}$ refers to the space of the $d \times d$ matrix-valued function. Let $(\cdot,\cdot)$ denotes the usual inner product for vector-valued functions in L^2. Similarly, if A and B are matrix-valued functions, we denote that

$$(A : B) := \int_{\mathbb{R}^d} \text{tr}(AB)\mathrm{d}x.$$

Note that $\text{tr}(AB) = A_{\alpha\beta}B_{\alpha\beta}$ for A, $B \in \mathcal{S}^d_0$. Moreover, for $Q \in \mathcal{S}^d_0$, we use the Frobenius norm of Q, i.e.,

$$|Q| := \sqrt{\text{tr}(Q^2)} = \sqrt{Q_{\alpha\beta}Q_{\alpha\beta}}.$$

For $t > \epsilon > 0$, we also define that

$$f_\epsilon(s) = \int_0^t \eta_\epsilon(s-\tau)f(\tau)\mathrm{d}\tau,$$

where η_ϵ is an even, positive, infinitely differentiable function with support in $(-\epsilon,\epsilon)$, and $\int_{\mathbb{R}} \eta_\epsilon \mathrm{d}\tau = 1$. We also remark that, for any $f \in L^q(0,T;L^p)$ with $1 \le q < \infty$, $f_\epsilon \in C^k(0,T;L^p)$ for all $k \ge 0$, and

$$\lim_{\epsilon\to 0} \|f_\epsilon - f\|_{L^q(0,T;L^p)} = 0.$$

Furthermore, if $\{f^i\}^\infty_{i=1}$ converges to f in $L^q(0,T;L^p)$, then

$$\lim_{i\to\infty} \|(f^i)_\epsilon - f_\epsilon\|_{L^q(0,T;L^p)} = 0.$$

Next, we are going to present the energy inequality for sufficiently regular solutions of (1.1).

Lemma 2.1 Let $(Q,u)\in C^\infty(\mathbb{R}_+\times\mathbb{R}^3;\mathcal{S}_0^3\times\mathbb{R}^3)$ be a smooth solution of (1.1). Then the energy inequality (1.4) holds.

Proof Noticing that div $u=0$, we take the summation of the first equation in (1.1) multiplied by $Q-\Delta Q$ and the second equation in (1.1) multiplied by u, take the trace, and then integrate by parts over $[0,t]\times\mathbb{R}^3$ to find

$$\begin{aligned}
&(u(t),u(t))+(Q(t),Q(t))+(\nabla Q(t),\nabla Q(t))+2\mu\int_0^t(\nabla u,\nabla u)\mathrm{d}s\\
&+2a\Gamma\int_0^t(Q,Q)\mathrm{d}s+2(a+1)\Gamma\int_0^t(\nabla Q,\nabla Q)\mathrm{d}s+2\Gamma\int_0^t(\Delta Q,\Delta Q)\mathrm{d}s\\
=&(u_0,u_0)+(Q_0,Q_0)+(\nabla Q_0,\nabla Q_0)+2\int_0^t(\lambda|Q|D:Q)\mathrm{d}s-2\int_0^t(a\lambda|Q|Q+\kappa Q,\nabla u)\mathrm{d}s\\
&\underbrace{+2\int_0^t(u\cdot\nabla Q,\Delta Q)\mathrm{d}s-2\int_0^t(\nabla\cdot(\nabla Q\odot\nabla Q),u)\,\mathrm{d}s}_{\mathcal{I}_1}\underbrace{-2\int_0^t(Q\Omega-\Omega Q,Q)\mathrm{d}s}_{\mathcal{I}_2}\\
&\underbrace{+2\int_0^t(Q\Omega-\Omega Q,\Delta Q)\mathrm{d}s-2\int_0^t(Q\Delta Q-\Delta QQ,\nabla u)\,\mathrm{d}s}_{\mathcal{I}_3}\\
&\underbrace{+2\int_0^t(\lambda|Q|\Delta Q,\nabla u)\mathrm{d}s-2\int_0^t(\lambda|Q|D,\Delta Q)\mathrm{d}s}_{\mathcal{I}_4}\\
&+2\Gamma\int_0^t\left(b\left[Q^2-\frac{\mathrm{tr}(Q^2)}{3}I_3\right]-cQ\mathrm{tr}(Q^2),Q-\Delta Q\right)\mathrm{d}s\\
&+2\lambda\int_0^t\left(|Q|\left\{b\left[Q^2-\frac{\mathrm{tr}(Q^2)}{3}I_3\right]-cQ\mathrm{tr}(Q^2)\right\},\nabla u\right)\mathrm{d}s. \qquad (2.3)
\end{aligned}$$

By the cancellation rules stated in Lemma A.1, we conclude that $\mathcal{I}_1=\mathcal{I}_2=\mathcal{I}_3=\mathcal{I}_4=0$. Therefore, the energy inequality (1.4) follows. □

In the last part of this section, we state the following result for Leray-Hopf type weak solutions to (1.1), which will be useful in the continuing sections (actually, such feature was well-known for Leray-Hopf weak solutions of Navier-Stokes equations; see also [19, 39]):

Proposition 2.2 Let (Q,u) be a Leray-Hopf type weak solution to (1.1) with initial data $(Q_0,u_0)\in H^1\times L^2_\sigma$. Then (Q,u) can be redefined on a set of zero Lebesgue measure in such a way that $(Q(t),u(t))\in H^1\times L^2$ for all $t\geq 0$ and satisfies the identities

$$\begin{aligned}
&\int_0^t(Q:\partial_t\phi)+\Gamma(\Delta Q:\phi)+(Q:u\cdot\nabla\phi)-(Q\Omega-\Omega Q-\lambda|Q|D:\phi)\mathrm{d}s\\
=&\,\Gamma\int_0^t\left(aQ-b\left[Q^2-\frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right]+cQ\mathrm{tr}(Q^2):\phi\right)\mathrm{d}s+(Q(t):\phi(t))-(Q_0:\phi_0) \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t(u,\partial_t\psi)+(u,u\cdot\nabla\psi)-\mu(\nabla u,\nabla\psi)\mathrm{d}s+\int_0^t\left(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla\psi^T\right)\mathrm{d}s\\
=&\int_0^t\left(\kappa Q-\lambda|Q|H[Q]:\nabla\psi^T\right)\mathrm{d}s+(u(t),\psi(t))-(u_0,\psi_0) \qquad (2.5)
\end{aligned}$$

for all $t\geq 0$, and all $\phi\in C_c^\infty([0,\infty)\times\mathbb{R}^3;\mathcal{S}_0^3)$, $\psi\in C_c^\infty([0,\infty)\times\mathbb{R}^3;\mathbb{R}^3)$.

Proof The proof closely follows one of Serrin's [39]. Let $\theta \in C^1(\mathbb{R})$ be a monotonic, non-negative function such that

$$\theta(\xi)=\begin{cases}1, & \text{if } \xi\le 1,\\ 0, & \text{if } \xi\ge 2.\end{cases}$$

For any fixed $t\in[0,\infty)$ and $h>0$ with $t+h<\infty$, we define that

$$\theta_h(s):=\theta\left(\frac{s-t+h}{h}\right).$$

Obviously, one can see that

$$\int_t^{t+h}\partial_s\theta_h\mathrm{d}s=-1 \quad\text{and}\quad -\frac{C}{h}\le\partial_s\theta_h\le 0, \tag{2.6}$$

where C is a positive constant. Substituting $\psi(x,s)$ by $\theta_h(s)\psi(x,s)$ in (1.3), we find that

$$\begin{aligned}&\int_0^{t+h}\theta_h(s)\Big\{(u,\partial_t\psi)+(u,u\cdot\nabla\psi)-\mu(\nabla u,\nabla\psi)+\big(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla\psi^T\big)\\&\qquad\qquad+\big(\lambda|Q|H[Q]-\kappa Q:\nabla\psi^T\big)\Big\}\mathrm{d}s\\&=-\int_0^{t+h}\partial_s\theta_h(s)(u,\psi)\mathrm{d}s-(u_0(x),\psi(0,x)).\end{aligned}\tag{2.7}$$

Now, we want to send h to zero in (2.7). For the left-hand side, by Lebesgue's dominated convergence theorem, we deduce that it tends to

$$\begin{aligned}&\int_0^{t+h}\Big\{(u,\partial_t\psi)+(u,u\cdot\nabla\psi)-\mu(\nabla u,\nabla\psi)+\big(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla\psi^T\big)\\&\qquad+\big(\lambda|Q|H[Q]-\kappa Q:\nabla\psi^T\big)\Big\}\mathrm{d}s\end{aligned}$$

as $h\to 0$. Hence, it remains to investigate the limit of the integral on the right-hand side of (2.7). In fact, for every fixed t, we have

$$\begin{aligned}&\left|-\int_0^{t+h}\partial_s\theta_h(s)(u,\psi)\mathrm{d}s-(u(t),\psi(t))\right|\\&=\left|\int_0^{t+h}\partial_s\theta_h(s)\left\{(u(s),\psi(s))-(u(t),\psi(t))\right\}\mathrm{d}s\right|\\&=\left|\int_0^{t+h}\partial_s\theta_h(s)\left\{(u(s)-u(t),\psi(t))+(u(s),\psi(s)-\psi(t))\right\}\mathrm{d}s\right|\\&\le C\|\psi(t)\|_{L^2}\left(\frac1h\int_t^{t+h}\|u(s)-u(t)\|_{L^2}\mathrm{d}s\right)\\&\quad+C\max_{s\in[t,t+h]}\|\psi(t)-\psi(s)\|_{L^2}\left(\frac1h\int_t^{t+h}\|u(s)\|_{L^2}\mathrm{d}s\right)\\&\le C\|\psi(t)\|_{L^2}\left(\frac1h\int_t^{t+h}\|u(s)-u(t)\|_{L^2}\mathrm{d}s\right)+CM\max_{s\in[t,t+h]}\|\psi(t)-\psi(s)\|_{L^2},\end{aligned}\tag{2.8}$$

where we used the relation (2.6) and $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2)$. Notice that

$$\lim_{h\to0}\max_{s\in[t,t+h]}\|\psi(t)-\psi(s)\|_{L^2}=0.$$

By applying Lebesgue's differentiation theorem, we obtain that

$$\lim_{h\to 0}\int_0^{t+h}\partial_s\theta_h(s)(u,\psi)\mathrm{d}s=-(u(t),\psi(t)),\quad \forall t\in\mathcal{L}(u),$$

where $\mathcal{L}(u)$ refers to the Lebesgue set of u. This implies that (2.5) holds for almost all values of t.

