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GENERALIZED FORELLI-RUDIN TYPE OPERATORS BETWEEN SEVERAL FUNCTION SPACES ON THE UNIT BALL OF $\mathbb{C}^{\mathrm{N}}$

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Abstract In this paper, we investigate sufficient and necessary conditions such that generalized Forelli-Rudin type operators $T_{\lambda,\tau,k}$, $S_{\lambda,\tau,k}$, $Q_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are bounded between Lebesgue type spaces. In order to prove the main results, we first give some bidirectional estimates for several typical integrals.

Key words Forelli-Rudin type operator; $L^{p,q,s,k}(B_n)$ space; boundedness; unit ball

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1 Introduction

In this paper, we write "$E \gtrsim G$" (or "$E \lesssim G$") if there exists a constant $c > 0$ such that $E \geq cG$ (or $E \leq cG$). We say that E and G are equivalent if "$E \gtrsim G$" and "$E \lesssim G$", written as "$E \asymp G$". All logarithmic and power functions take the main branch, that is, $\log 1 = 0$, $1^k = 1$ for real k.

Let B_n be the unit ball in $\mathbb{C}^n$ (we write as D when $n = 1$). The class of holomorphic functions on B_n is denoted by $H(B_n)$. Suppose that $\mathrm{d}v$ denotes the Lebesgue measure on B_n such that $v(B_n)$=1, and $\mathrm{d}\sigma$ denotes the measure on the boundary S_n of B_n such that $\sigma(S_n)$=1. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, the inner product of z and w is defined by

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}.$$

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For $\alpha \geq 0$, the growth space $\mathcal{G}_\alpha(B_n)$ is the set of all functions f on B_n such that

$$||f||_\alpha = \sup_{z\in B_n} (1-|z|^2)^\alpha |f(z)| < \infty.$$

For any $a \in B_n$, the Möbius transform of B_n is defined by

$$\varphi_a(z) = \frac{a - \frac{\langle z,a\rangle a}{|a|^2} - \sqrt{1-|a|^2}\left(z - \frac{\langle z,a\rangle a}{|a|^2}\right)}{1-\langle z,a\rangle} \quad (a \neq 0),$$

and $\varphi_0(z) = -z$. It is clear that φ_a has the following properties: $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a = \varphi_a^{-1}$. It follows from Lemma 1.3 in [1] that

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1-|a|^2)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)} \quad (z, w \in \overline{B_n}). \tag{1.1}$$

In particular, if $w = z$ or $w = 0$, then we have that

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}, \quad 1 - \langle \varphi_a(z), a\rangle = \frac{1-|a|^2}{1-\langle z,a\rangle}. \tag{1.2}$$

For $p > 0$, $s \geq 0$, $q + n \geq 0$, $q + s \geq 0$, if f is a Lebesgue measurable function on B_n and $||f||_{p,q,s} = \sup_{0\leq r<1} M_{p,q,s}(r, f) < \infty$, then we say that $f \in L^{p,q,s}(B_n)$, where

$$M^p_{p,q,s}(r, f) = \sup_{a\in B_n} (1-r^2)^q \int_{S_n} |f(r\xi)|^p \ (1-|\varphi_a(r\xi)|^2)^s \mathrm{d}\sigma(\xi).$$

The space $L^{p,q,s}(B_n)$ is a Banach space under the norm $||.||_{p,q,s}$ when $p \geq 1$. If $0 < p < 1$, then $L^{p,q,s}(B_n)$ is a complete metric space under the distance

$$\rho(f, g) = ||f - g||^p_{p,q,s}.$$

In particular, $H^{p,q,s}(B_n) = L^{p,q,s}(B_n) \bigcap H(B_n)$ is called the general Hardy type space. In fact, the space $H^{p,q,s}(B_n)$ comes from some practical applications. For example, in 2010, Stević and Ueki [2] proved that the multiplier operator M_u is bounded from $A^p_\alpha(B_n)$ to $H^q_\beta(B_n)$ if and only if $u \in H(B_n)$ and

$$\sup_{0\leq r<1} \sup_{a\in B_n} (1-r^2)^{\beta - \frac{q(\alpha+n+1)}{p}} \int_{S_n} |u(r\xi)|^q (1-|\varphi_a(r\xi)|^2)^{\frac{q(\alpha+n+1)}{p}} \mathrm{d}\sigma(\xi) < \infty.$$

There are also some similar applications in [3, 4]. Recently, we considered several basic problems of $H^{p,q,s}(B_n)$ in [5–7]. If $q = s = 0$, then $H^{p,q,s}(B_n)$ is just the Hardy space $H^p(B_n)$. Therefore, $H^{p,q,s}(B_n)$ is a generalization of the Hardy space. Furthermore, $H^{p,q,s}(B_n)$ contains several classical function spaces (see [5]).

Given $r > 0$, the Bergman ball with a as the center and r as the radius is the set

$$D(a, r) = \{z \in B_n : \ \beta(z, a) < r\}, \quad \text{where} \quad \beta(z, a) = \frac{1}{2} \log \frac{1 + |\varphi_a(z)|}{1 - |\varphi_a(z)|}.$$

For $p > 0$, $s \geq 0$, $q + n \geq 0$, $q + s \geq 0$ and a real number k, we define that $L^{p,q,s,k}(B_n) = \{f : ||f||_{p,q,s,k} < \infty\}$, where

$$||f||^p_{p,q,s,k} = \sup_{0\leq r<1} \sup_{a\in B_n} (1-r^2)^q \int_{S_n} \left| f(r\xi) \log^k \frac{\mathrm{e}}{1-|r\xi|^2} \right|^p \ (1-|\varphi_a(r\xi)|^2)^s \mathrm{d}\sigma(\xi).$$

For $f \in L^{p,q,s,k}(B_n)$ and $t > 0$, the function $|f|^t$ is usually not subharmonic on B_n. In order to discuss the operator problem from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$ for $0 < p < 1$, we need to add a condition. For any $t>0$, if $f\in L^{p,q,s,k}(B_n)$ and

$$|f(z)|^t \lesssim \frac{1}{(1-|z|^2)^{n+1}}\int_{D(z,r)}|f(w)|^t\mathrm{d}v(w) \ \text{ for all } z\in B_n, \tag{1.3}$$

then we say that $f\in\mathcal{H}^{p,q,s,k}(B_n)$ (we say that $f\in\mathcal{H}^{p,q,s}(B_n)$ when $k=0$), and the control constant in (1.3) relies only on n, t and r. Similarly, if $f\in\mathcal{G}_\alpha(B_n)$ and (1.3) is satisfied, then we say that $f\in\mathcal{H}^\infty_\alpha(B_n)$. For a real number k, let $\mathcal{H}^\infty_{\alpha,k}(B_n)=\{f: ||f||_{\alpha,k}<\infty$ and let f satisfy (1.3)