On the other hand, for any $t\in[0,+\infty)$, there exists a sequence $\{t_n\}_n\in\mathcal{L}(u)$ such that $t_n\to t$ as $n\to+\infty$. Since $\{u(t_n)\}_n$ is bounded in L^2_σ, it follows from the weak compactness of L^2 that there exists a subsequence $\{t_{n_k}\}_k$ such that $t_{n_k}\to t$ and $u(t_{n_k})\rightharpoonup U(t)\in L^2_\sigma$ in L^2 as $k\to+\infty$. Recall that (2.5) holds for all $t_{n_k}\in\mathcal{L}(u)$. Hence, letting $k\to+\infty$, we have

$$\begin{aligned}&\int_0^t(u,\partial_t\psi)+(u,u\cdot\nabla\psi)-\mu(\nabla u,\nabla\psi)\mathrm{d}s+\int_0^t\left(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla\psi^T\right)\mathrm{d}s\\&=\int_0^t\left(\kappa Q-\lambda|Q|H[Q]:\nabla\psi^T\right)\mathrm{d}s+(U(t),\psi(t))-(u_0,\psi_0).\end{aligned}$$

Obviously, if $t\in\mathcal{L}(u)$, we see that $(U(t),\psi(t))=(u(t),\psi(t))$. This implies that $U(t)$ must be identical to $u(t)$ on $\mathcal{L}(u)$. Therefore, U is the redefinition of u on a set of values of t of zero Lebesgue measure, and (2.5) holds for U and all $t\geq 0$.

Finally, a similar argument shows that (2.4) holds for $\widetilde{Q}$ and all $t\geq 0$. Here $\widetilde{Q}$ is a redefinition of Q on a set of values of t of measure zero and $\widetilde{Q}\equiv Q$ on $\mathcal{L}(Q)$. This completes the proof. □

Remark 2.3 In what follows, whenever we speak of a Leray-Hopf type weak solution to (1.1), we assume that every weak solution has been modified on a set of zero Lebesgue measure in such a way that (2.4) and (2.5) hold for all $t\geq 0$.

3 Engrey Equality of Weak Solutions

In this section, we show that weak solution of (1.1) verifies energy equality under the condition (1.5).

Proof of Theorem 1.2 (1) The first objective here is to show (1.6). As in [39], let us extract the following subsequences $\{(Q^i,u^i)\}$ and $\{(R^i,v^i)\}$ such that

$$\begin{aligned}&\{u^i\}\subset C^1_c([0,T],C^\infty_c)\longrightarrow u\quad\text{in}\quad L^2(0,T;H^1),\\&\{\nabla u^i\}\subset C^1_c([0,T],C^\infty_c)\longrightarrow \nabla u\quad\text{in}\quad L^q(0,T;L^p),\\&\{Q^i\}\subset C^1_c([0,T]),C^\infty_c)\longrightarrow Q\quad\text{in}\quad L^2(0,T;H^2),\\&\{\Delta Q^i\}\subset C^1_c([0,T],C^\infty_c)\longrightarrow \Delta Q\quad\text{in}\quad L^q(0,T;L^p).\\&\{v^i\}\subset C^1_c([0,T],C^\infty_c)\longrightarrow v\quad\text{in}\quad L^2(0,T;H^1),\\&\{R^i\}\subset C^1_c([0,T]),C^\infty_c)\longrightarrow R\quad\text{in}\quad L^2(0,T;H^2).\end{aligned}$$

Such subsequences exist because of assumption (1.5) and Definition 1.1. Letting $t\in[0,T]$ be fixed and choosing in (2.5) $\psi=v^i_\epsilon:=(v^i)_\epsilon$, one obtains that

$$\begin{aligned}&\int_0^t\left(u,\partial_t v^i_\epsilon\right)+(u,u\cdot\nabla v^i_\epsilon)-\mu(\nabla u,\nabla v^i_\epsilon)\mathrm{d}s+\int_0^t\left(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla v^i_\epsilon\right)\mathrm{d}s\\&=\int_0^t\left(\kappa Q-\lambda|Q|H[Q]:\nabla v^i_\epsilon\right)\mathrm{d}s+(u(t),v^i_\epsilon(t))-(u_0,v^i_\epsilon(0)).\end{aligned}\tag{3.1}$$

Observe that

$$
\begin{aligned}
&\Big|\int_0^t (u, u\cdot\nabla(v_\epsilon^i - v_\epsilon))\mathrm{d}s\Big| \\
\le& \int_0^t \|u\|_{L^s}\|u\|_{L^{\frac{2s}{s-2}}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2}\mathrm{d}s \\
\le& \int_0^t \|u\|_{L^s}\|u\|_{L^2}^{\frac{s-3}{s}}\|u\|_{L^6}^{\frac{3}{s}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2}\mathrm{d}s \\
\le& \left(\int_0^t \left[\|u\|_{L^s}^2(\|u\|_{L^2}^2)^{\frac{s-3}{s}}\right](\|\nabla u\|_{L^2}^2)^{\frac{3}{s}}\mathrm{d}s\right)^{\frac12}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)} \\
\le& \left(\int_0^t \|u\|_{L^s}^{\frac{2s}{s-3}}\|u\|_{L^2}^2\mathrm{d}s\right)^{\frac{s-3}{2s}}\cdot\left(\int_0^t \|\nabla u\|_{L^2}^2\mathrm{d}s\right)^{\frac{3}{2s}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)} \\
\le& \|u\|_{L^\infty(0,T;L^2)}^{\frac{s-3}{s}}\|\nabla u\|_{L^2(0,T;L^2)}^{\frac{3}{s}}\left(\int_0^t \|u\|_{L^s}^{\frac{2s}{s-3}}\mathrm{d}s\right)^{\frac{s-3}{2s}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)} \\
\le& \|u\|_{L^\infty(0,T;L^2)}^{\frac{2p-3}{p}}\|\nabla u\|_{L^2(0,T;L^2)}^{\frac{3-p}{p}}\|\nabla u\|_{L^q(0,T;L^p)}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)}.
\end{aligned}
\tag{3.2}
$$

Here we used the Sobolev embedding inequality

$$
\|u\|_{L^s} \le \|\nabla u\|_{L^p}, \quad \frac{1}{s} = \frac{1}{p} - \frac{1}{3}, \tag{3.3}
$$

and

$$
\frac{2s}{s-3} = \frac{2p}{2p-3}. \tag{3.4}
$$

Similarly, by the L^p interpolation and the Sobolev embedding $W^{1,\frac{6p}{5p-6}} \hookrightarrow L^{\frac{2p}{p-2}}$, we obtain the following estimates for the rest of the nonlinear terms:

$$
\begin{aligned}
&\Big|\int_0^t (\nabla Q\odot\nabla Q : \nabla(v_\epsilon^i - v_\epsilon))\mathrm{d}s\Big| \\
\le& \|\nabla Q\|_{L^\infty(0,T;L^2)}^{\frac{2p-3}{p}}\|D^2Q\|_{L^q(0,T;L^p)}\|D^2Q\|_{L^2(0,T;L^2)}^{\frac{3-p}{p}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)} \\
\le& C\|Q\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|\Delta Q\|_{L^q(0,T;L^p)}\|Q\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)},
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&\Big|\int_0^t (Q\Delta Q - \Delta Q Q : \nabla(v_\epsilon^i - v_\epsilon))\mathrm{d}s\Big| \\
\le& \int_0^t \|\Delta Q\|_{L^p}\|Q\|_{L^{\frac{2p}{p-2}}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2}\mathrm{d}s \\
\le& \int_0^t \|\Delta Q\|_{L^p}\|Q\|_{W^{1,\frac{6p}{5p-6}}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2}\mathrm{d}s \\
\le& \int_0^t \|\Delta Q\|_{L^p}\|Q\|_{H^1}^{\frac{2p-3}{p}}\|Q\|_{W^{1,6}}^{\frac{3-p}{p}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2}\mathrm{d}s \\
\le& \left(\int_0^t \|\Delta Q\|_{L^p}^2(\|Q\|_{H^1}^2)^{\frac{2p-3}{p}}(\|Q\|_{H^2}^2)^{\frac{3-p}{p}}\mathrm{d}s\right)^{\frac12}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)} \\
\le& \left(\int_0^t \|\Delta Q\|_{L^p}^{\frac{2p}{2p-3}}\|Q\|_{H^1}^2\right)^{\frac{2p-3}{2p}}\left(\int_0^t \|Q\|_{H^2}^2\mathrm{d}s\right)^{\frac{3-p}{2p}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)} \\
\le& \|Q\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|\Delta Q\|_{L^q(0,T;L^p)}\|Q\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla(v_\epsilon^i - v_\epsilon)\|_{L^2(0,T;L^2)}
\end{aligned}
\tag{3.6}
$$

and

$$\begin{aligned}
&\left|\int_0^t \left(\lambda|Q|H[Q]:\nabla(v_\epsilon^i-v_\epsilon)\right)\mathrm{d}s\right| \\
&=\lambda\left|\int_0^t \left(|Q|\left(\Delta Q-aQ+b\left[Q^2-\frac{\operatorname{tr}(Q^2)}{3}\mathcal{I}_3\right]-cQ\operatorname{tr}(Q^2)\right):\nabla(v_\epsilon^i-v_\epsilon)\right)\mathrm{d}s\right| \\
&\le C\int_0^t \left(\|\Delta Q\|_{L^p}\|Q\|_{L^{\frac{2p}{p-2}}}+\|Q\|_{L^4}^2+\|Q\|_{L^6}^3+\|Q\|_{L^8}^4\right)\|\nabla(v_\epsilon^i-v_\epsilon)\|_{L^2}\mathrm{d}s \\
&\le C\|Q\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|\Delta Q\|_{L^q(0,T;L^p)}\|Q\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla(v_\epsilon^i-v_\epsilon)\|_{L^2(0,T;L^2)} \\
&\quad +C\left(\int_0^t \|Q\|_{H^1}^4+\|Q\|_{H^1}^6+\|D^2Q\|_{L^2}\|Q\|_{L^6}^7\mathrm{d}s\right)^{\frac12}\|\nabla(v_\epsilon^i-v_\epsilon)\|_{L^2(0,T;L^2)} \\
&\le C\|Q\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|\Delta Q\|_{L^q(0,T;L^p)}\|Q\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla(v_\epsilon^i-v_\epsilon)\|_{L^2(0,T;L^2)} \\
&\quad +C\left(\sum_{n=1}^3\|Q\|_{L^\infty(0,T;H^1)}^n\right)\|Q\|_{L^2(0,T;H^2)}\|\nabla(v_\epsilon^i-v_\epsilon)\|_{L^2(0,T;L^2)}. \qquad (3.7)
\end{aligned}$$

Notice that in the previous estimates of (3.6), the constraint $2\le p\le 3$ must be imposed. With the above results, we can send i to infinity in (3.1) and conclude that

$$\begin{aligned}
&\int_0^t (u,\partial_t v_\epsilon)+(u,u\cdot\nabla v_\epsilon)-\mu(\nabla u,\nabla v_\epsilon)\mathrm{d}s+\int_0^t (\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla v_\epsilon)\,\mathrm{d}s \\
&=\int_0^t (\kappa Q-\lambda|Q|H[Q]:\nabla v_\epsilon)\,\mathrm{d}s+(u(t),v_\epsilon(t))-(u_0,v_\epsilon(0)). \qquad (3.8)
\end{aligned}$$

On the other hand, substitute u, Q and v_ϵ^i by v, R and u_ϵ^i in (3.1), respectively, we also have

$$\begin{aligned}
&\int_0^t \left(v,\partial_t u_\epsilon^i\right)+(v,v\cdot\nabla u_\epsilon^i)-\mu(\nabla v,\nabla u_\epsilon^i)\mathrm{d}s+\int_0^t \left(\nabla R\odot\nabla R-R\Delta R+\Delta RR:\nabla u_\epsilon^i\right)\mathrm{d}s \\
&=\int_0^t \left(\kappa R-\lambda|R|H[R]:\nabla u_\epsilon^i\right)\mathrm{d}s+(v(t),u_\epsilon^i(t))-(v_0,u_\epsilon^i(0)). \qquad (3.9)
\end{aligned}$$

Obviously, we still want to send i to infinity in (3.9). Although the nonlinear terms above are quite similar, they should be estimated as follows:

$$\begin{aligned}
&\left|\int_0^t (v,v\cdot\nabla(u_\epsilon^i-u_\epsilon))\mathrm{d}s\right| \\
&\le\int_0^t \|v\|_{L^6}\|v\|_{L^{\frac{6p}{5p-6}}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}\mathrm{d}s \\
&\le\int_0^t \|\nabla v\|_{L^2}\|v\|_{L^2}^{\frac{2p-3}{p}}\|v\|_{L^6}^{\frac{3-p}{p}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}\mathrm{d}s \\
&\le\left(\int_0^t \left[(\|v\|_{L^2}^2)^{\frac{2p-3}{p}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}^2\right](\|\nabla v\|_{L^2}^2)^{\frac{3-p}{p}}\mathrm{d}s\right)^{\frac12}\|\nabla v\|_{L^2(0,T;L^2)} \\
&\le\left(\left(\int_0^t \|v\|_{L^2}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}^{\frac{2p}{2p-3}}\right)^{\frac{2p-3}{p}}\cdot\left(\int_0^t \|\nabla v\|_{L^2}^2\mathrm{d}s\right)^{\frac{3-p}{p}}\right)^{\frac12}\|\nabla v\|_{L^2(0,T;L^2)} \\
&\le\|v\|_{L^\infty(0,T;L^2)}^{\frac{2p-3}{p}}\|\nabla v\|_{L^2(0,T;L^2)}^{\frac{3}{p}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^q(0,T;L^p)}, \qquad (3.10)
\end{aligned}$$

$$\left|\int_0^t (R\Delta R-\Delta RR:\nabla(u_\epsilon^i-u_\epsilon))\mathrm{d}s\right|$$

$$
\begin{aligned}
&\le \int_0^t \|\Delta R\|_{L^2}\|R\|_{L^{\frac{2p}{p-2}}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}\mathrm{d}s\\
&\le \int_0^t \|\Delta R\|_{L^2}\|R\|_{W^{1,\frac{6p}{5p-6}}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}\mathrm{d}s\\
&\le \int_0^t \|\Delta R\|_{L^2}\|R\|_{H^1}^{\frac{2p-3}{p}}\|R\|_{H^2}^{\frac{3-p}{p}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}\mathrm{d}s\\
&\le \left(\int_0^t \|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}^{\frac{2p}{2p-3}}\|R\|_{H^1}^2\mathrm{d}s\right)^{\frac{2p-3}{2p}}\left(\int_0^t\|R\|_{H^2}^2\mathrm{d}s\right)^{\frac{3-p}{2p}}\|\Delta R\|_{L^2(0,T;L^2)}\\
&\le \|R\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|\Delta R\|_{L^2(0,T;L^2)}\|R\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^q(0,T;L^p)},
\end{aligned}\tag{3.11}
$$

$$
\begin{aligned}
&\left|\int_0^t \left(\lambda|R|H[R]:\nabla(u_\epsilon^i-u_\epsilon)\right)\mathrm{d}s\right|\\
&\le C\int_0^t \|\Delta R\|_{L^2}\|R\|_{L^{\frac{2p}{p-2}}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^p}+\left(\|R\|_{L^4}^2+\|R\|_{L^6}^3+\|R\|_{L^8}^4\right)\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^2}\mathrm{d}s\\
&\le C\|R\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|\Delta R\|_{L^2(0,T;L^2)}\|R\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^q(0,T;L^p)}\\
&\quad + C\left(\int_0^t \|R\|_{H^1}^4+\|R\|_{H^1}^6+\|D^2R\|_{L^2}\|R\|_{L^6}^7\mathrm{d}s\right)^{\frac12}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^2(0,T;L^2)}\\
&\le C\|R\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|\Delta R\|_{L^2(0,T;L^2)}\|R\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^q(0,T;L^p)}\\
&\quad + C\left(\sum_{n=1}^3\|R\|_{L^\infty(0,T;H^1)}^n\right)\|R\|_{L^2(0,T;H^2)}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^2(0,T;L^2)},
\end{aligned}\tag{3.12}
$$

$$
\begin{aligned}
&\left|\int_0^t(\nabla R\odot\nabla R:\nabla(u_\epsilon^i-u_\epsilon))\mathrm{d}s\right|\\
&\le \|\nabla R\|_{L^\infty(0,T;L^2)}^{\frac{2p-3}{p}}\|D^2R\|_{L^2(0,T;L^2)}^{\frac3p}\|\nabla(u_\epsilon^i-u_\epsilon)\|_{L^q(0,T;L^p)}.
\end{aligned}\tag{3.13}
$$

Therefore, letting $i\to+\infty$ in (3.9), we have

$$
\begin{aligned}
&\int_0^t (v,\partial_t u_\epsilon)+(v,v\cdot\nabla u_\epsilon)-\mu(\nabla v,\nabla u_\epsilon)\mathrm{d}s+\int_0^t\left(\nabla R\odot\nabla R-R\Delta R+\Delta RR:\nabla u_\epsilon\right)\mathrm{d}s\\
&=\int_0^t\left(\kappa R-\lambda|R|H[R]:\nabla u_\epsilon\right)\mathrm{d}s+(v(t),u_\epsilon(t))-(v_0,u_\epsilon(0)).
\end{aligned}\tag{3.14}
$$

(2) With the above results, one can sum up (3.8) and (3.14) to find that

$$
\begin{aligned}
&\int_0^t (v,\partial_t u_\epsilon)+(u,\partial_t v_\epsilon)\,\mathrm{d}s-\mu\int_0^t(\nabla v,\nabla u_\epsilon)+(\nabla u,\nabla v_\epsilon)\mathrm{d}s\\
&\quad+\int_0^t(v,v\cdot\nabla u_\epsilon)+(u,u\cdot\nabla v_\epsilon)\mathrm{d}s+\int_0^t\left(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla v_\epsilon\right)\mathrm{d}s\\
&\quad+\int_0^t\left(\nabla R\odot\nabla R-R\Delta R+\Delta RR:\nabla u_\epsilon\right)\mathrm{d}s\\
&=\int_0^t\left(\kappa Q-\lambda|Q|H[Q]:\nabla v_\epsilon\right)\mathrm{d}s+\int_0^t\left(\kappa R-\lambda|R|H[R]:\nabla u_\epsilon\right)\mathrm{d}s\\
&\quad+(u(t),v_\epsilon(t))+(v(t),u_\epsilon(t))-(u_0,v_\epsilon(0))-(v_0,u_\epsilon(0)).
\end{aligned}\tag{3.15}
$$

We wish now to prove (1.6) by taking the limit in (3.15) as ϵ goes to zero. Observing that the first term here vanishes because η_ϵ is chosen to be even, we have

$$\begin{aligned}\int_0^t (u,\partial_t v_\epsilon)\,\mathrm{d}s &= \int_0^t\int_0^t \dot\eta_\epsilon(s-\tau)\,(u(s),v(\tau))\,\mathrm{d}\tau\mathrm{d}s\\ &= -\int_0^t\int_0^t \dot\eta_\epsilon(\tau-s)\,(u(s),v(\tau))\,\mathrm{d}\tau\mathrm{d}s\\ &= -\int_0^t\int_0^t \dot\eta_\epsilon(\tau-s)\,(v(\tau),u(s))\,\mathrm{d}s\mathrm{d}\tau\\ &= -\int_0^t (v,\partial_t u_\epsilon)\,\mathrm{d}s.\end{aligned}$$

By the usual properties of mollifiers, we have the following convergence for the second term:

$$\lim_{\epsilon\to 0}\mu\int_0^t (\nabla v,\nabla u_\epsilon)+(\nabla u,\nabla v_\epsilon)\mathrm{d}s = 2\mu\int_0^t(\nabla v,\nabla u)\mathrm{d}s.$$

For the third term in (3.15), we have that

$$\int_0^t (v,v\cdot\nabla u_\epsilon)\mathrm{d}s-\int_0^t(v,v\cdot\nabla u)\mathrm{d}s=\int_0^t(v,v\cdot\nabla u_\epsilon-\nabla u)\mathrm{d}s$$

and

$$\int_0^t (u,u\cdot\nabla v_\epsilon)\mathrm{d}s-\int_0^t(u,u\cdot\nabla v)\mathrm{d}s=\int_0^t(u,u\cdot\nabla v_\epsilon-\nabla v)\mathrm{d}s.$$

The same as in (3.2) and (3.10), it is easy to see that things are bounded by

$$\|v\|_{L^\infty(0,T;L^2)}^{\frac{2p-3}{p}}\|\nabla v\|_{L^2(0,T;L^2)}^{\frac{3}{p}}\|\nabla u_\epsilon-\nabla u\|_{L^q(0,T;L^p)}$$

and

$$\|u\|_{L^\infty(0,T;L^2)}^{\frac{2p-3}{p}}\|\nabla u\|_{L^2(0,T;L^2)}^{\frac{3-p}{p}}\|\nabla u\|_{L^q(0,T;L^p)}\|\nabla v_\epsilon-\nabla v\|_{L^2(0,T;L^2)},$$

respectively. Thus we deduce that

$$\lim_{\epsilon\to 0}\int_0^t (v,v\cdot\nabla u_\epsilon)+(u,u\cdot\nabla v_\epsilon)\mathrm{d}s=\int_0^t(v,v\cdot\nabla u)+(u,u\cdot\nabla v)\mathrm{d}s.$$

On the other hand, by (3.5), (3.6) and (3.7) with v_ϵ in place of v_ϵ^i and v in place of v_ϵ, respectively, we obtain that

$$\lim_{\epsilon\to 0}\int_0^t (\kappa Q-\lambda|Q|H[Q]:\nabla v_\epsilon)\,\mathrm{d}s=\int_0^t(\kappa Q-\lambda|Q|H[Q]:\nabla v)\,\mathrm{d}s$$

and

$$\lim_{\epsilon\to 0}\int_0^t (\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla v_\epsilon)\,\mathrm{d}s=\int_0^t(\nabla Q\odot\nabla Q-Q\Delta Q+\Delta QQ:\nabla v)\,\mathrm{d}s.$$

A similar argument as (3.11), (3.12) and (3.13) implies that

$$\lim_{\epsilon\to 0}\int_0^t (\kappa R-\lambda|R|H[R]:\nabla u_\epsilon)\,\mathrm{d}s=\int_0^t(\kappa R-\lambda|R|H[R]:\nabla u)\,\mathrm{d}s$$

and

$$\lim_{\epsilon\to 0}\int_0^t (\nabla R\odot\nabla R-R\Delta R+\Delta RR:\nabla u_\epsilon)\,\mathrm{d}s=\int_0^t(\nabla R\odot\nabla R-R\Delta R+\Delta RR:\nabla u)\,\mathrm{d}s.$$

Finally, defining that $u(t)=v(t)=0$ for $t<0$, if $\epsilon<t$, we find that

$$(u(t),v_\epsilon(t))=\int_0^t\eta_\epsilon(t-\tau)(u(t),v(\tau))\mathrm{d}\tau$$

$$
\begin{aligned}
&= \int_0^\epsilon \eta_\epsilon(\tau)(u(t), v(t-\tau))\mathrm{d}\tau \\
&= \int_0^\epsilon \eta_\epsilon(\tau)\left[(u(t), v(t)) + (u(t), v(t-\tau) - v(t))\right]\mathrm{d}\tau \\
&= \frac{1}{2}(u(t), v(t)) + \mathcal{O}(\epsilon),
\end{aligned} \tag{3.16}
$$

where we used the weak L^2 continuity of v and the $\int_0^\epsilon \eta_\epsilon(\tau)\mathrm{d}\tau = \frac{1}{2}$. Likewise, we obtain that

$$
\begin{aligned}
&(v(t), u_\epsilon(t)) = \frac{1}{2}(v(t), u(t)) + \mathcal{O}(\epsilon), \\
&(u_0, v_\epsilon(0)) = \frac{1}{2}\|u_0\|_{L^2}^2 + \mathcal{O}(\epsilon), \\
&(v_0, u_\epsilon(0)) = \frac{1}{2}\|u_0\|_{L^2}^2 + \mathcal{O}(\epsilon).
\end{aligned}
$$

Therefore, (1.6) follows by letting $\epsilon \to 0$ in (3.15).

(3) We shall prove (1.7) by the same procedure as for (1.6); we set that $\phi := R_\epsilon^i - \Delta R_\epsilon^i$ in (2.4) and integrate by parts to get that

$$
\begin{aligned}
&\int_0^t (Q : \partial_t R_\epsilon^i) + (\nabla Q : \partial_t \nabla R_\epsilon^i)\mathrm{d}s - (a+1)\Gamma\int_0^t (\nabla Q : \nabla R_\epsilon^i)\mathrm{d}s - \Gamma\int_0^t (\Delta Q : \Delta R_\epsilon^i)\mathrm{d}s \\
&\quad + \int_0^t (Q : u\cdot\nabla R_\epsilon^i)\mathrm{d}s + \int_0^t (\Delta R_\epsilon^i : u\cdot\nabla Q)\mathrm{d}s - \int_0^t (Q\Omega - \Omega Q - \lambda|Q|D : R_\epsilon^i - \Delta R_\epsilon^i)\mathrm{d}s \\
&= a\Gamma\int_0^t \left(Q : R_\epsilon^i\right)\mathrm{d}s - \Gamma\int_0^t \left(b\left[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right] - cQ\mathrm{tr}(Q^2) : R_\epsilon^i - \Delta R_\epsilon^i\right)\mathrm{d}s \\
&\quad + (Q(t) : R_\epsilon^i(t)) + (\nabla Q(t) : \nabla R_\epsilon^i(t)) - (Q_0 : R_\epsilon^i(0)) - (\nabla Q_0 : \nabla R_\epsilon^i(0)).
\end{aligned} \tag{3.17}
$$

Similarly, our first goal is to take the limit in (3.17) as $i \to +\infty$. In fact, we only focus on some terms that are not so easy to deal with. These can be estimated as

$$
\begin{aligned}
&\Big|\int_0^t (\Delta R_\epsilon^i - \Delta R_\epsilon : u\cdot\nabla Q)\mathrm{d}s\Big| \\
&\le \int_0^t \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2}\|\nabla Q\|_{L^{\frac{2s}{s-2}}}\|u\|_{L^s}\mathrm{d}s \\
&\le \int_0^t \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2}\|\nabla Q\|_{L^2}^{\frac{s-3}{s}}\|\nabla Q\|_{L^6}^{\frac{3}{s}}\|u\|_{L^s}\mathrm{d}s \\
&\le \left(\int_0^t \|u\|_{L^s}^2\left(\|\nabla Q\|_{L^2}^2\right)^{\frac{s-3}{s}}\left(\|D^2 Q\|_{L^2}^2\right)^{\frac{3}{s}}\mathrm{d}s\right)^{\frac{1}{2}}\|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2(0,T;L^2)} \\
&\le \left(\int_0^t \|u\|_{L^s}^{\frac{2s}{s-3}}\|\nabla Q\|_{L^2}^2\mathrm{d}s\right)^{\frac{s-3}{2s}}\left(\int_0^t \|D^2 Q\|_{L^2}^2\mathrm{d}s\right)^{\frac{3}{2s}}\|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2(0,T;L^2)} \\
&\le \|Q\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|Q\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla u\|_{L^q(0,T;L^p)}\|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2(0,T;L^2)},
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
&\Big|\int_0^t (Q\Omega - \Omega Q - \lambda|Q|D : \Delta R_\epsilon^i - \Delta R_\epsilon)\mathrm{d}s\Big| \\
&\le \int_0^t \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2}\|Q\|_{L^{\frac{2p}{p-2}}}\|\nabla u\|_{L^p}\mathrm{d}s \\
&\le \int_0^t \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2}\|Q\|_{W^{1,\frac{6p}{5p-6}}}\|\nabla p\|_{L^p}\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2} \|Q\|_{H^1}^{\frac{2p-3}{p}} \|Q\|_{W^{1,6}}^{\frac{3-p}{p}} \|\nabla u\|_{L^p} \mathrm{d}s \\
&\le \left(\int_0^t \|\nabla u\|_{L^p}^2 \left(\|Q\|_{H^1}^2\right)^{\frac{2p-3}{p}} \left(\|Q\|_{H^2}^2\right)^{\frac{3-p}{p}} \mathrm{d}s\right)^{\frac{1}{2}} \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2(0,T;L^2)} \\
&\le \left(\int_0^t \|\nabla u\|_{L^p}^{\frac{2p}{2p-3}} \|Q\|_{H^1}^2\right)^{\frac{2p-3}{2p}} \left(\int_0^t \|Q\|_{H^2}^2 \mathrm{d}s\right)^{\frac{3-p}{2p}} \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2(0,T;L^2)} \\
&\le \|Q\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}} \|Q\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}} \|\nabla u\|_{L^q(0,T;L^p)} \|\Delta R_\epsilon^i - \Delta R_\epsilon\|_{L^2(0,T;L^2)} \qquad (3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_0^t (Q : u\cdot\nabla(R_\epsilon^i - R_\epsilon))\mathrm{d}s\right| &\le \int_0^t \|Q\|_{L^3}\|u\|_{L^6}\|\nabla R_\epsilon^i - \nabla R_\epsilon\|_{L^2}\mathrm{d}s \\
&\le \left(\int_0^t \|Q\|_{H^1}^2 \|\nabla u\|_{L^2}^2 \mathrm{d}s\right)^{\frac{1}{2}} \|\nabla R_\epsilon^i - \nabla R_\epsilon\|_{L^2(0,T;L^2)} \\
&\le \|Q\|_{L^\infty(0,T;H^1)} \|u\|_{L^2(0,T;H^1)} \|\nabla R_\epsilon^i - \nabla R_\epsilon\|_{L^2(0,T;L^2)}. \qquad (3.20)
\end{aligned}
$$

One should note that in the previous estimates for (3.19), the constraint $2 \le p \le 3$ must be imposed. Moreover, since Q^2, $\mathrm{tr}(Q^2)$ and $Q\mathrm{tr}(Q^2)$ are bounded in $L^2(0,T;L^2)$, we have

$$
\lim_{i\to\infty} \Gamma \int_0^t \left(b\left[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right] - cQ\mathrm{tr}(Q^2) : R_\epsilon^i - R_\epsilon + \Delta R_\epsilon - \Delta R_\epsilon^i\right)\mathrm{d}s = 0
$$

Therefore, taking the limit in (3.17) as $i \to +\infty$, we obtain that

$$
\begin{aligned}
&\int_0^t (Q : \partial_t R_\epsilon) + (\nabla Q : \partial_t \nabla R_\epsilon)\mathrm{d}s - (a+1)\Gamma \int_0^t (\nabla Q : \nabla R_\epsilon)\mathrm{d}s \\
&- \Gamma \int_0^t (\Delta Q : \Delta R_\epsilon)\mathrm{d}s + \int_0^t (Q : u\cdot\nabla R_\epsilon)\mathrm{d}s \\
&+ \int_0^t (\Delta R_\epsilon : u\cdot\nabla Q)\mathrm{d}s - \int_0^t (Q\Omega - \Omega Q - \lambda|Q|D : R_\epsilon - \Delta R_\epsilon)\mathrm{d}s \\
= {}& a\Gamma \int_0^t (Q : R_\epsilon)\,\mathrm{d}s - \Gamma \int_0^t \left(b\left[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right] - cQ\mathrm{tr}(Q^2) : R_\epsilon - \Delta R_\epsilon\right)\mathrm{d}s \\
&+ (Q(t) : R_\epsilon(t)) + (\nabla Q(t) : \nabla R_\epsilon(t)) - (Q_0 : R_\epsilon(0)) - (\nabla Q_0 : \nabla R_\epsilon(0)). \qquad (3.21)
\end{aligned}
$$

Analogously, substituting ϕ by $Q_\epsilon^i - \Delta Q_\epsilon^i$ in the weak formula of R, one can also deduce that

$$
\begin{aligned}
&\int_0^t (R : \partial_t Q_\epsilon) + (\nabla R : \partial_t \nabla Q_\epsilon)\mathrm{d}s - (a+1)\Gamma \int_0^t (\nabla R : \nabla Q_\epsilon)\mathrm{d}s \\
&- \Gamma \int_0^t (\Delta R : \Delta Q_\epsilon)\mathrm{d}s + \int_0^t (R : v\cdot\nabla Q_\epsilon)\mathrm{d}s \\
&+ \int_0^t (\Delta Q_\epsilon : v\cdot\nabla R)\mathrm{d}s - \int_0^t (R\overline{\Omega} - \overline{\Omega}R - \lambda|R|\overline{D} : Q_\epsilon - \Delta Q_\epsilon)\mathrm{d}s \\
= {}& a\Gamma \int_0^t (R : Q_\epsilon)\,\mathrm{d}s - \Gamma \int_0^t \left(b\left[R^2 - \frac{\mathrm{tr}(R^2)}{3}\mathcal{I}_3\right] - cR\mathrm{tr}(R^2) : Q_\epsilon - \Delta Q_\epsilon\right)\mathrm{d}s \\
&+ (R(t) : Q_\epsilon(t)) + (\nabla R(t) : \nabla Q_\epsilon(t)) - (R_0 : Q_\epsilon(0)) - (\nabla R_0 : \nabla Q_\epsilon(0)), \qquad (3.22)
\end{aligned}
$$

where

$$
\overline{D} := \frac{1}{2}\left(\nabla v + \nabla v^T\right) \quad \text{and} \quad \overline{\Omega} := \frac{1}{2}\left(\nabla v - \nabla v^T\right).
$$

Since the procedure is the same as in (3.21), we omit the details here. More precisely, the following estimates of the nonlinear terms are enough to show things:

$$
\begin{aligned}
&\Big|\int_0^t (\Delta Q_\epsilon^i - \Delta Q_\epsilon : v\cdot\nabla R)\mathrm{d}s\Big| \\
\le& \int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}\|\nabla R\|_{L^{\frac{6p}{5p-6}}}\|v\|_{L^6}\mathrm{d}s \\
\le& \int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}\|\nabla R\|_{L^2}^{\frac{2p-3}{p}}\|\nabla R\|_{L^6}^{\frac{3-p}{p}}\|\nabla v\|_{L^2}\mathrm{d}s \\
\le& \left(\int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}^2\left(\|\nabla R\|_{L^2}^2\right)^{\frac{2p-3}{p}}\left(\|D^2R\|_{L^2}^2\right)^{\frac{3-p}{p}}\mathrm{d}s\right)^{\frac12}\|\nabla v\|_{L^2(0,T;L^2)} \\
\le& \left(\int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}^{\frac{2p}{2p-3}}\|\nabla R\|_{L^2}^2\right)^{\frac{2p-3}{2p}}\left(\int_0^t\|D^2R\|_{L^2}^2\mathrm{d}s\right)^{\frac{3-p}{2p}}\|\nabla v\|_{L^2(0,T;L^2)} \\
\le& \|R\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|R\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla v\|_{L^2(0,T;L^2)}\|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^q(0,T;L^p)},
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\Big|\int_0^t (R : v\cdot\nabla Q_\epsilon^i - \nabla Q_\epsilon)\mathrm{d}s\Big| \le& \int_0^t \|R\|_{L^3}\|v\|_{L^6}\|\nabla Q_\epsilon^i - \nabla Q_\epsilon\|_{L^2}\mathrm{d}s \\
\le& \left(\int_0^t \|R\|_{H^1}^2\|\nabla v\|_{L^2}^2\mathrm{d}s\right)^{\frac12}\|\nabla Q_\epsilon^i - \nabla Q_\epsilon\|_{L^2(0,T;L^2)} \\
\le& \|R\|_{L^\infty(0,T;H^1)}\|v\|_{L^2(0,T;H^1)}\|\nabla Q_\epsilon^i - \nabla Q_\epsilon\|_{L^2(0,T;L^2)}
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
&\Big|\int_0^t (R\overline{\Omega} - \overline{\Omega}R - \lambda|R|\overline{D} : \Delta Q_\epsilon^i - \Delta Q_\epsilon)\mathrm{d}s\Big| \\
\le& \int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}\|R\|_{L^{\frac{2p}{p-2}}}\|\nabla v\|_{L^2}\mathrm{d}s \\
\le& \int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}\|R\|_{W^{1,\frac{6p}{5p-6}}}\|\nabla v\|_{L^2}\mathrm{d}s \\
\le& \int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}\|R\|_{H^1}^{\frac{2p-3}{p}}\|R\|_{W^{1,6}}^{\frac{3-p}{p}}\|\nabla v\|_{L^2}\mathrm{d}s \\
\le& \left(\int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}^2\left(\|R\|_{H^1}^2\right)^{\frac{2p-3}{p}}\left(\|R\|_{H^2}^2\right)^{\frac{3-p}{p}}\mathrm{d}s\right)^{\frac12}\|\nabla v\|_{L^2(0,T;L^2)} \\
\le& \left(\int_0^t \|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^p}^{\frac{2p}{2p-3}}\|R\|_{H^1}^2\right)^{\frac{2p-3}{2p}}\left(\int_0^t\|R\|_{H^2}^2\mathrm{d}s\right)^{\frac{3-p}{2p}}\|\nabla v\|_{L^2(0,T;L^2)} \\
\le& \|R\|_{L^\infty(0,T;H^1)}^{\frac{2p-3}{p}}\|R\|_{L^2(0,T;H^2)}^{\frac{3-p}{p}}\|\nabla v\|_{L^2(0,T;L^2)}\|\Delta Q_\epsilon^i - \Delta Q_\epsilon\|_{L^q(0,T;L^p)}.
\end{aligned}
\tag{3.25}
$$

(4) Now, summing up (3.21) and (3.22), by a similar procedure as the proof of (1.6), we obtain the desired (1.7) by taking the limit as $\epsilon \to 0$.

(5) Finally, we show the energy equality (1.8) for Q and u. Since (R, v) in (1.6) and (1.7) can be replaced by (Q, u), a direct summation yields

$$
\|u(t)\|_{L^2} + \|Q(t)\|_{H^1}^2 = \|u_0\|_{L^2}^2 + \|Q_0\|_{H^1}^2 - 2\mu\int_0^t \|\nabla u\|_{L^2}^2\mathrm{d}s + \underbrace{2\int_0^t (u, u\cdot\nabla u)\mathrm{d}s}_{\mathcal{K}_1}
$$

$$
\begin{aligned}
&-2a\Gamma\int_0^t\|Q\|_{L^2}^2\mathrm{d}s-2(a+1)\Gamma\int_0^t\|\nabla Q\|_{L^2}^2\mathrm{d}s-2\Gamma\int_0^t\|\Delta Q\|_{L^2}^2\mathrm{d}s\\
&+\underbrace{2\int_0^t(\nabla Q\odot\nabla Q:\nabla u)+(\Delta Q:u\cdot\nabla Q)\mathrm{d}s}_{\mathcal{K}_2}+\underbrace{2\int_0^t(Q:u\cdot\nabla Q)\mathrm{d}s}_{\mathcal{K}_3}\\
&+\underbrace{2\int_0^t(Q\Omega-\Omega Q:\Delta Q)-(Q\Delta Q-\Delta QQ:\nabla u)\,\mathrm{d}s}_{\mathcal{K}_4}\\
&-\underbrace{2\int_0^t(Q\Omega-\Omega Q:Q)\mathrm{d}s}_{\mathcal{K}_5}+2\int_0^t(\lambda|Q|D:Q)\mathrm{d}s-2\int_0^t(\kappa Q:\nabla u)\,\mathrm{d}s\\
&+\underbrace{2\int_0^t(\lambda|Q|\Delta Q:\nabla u)-(\lambda|Q|D:\Delta Q)\mathrm{d}s}_{\mathcal{K}_6}-2\int_0^t(a\lambda|Q|Q:\nabla u)\,\mathrm{d}s\\
&+2\lambda\int_0^t\left(|Q|\left\{b\left[Q^2-\frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right]-cQ\mathrm{tr}(Q^2)\right\}:\nabla u\right)\mathrm{d}s\\
&+2\Gamma\int_0^t\left(b\left[Q^2-\frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right]-cQ\mathrm{tr}(Q^2):Q-\Delta Q\right)\mathrm{d}s. \qquad (3.26)
\end{aligned}
$$

Now, by the cancellation rules proven in Lemma A.1, we see that $\mathcal{K}_i=0$, $1\le i\le 6$. With these facts in hand, (3.26) turns into the desired energy equality (1.8).

This completes the proof of Theorem 1.2. □

4 Weak-Strong Uniqueness

In this section, we prove Theorem 1.5, which implies the weak-strong uniqueness for the Leray-Hopf type weak solutions to system (1.1).

Proof of Theorem 1.5 Let us denote that

$$G:=R-Q,\quad \omega:=v-u\quad\text{and}\quad \widetilde{\Omega}:=\overline{\Omega}-\Omega.$$

Notice that

$$
\begin{aligned}
\|\omega(t)\|_{L^2}^2+\|G(t)\|_{H^1}^2=&\|v(t)\|_{L^2}^2+\|R(t)\|_{H^1}^2+\|u(t)\|_{L^2}^2+\|Q(t)\|_{H^1}^2\\
&-2(v(t),u(t))-2(R(t),Q(t))-2(\nabla R(t),\nabla Q(t)). \qquad (4.1)
\end{aligned}
$$

We substitute (1.4), (1.6), (1.7) and (1.8) into the above equation and apply integration by parts to obtain

$$
\begin{aligned}
&\|\omega(t)\|_{L^2}^2+\|G(t)\|_{H^1}^2\\
\le&-2\mu\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+\underbrace{2\int_0^t(u,\omega\cdot\nabla v)\mathrm{d}s-2\kappa\int_0^t(G:\nabla\omega)\,\mathrm{d}s}_{\mathcal{Q}_0}\\
&-2a\Gamma\int_0^t\|G\|_{L^2}^2\mathrm{d}s-2(a+1)\Gamma\int_0^t\|\nabla G\|_{L^2}^2\mathrm{d}s-2\Gamma\int_0^t\|\Delta G\|_{L^2}^2\mathrm{d}s
\end{aligned}
$$

$$\underbrace{-2\int_0^t (\nabla Q\odot\nabla Q:\nabla v)+(\nabla R\odot\nabla R:\nabla u)\,\mathrm{d}s}_{\mathcal{Q}_1}$$

$$\underbrace{-2\int_0^t (Q:u\cdot\nabla R)+(\Delta R:u\cdot\nabla Q)+(R:v\cdot\nabla Q)+(\Delta Q:v\cdot\nabla R)\mathrm{d}s}_{\mathcal{Q}_2}$$

$$\underbrace{+2\int_0^t (Q\Delta Q-\Delta QQ:\nabla v)+(R\Delta R-\Delta RR:\nabla u)\,\mathrm{d}s}_{\mathcal{Q}_3}$$

$$\underbrace{+2\int_0^t (Q\Omega-\Omega Q:R-\Delta R)+(R\overline{\Omega}-\overline{\Omega}R:Q-\Delta Q)\mathrm{d}s}_{\mathcal{Q}_4}$$

$$\underbrace{+2\lambda\int_0^t \big(|R|\overline{D}-|Q|D:G\big)\,\mathrm{d}s+2a\lambda\int_0^t (|Q|Q-|R|R:\nabla\omega)\,\mathrm{d}s}_{\mathcal{Q}_5}$$

$$\underbrace{+2\lambda\int_0^t (|Q|D:\Delta R)+(|R|\overline{D}:\Delta Q)-(|Q|\Delta Q:\nabla v)-(|R|\Delta R:\nabla u)\,\mathrm{d}s}_{\mathcal{Q}_6}$$

$$\underbrace{+2\int_0^t \Gamma\left(\mathcal{M}[R]-\mathcal{M}[Q]:G-\Delta G\right)+\lambda\left(|R|\mathcal{M}[R]-|Q|\mathcal{M}[Q]:\nabla\omega\right)\mathrm{d}s,}_{\mathcal{Q}_7} \tag{4.2}$$

where

$$\mathcal{M}[Q]:=b\left[Q^2-\frac{\mathrm{tr}(Q^2)}{3}\mathcal{I}_3\right]-cQ\mathrm{tr}(Q^2).$$

Recalling that $v=\omega+u$ and $\int_0^t(u,\omega\cdot\nabla u)\mathrm{d}s=0$, we deduce that

$$\int_0^t(u,\omega\cdot\nabla v)\mathrm{d}s=\int_0^t(u,\omega\cdot\nabla\omega)\mathrm{d}s.$$

Thus, applying (3.3), we have

$$\begin{aligned}
|\mathcal{Q}_0| &\le \int_0^t\int_{\mathbb{R}^3}|u||\omega||\nabla\omega|+|G||\nabla\omega|\mathrm{d}x\mathrm{d}s\\
&\le \int_0^t \|u\|_{L^s}\|\omega\|_{L^{\frac{2s}{s-2}}}\|\nabla\omega\|_{L^2}+\|G\|_{L^2}\|\nabla\omega\|_{L^2}\mathrm{d}s\\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\|G\|_{L^2}^2\mathrm{d}s+\int_0^t\|u\|_{L^s}\|\omega\|_{L^2}^{\frac{s-3}{s}}\|\omega\|_{L^6}^{\frac{3}{s}}\|\nabla\omega\|_{L^2}\mathrm{d}s\\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\|G\|_{L^2}^2\mathrm{d}s+\int_0^t\|u\|_{L^s}\|\omega\|_{L^2}^{\frac{s-3}{s}}\|\nabla\omega\|_{L^2}^{\frac{s+3}{s}}\mathrm{d}s\\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\|G\|_{L^2}^2\mathrm{d}s+\int_0^t\left[\|u\|_{L^s}\left(\|\omega\|_{L^2}^2\right)^{\frac{s-3}{2s}}\right]\cdot\left(\|\nabla\omega\|_{L^2}^2\right)^{\frac{s+3}{2s}}\mathrm{d}s\\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\|G\|_{L^2}^2\mathrm{d}s+\int_0^t\|u\|_{L^s}^{\frac{2s}{s-3}}\|\omega\|_{L^2}^2\mathrm{d}s\\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\underbrace{\left(1+\|\nabla u\|_{L^p}^{\frac{2p}{2p-3}}\right)}_{\mathcal{N}_1}\left(\|G\|_{L^2}^2+\|\omega\|_{L^2}^2\right)\mathrm{d}s.
\end{aligned} \tag{4.3}$$

On the other hand, by applying Lemma A.2, one can estimate $\mathcal{Q}_1+\mathcal{Q}_2$ as follows:

$$\begin{aligned}
|\mathcal{Q}_1+\mathcal{Q}_2| &= \left|2\int_0^t (\Delta G, u\cdot\nabla G)-(\Delta Q,\omega\cdot\nabla G)-(G:\omega\cdot\nabla Q)\mathrm{d}s\right| \\
&\le \int_0^t \|u\|_{L^s}\|\nabla G\|_{L^{\frac{2s}{s-2}}}\|\Delta G\|_{L^2}\mathrm{d}s+\int_0^t \|\Delta Q\|_{L^p}\|\nabla G\|_{L^{\frac{6p}{5p-6}}}\|\omega\|_{L^6}\mathrm{d}s \\
&\quad +\int_0^t \|G\|_{L^2}\|\omega\|_{L^6}\|\nabla Q\|_{L^3}\mathrm{d}s \\
&\le \int_0^t \|u\|_{L^s}\|\nabla G\|_{L^2}^{\frac{s-3}{s}}\|\nabla G\|_{L^6}^{\frac{3}{s}}\|\Delta G\|_{L^2}\mathrm{d}s \\
&\quad +\int_0^t \|\Delta Q\|_{L^p}\|\nabla G\|_{L^2}^{\frac{2p-3}{p}}\|\nabla G\|_{L^6}^{\frac{3-p}{p}}\|\nabla\omega\|_{L^2}\mathrm{d}s+\int_0^t\|\nabla\omega\|_{L^2}\|G\|_{L^2}\|\nabla Q\|_{H^1}\mathrm{d}s \\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\|Q\|_{H^2}^2\|G\|_{L^2}^2\mathrm{d}s+\int_0^t\|u\|_{L^s}\|\nabla G\|_{L^2}^{\frac{s-3}{s}}\|D^2G\|_{L^2}^{\frac{s+3}{s}}\mathrm{d}s \\
&\quad +\int_0^t \|\Delta Q\|_{L^p}\|\nabla G\|_{L^2}^{\frac{2p-3}{p}}\|D^2G\|_{L^2}^{\frac{3-p}{p}}\|\nabla\omega\|_{L^2}\mathrm{d}s \\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\|Q\|_{H^2}^2\|G\|_{L^2}^2\mathrm{d}s \\
&\quad +\int_0^t\left[\|u\|_{L^s}\left(\|\nabla G\|_{L^2}^2\right)^{\frac{s-3}{2s}}\right]\cdot\left(\|D^2G\|_{L^2}^2\right)^{\frac{s+3}{2s}}\mathrm{d}s \\
&\quad +\int_0^t\left[\|\Delta Q\|_{L^p}^2\left(\|\nabla G\|_{L^2}^2\right)^{\frac{2p-3}{p}}\right]\cdot\left(\|D^2G\|_{L^2}^2\right)^{\frac{3-p}{p}}\mathrm{d}s \\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2+\|D^2G\|_{L^2}^2\mathrm{d}s \\
&\quad +C(\epsilon)\int_0^t\|Q\|_{H^2}^2\|G\|_{L^2}^2+\|u\|_{L^s}^{\frac{2s}{s-3}}\|\nabla G\|_{L^2}^2+\|\Delta Q\|_{L^p}^{\frac{2p}{2p-3}}\|\nabla G\|_{L^2}^2\mathrm{d}s \\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2+\|D^2G\|_{L^2}^2\mathrm{d}s \\
&\quad +C(\epsilon)\int_0^t\underbrace{\left(\|Q\|_{H^2}^2+\|\nabla u\|_{L^p}^{\frac{2p}{2p-3}}+\|\Delta Q\|_{L^p}^{\frac{2p}{2p-3}}\right)}_{\mathcal{N}_2}\|G\|_{H^1}^2\mathrm{d}s. \qquad (4.4)
\end{aligned}$$

Similarly, using the identities shown in Lemma A.2, one has following estimates:

$$\begin{aligned}
|\mathcal{Q}_3+\mathcal{Q}_4| &= \left|2\int_0^t\left(\widetilde{\Omega}Q-Q\widetilde{\Omega}:G\right)+(G\Delta G-\Delta GG,\nabla u)-(G\Delta Q-\Delta QG,\nabla\omega)\mathrm{d}s\right| \\
&\le \int_0^t\|Q\|_{L^3}\|\nabla\omega\|_{L^2}\|G\|_{L^6}\mathrm{d}s+\int_0^t\|\nabla u\|_{L^p}\|G\|_{L^{\frac{2p}{p-2}}}\|\Delta G\|_{L^2}\mathrm{d}s \\
&\quad +\int_0^t\|\Delta Q\|_{L^p}\|G\|_{L^{\frac{2p}{p-2}}}\|\nabla\omega\|_{L^2}\mathrm{d}s \\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}s+C(\epsilon)\int_0^t\|Q\|_{H^2}^2\|\nabla G\|_{L^2}^2\mathrm{d}s \\
&\quad +\int_0^t\|\nabla u\|_{L^p}\|G\|_{W^{1,\frac{6p}{5p-6}}}\|\Delta G\|_{L^2}\mathrm{d}s+\int_0^t\|\Delta Q\|_{L^p}\|G\|_{W^{1,\frac{6p}{5p-6}}}\|\nabla\omega\|_{L^2}\mathrm{d}s \\
&\le \epsilon\int_0^t\|\nabla\omega\|_{L^2}^2\mathrm{d}\tau+C(\epsilon)\int_0^t\|Q\|_{H^2}^2\|\nabla G\|_{L^2}^2\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|\nabla u\|_{L^p}\|G\|_{H^1}^{\frac{2p-3}{p}}\|G\|_{W^{1,6}}^{\frac{3-p}{p}}\|\Delta G\|_{L^2}\mathrm{d}\tau \\
&+\int_0^t \|\Delta Q\|_{L^p}\|G\|_{H^1}^{\frac{2p-3}{p}}\|G\|_{W^{1,6}}^{\frac{3-p}{p}}\|\nabla \omega\|_{L^2}\mathrm{d}\tau \\
\leq{}& \epsilon\int_0^t \|\nabla\omega\|_{L^2}^2\mathrm{d}s + C(\epsilon)\int_0^t \|Q\|_{H^2}^2\|\nabla G\|_{L^2}^2\mathrm{d}s \\
&+\int_0^t \|\nabla u\|_{L^p}\|G\|_{H^1}^{\frac{2p-3}{p}}\|G\|_{H^2}^{\frac{3-p}{p}}\|\Delta G\|_{L^2}\mathrm{d}s \\
&+\int_0^t \|\Delta Q\|_{L^p}\|G\|_{H^1}^{\frac{2p-3}{p}}\|G\|_{H^2}^{\frac{3-p}{p}}\|\nabla \omega\|_{L^2}\mathrm{d}s \\
\leq{}& \epsilon\int_0^t \|\nabla\omega\|_{L^2}^2 + \|\Delta G\|_{L^2}^2\mathrm{d}s + C(\epsilon)\int_0^t \|Q\|_{H^2}^2\|\nabla G\|_{L^2}^2\mathrm{d}s \\
&+C(\epsilon)\int_0^t \left[\|\nabla u\|_{L^p}^2\left(\|G\|_{H^1}^2\right)^{\frac{2p-3}{p}}\right]\cdot\left(\|G\|_{H^2}^2\right)^{\frac{3-p}{p}} \\
&+\left[\|\Delta Q\|_{L^p}^2\left(\|G\|_{H^1}^2\right)^{\frac{2p-3}{p}}\right]\cdot\left(\|G\|_{H^2}^2\right)^{\frac{3-p}{p}}\mathrm{d}s \\
\leq{}& \epsilon\int_0^t \|\nabla\omega\|_{L^2}^2 + \|G\|_{H^2}^2\mathrm{d}s \\
&+C(\epsilon)\int_0^t \underbrace{\left(\|Q\|_{H^2}^2 + \|\nabla u\|_{L^p}^{\frac{2p}{2p-3}} + \|\Delta Q\|_{L^p}^{\frac{2p}{2p-3}}\right)}_{\mathcal{N}_3}\|G\|_{H^1}^2\mathrm{d}s,
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
|\mathcal{Q}_5+\mathcal{Q}_6| \leq{}& \int_0^t\int_{\mathbb{R}^3} |G||\nabla u||\Delta G| + |G|^2|\nabla u| + |G||\Delta Q||\nabla\omega| + |R||G||\nabla\omega| + |Q||G||\nabla\omega|\mathrm{d}x\mathrm{d}s \\
\leq{}& \int_0^t \|\nabla u\|_{L^p}\|G\|_{L^{\frac{2p}{p-2}}}\|\Delta G\|_{L^2}\mathrm{d}s + \int_0^t \|\Delta Q\|_{L^p}\|G\|_{L^{\frac{2p}{p-2}}}\|\nabla\omega\|_{L^2}\mathrm{d}s \\
&+\int_0^t \|\nabla u\|_{L^2}\|G\|_{L^3}\|G\|_{L^6} + \|\nabla\omega\|_{L^2}\|R\|_{L^3}\|G\|_{L^6} + \|\nabla\omega\|_{L^2}\|Q\|_{L^3}\|G\|_{L^6}\mathrm{d}s \\
\leq{}& \epsilon\int_0^t \|\nabla\omega\|_{L^2}^2 + \|G\|_{H^2}^2\mathrm{d}s + C(\epsilon)\int_0^t\left(\|\nabla u\|_{L^p}^{\frac{2p}{2p-3}} + \|\Delta Q\|_{L^p}^{\frac{2p}{2p-3}}\right)\|G\|_{H^1}^2\mathrm{d}s \\
&+\int_0^t \|\nabla u\|_{L^2}\|G\|_{H^1}\|\nabla G\|_{L^2} + \|\nabla\omega\|_{L^2}\|R\|_{H^1}\|\nabla G\|_{L^2} + \|\nabla\omega\|_{L^2}\|Q\|_{H^1}\|\nabla G\|_{L^2}\mathrm{d}s \\
\leq{}& \epsilon\int_0^t \|\nabla\omega\|_{L^2}^2 + \|G\|_{H^2}^2\mathrm{d}s \\
&+C(\epsilon)\int_0^t \underbrace{\left(\|u\|_{H^1}^2 + \|Q\|_{H^2}^2 + \|R\|_{H^2}^2 + \|\nabla u\|_{L^p}^{\frac{2p}{2p-3}} + \|\Delta Q\|_{L^p}^{\frac{2p}{2p-3}}\right)}_{\mathcal{N}_4}\|G\|_{H^1}^2\mathrm{d}s,
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
|\mathcal{Q}_7| \leq{}& \int_0^t\int_{\mathbb{R}^3} (|R||G| + |Q||G|)(|G| + |\Delta G|)\mathrm{d}x\mathrm{d}s \\
&+\int_0^t\int_{\mathbb{R}^3} (|R|^2|G| + |R||Q||G| + |G||Q|^2)(|\nabla\omega| + |G| + |\Delta G|)\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}^3}(|Q|^3|G|+|R|^2|Q||G|+|R||G||Q|^2+|R|^3|G|)|\nabla\omega|\mathrm{d}x\mathrm{d}s\\
\le{}&\epsilon\int_0^t\|\nabla\omega\|_{L^2}^2+\|G\|_{L^2}^2+\|\Delta G\|_{L^2}^2\mathrm{d}\tau+C(\epsilon)\int_0^t(\|R\|_{L^4}^2+\|Q\|_{L^4}^2)\|G\|_{L^4}^2\mathrm{d}s\\
&+C(\epsilon)\int_0^t(\|R\|_{L^6}^4+\|R\|_{L^6}^2\|Q\|_{L^6}^2+\|Q\|_{L^6}^4)\|G\|_{L^6}^2\mathrm{d}s\\
&+C(\epsilon)\int_0^t(\|R\|_{L^9}^6+\|R\|_{L^9}^4\|Q\|_{L^9}^2+\|R\|_{L^9}^2\|Q\|_{L^9}^4+\|Q\|_{L^9}^6)\|G\|_{L^6}^2\mathrm{d}s\\
\le{}&\epsilon\int_0^t\|\nabla\omega\|_{L^2}^2+\|G\|_{L^2}^2+\|\Delta G\|_{L^2}^2\mathrm{d}\tau+C(\epsilon)\int_0^t(\|Q\|_{H^1}^2+\|R\|_{H^1}^2)\|G\|_{H^1}^2\mathrm{d}s\\
&+C(\epsilon)\int_0^t(\|\nabla R\|_{L^2}^4+\|\nabla Q\|_{L^2}^4+\|Q\|_{L^9}^6+\|R\|_{L^9}^6)\|\nabla G\|_{L^2}^2\mathrm{d}s\\
\le{}&\epsilon\int_0^t\|\nabla\omega\|_{L^2}^2+\|G\|_{L^2}^2+\|\Delta G\|_{L^2}^2\mathrm{d}s\\
&+C(\epsilon)\int_0^t(1+\|Q\|_{H^1}^2+\|R\|_{H^1}^2)(\|Q\|_{H^2}^2+\|R\|_{H^2}^2)\|G\|_{H^1}^2\mathrm{d}s\\
&+C(\epsilon)\int_0^t(\|D^2Q\|_{L^2}\|\nabla Q\|_{L^2}^5+\|D^2R\|_{L^2}\|\nabla R\|_{L^2}^5)\|\nabla G\|_{L^2}^2\mathrm{d}s\\
\le{}&\epsilon\int_0^t\|\nabla\omega\|_{L^2}^2+\|G\|_{L^2}^2+\|\Delta G\|_{L^2}^2\mathrm{d}s\\
&+C(\epsilon)\int_0^t\underbrace{(1+\|Q\|_{H^1}^2+\|R\|_{H^1}^2+\|Q\|_{H^1}^4+\|R\|_{H^1}^4)(\|Q\|_{H^2}^2+\|R\|_{H^2}^2)}_{\mathcal{N}_5}\|G\|_{H^1}^2\mathrm{d}s.
\end{aligned}
\tag{4.7}
$$

Now, let

$$\mathcal{E}(t):=\|\omega(t)\|_{L^2}^2+\|G(t)\|_{H^1}^2.$$

Then $\mathcal{E}(0)=0$. From the above results, along with a sufficiently small ϵ, we have that

$$\mathcal{E}(t)\le\int_0^t A(s)\mathcal{E}(s)\mathrm{d}s, \tag{4.8}$$

where $A:=\sum_{i=1}^{5}\mathcal{N}_i$. Clearly, by the regularity of (Q,u) and (R,v), A is integrable on $[0,T]$. Therefore, by Grönwall's inequality, we conclude that $\mathcal{E}(t)\equiv 0$ for all $t\in[0,T]$. This implies the uniqueness, and the proof of Theorem 1.5 is complete. □

Conflict of Interest The authors declare no conflict of interest.

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Appendix A Some Basic Theories and Lemmas

In this section, we state some lemmas that have been used extensively in this paper. First, let us prove the following important cancellations rules (see also [11, 37] and references therein):

Lemma A.1 Let $u \in L^2_\sigma(\mathbb{R}^d)$ and $Q \in H^2(\mathbb{R}^d) \cap S^d_0$. Then

(1) $(u \cdot \nabla u, u) = (u \cdot \nabla Q, Q) = 0$;

(2) $(Q\Omega - \Omega Q, Q) = 0$;

(3) $(u \cdot \nabla Q, \Delta Q) - (\nabla \cdot (\nabla Q \odot \nabla Q), u) = 0$;

(4) $(Q\Omega - \Omega Q, \Delta Q) - (Q\Delta Q - \Delta Q Q, \nabla u) = 0$;

(5) $(|Q|\Delta Q, \nabla u) - (|Q|D, \Delta Q) = 0$.

Proof We begin with the proof of item (1). In fact, we only need to show that

$$(u \cdot \nabla f, f) = 0, \quad \forall f \in H^1.$$

Since there exists a sequence $\{f_k\} \subset C_0^\infty(\mathbb{R}^d)$ that converges to f in H^1, it follows that

$$\lim_{k\to\infty} (u \cdot \nabla f_k, f_k) = (u \cdot \nabla f, f).$$

Now, by integration by parts, and recalling that u is divergence free, we have

$$(u \cdot \nabla f_k, f_k) = 0, \quad \forall k \in \mathbb{N},$$

which furnishes our first statement. On the other hand, since the trace is invariant under circular shifts, it follows that

$$(Q\Omega - \Omega Q, Q) = 0.$$

Finally, the items (3), (4) and (5) follow from [11, Proposition 2.1] and [11, Lemma A.1]. □

Next, we state the following Lemma, which simplifies our proof in Section 4:

Lemma A.2 Let (Q,u) and (R,v) be two Leray-Hopf weak solutions in Theorem 1.5. Then we have

$$\mathcal{Q}_1+\mathcal{Q}_2=2\int_0^t(\Delta G,u\cdot\nabla G)-(\Delta Q,\omega\cdot\nabla G)-(G:\omega\cdot\nabla Q)\mathrm{d}s, \tag{A.1}$$

$$\mathcal{Q}_3+\mathcal{Q}_4=2\int_0^t\left(\widetilde{\Omega}Q-Q\widetilde{\Omega}:G\right)+(G\Delta G-\Delta GG,\nabla u)-(G\Delta Q-\Delta QG,\nabla\omega)\mathrm{d}s, \tag{A.2}$$

$$\begin{aligned}\mathcal{Q}_5+\mathcal{Q}_6=2\lambda\int_0^t&((|Q|-|R|)D:\Delta G)-((|Q|-|R|)D:G)+((|R|-|Q|)\Delta Q,\nabla\omega)\mathrm{d}s\\&+2\lambda\int_0^t(|R|G,\nabla\omega)\mathrm{d}s-2a\lambda\int_0^t((|R|-|Q|)R,\nabla\omega)\mathrm{d}s-2a\lambda\int_0^t(|Q|G,\nabla\omega)\mathrm{d}s,\end{aligned} \tag{A.3}$$

$$\mathcal{M}[R]-\mathcal{M}[Q]=b\left(RG+QG\right)-b\left(\frac{\mathrm{tr}(RG+QG)}{3}I_3\right)-c\left(Q\mathrm{tr}(RG+QG)+G\mathrm{tr}(R^2)\right) \tag{A.4}$$

and

$$\begin{aligned}&|R|\mathcal{M}[R]-|Q|\mathcal{M}[Q]\\=&\,b\left(|R|(RG+QG)+(|R|-|Q|)Q^2\right)-b\left(\frac{|R|\mathrm{tr}(RG+QG)+(|R|-|Q|)\mathrm{tr}(Q^2)}{3}I_3\right)\\&+c\left((|Q|-|R|)Q\mathrm{tr}(Q^2)-|R|Q\mathrm{tr}(RG+QG)-|R|G\mathrm{tr}(R^2)\right).\end{aligned} \tag{A.5}$$

Proof First, as was shown in Lemma A.1, it is not difficlut to have that

$$\int_0^t(Q:u\cdot\nabla R)+(R:u\cdot\nabla Q)\mathrm{d}\tau=0\quad\text{and}\quad\int_0^t(Q:\omega\cdot\nabla Q)\mathrm{d}\tau=0.$$

Thus we obtain that

$$\begin{aligned}\mathcal{Q}_1+\mathcal{Q}_2&=2\int_0^t(\Delta Q,v\cdot\nabla Q)+(\Delta R,u\cdot\nabla R)-(\Delta R,u\cdot\nabla Q)-(\Delta Q,v\cdot\nabla R)\mathrm{d}s\\&\quad-2\int_0^t(Q:u\cdot\nabla R)+(R:v\cdot\nabla Q)\mathrm{d}s\\&=2\int_0^t(\Delta R,u\cdot\nabla G)-(\Delta Q,v\cdot\nabla G)\mathrm{d}s-2\int_0^t(R:\omega\cdot\nabla Q)\mathrm{d}s\\&=2\int_0^t(\Delta R,u\cdot\nabla G)-(\Delta Q,v\cdot\nabla G)\mathrm{d}s-2\int_0^t(G:\omega\cdot\nabla Q)\mathrm{d}s\\&=2\int_0^t(\Delta G,u\cdot\nabla G)-(\Delta Q,\omega\cdot\nabla G)-(G:\omega\cdot\nabla Q)\mathrm{d}s.\end{aligned}$$

Similarly, by Lemma A.1, one can find the above identities for $\mathcal{Q}_3+\mathcal{Q}_4$ and $\mathcal{Q}_5+\mathcal{Q}_6$. Moreover, a direct calculation gives

$$\begin{aligned}\mathcal{M}[R]-\mathcal{M}[Q]&=b\left(R^2-Q^2\right)-b\left(\frac{\mathrm{tr}(R^2)-\mathrm{tr}(Q^2)}{3}I_3\right)+c\left(Q\mathrm{tr}(Q^2)-R\mathrm{tr}(R^2)\right)\\&=b\left(RG+QG\right)-b\left(\frac{\mathrm{tr}(RG+QG)}{3}I_3\right)-c\left(Q\mathrm{tr}(RG+QG)+G\mathrm{tr}(R^2)\right)\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}&|R|\mathcal{M}[R]-|Q|\mathcal{M}[Q]\\=&\,b\left(|R|R^2-|Q|Q^2\right)-b\left(\frac{|R|\mathrm{tr}(R^2)-|Q|\mathrm{tr}(Q^2)}{3}I_3\right)+c\left(|Q|Q\mathrm{tr}(Q^2)-|R|R\mathrm{tr}(R^2)\right)\end{aligned}$$

$$
\begin{aligned}
&= b\left(|R|R^2 - |R|Q^2 + |R|Q^2 - |Q|Q^2\right) \\
&\quad - b\left(\frac{|R|\mathrm{tr}(R^2) - |R|\mathrm{tr}(Q^2) + |R|\mathrm{tr}(Q^2) - |Q|\mathrm{tr}(Q^2)}{3} I_3\right) \\
&\quad + c\left(|Q|Q\mathrm{tr}(Q^2) - |R|Q\mathrm{tr}(Q^2) + |R|Q\mathrm{tr}(Q^2) - |R|Q\mathrm{tr}(R^2) - |R|G\mathrm{tr}(R^2)\right) \\
&= b\left(|R|(RG+QG) + (|R|-|Q|)Q^2\right) - b\left(\frac{|R|\mathrm{tr}(RG+QG) + (|R|-|Q|)\mathrm{tr}(Q^2)}{3} I_3\right) \\
&\quad + c\left((|Q|-|R|)Q\mathrm{tr}(Q^2) - |R|Q\mathrm{tr}(RG+QG) - |R|G\mathrm{tr}(R^2)\right). \qquad \text{(A.7)}
\end{aligned}
$$

This completes the proof. □

Finally, we close this section by presenting the following well-known interpolation inequality:

Lemma A.3 (Gagliardo-Nirenberg inequality) Let $1 \le q, r \le \infty$ and be a natural number m. Suppose that a real number α and a natural number $j < m$ are such that

$$
\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q}
$$

and

$$
\frac{j}{m} \le \alpha \le 1.
$$

In particular, under the folowing two exceptional cases:

(i) if $j = 0, mr < d$ and $q = \infty$, then it is necessary to make the additional assumption that either u tends to zero at infinity or that u lies in L^s for some finite $s > 0$;

(iii) if $1 < r < \infty$ and $m - j - \frac{d}{r}$ is a non-negative integer, then it is necessary to assume also that $\alpha \ne 1$,

the following inequalities hold:

$$
\|\mathrm{D}^j u\|_{L^p} \le C\|\mathrm{D}^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \tag{A.8}
$$

where C is a constant depending only on d, m, j, q, r and α.

As a corollary to this Lemma, we have

Corollary A.4 For any function $f \in H^2(\mathbb{R}^3)$, we have

$$
\|f\|_{L^4} \le \|\nabla f\|_{L^2}^{\frac{3}{4}}\|f\|_{L^2}^{\frac{1}{4}} \le \|f\|_{H^1}; \tag{A.9}
$$

$$
\|f\|_{L^8} \le \|D^2 f\|_{L^2}^{\frac{1}{8}}\|f\|_{L^6}^{\frac{7}{8}} \le \|D^2 f\|_{L^2}^{\frac{1}{8}}\|\nabla f\|_{L^2}^{\frac{7}{8}}; \tag{A.10}
$$

$$
\|f\|_{L^9} \le \|D^2 f\|_{L^2}^{\frac{1}{6}}\|f\|_{L^6}^{\frac{5}{6}} \le \|D^2 f\|_{L^2}^{\frac{1}{6}}\|\nabla f\|_{L^2}^{\frac{5}{6}}. \tag{A.11}
$$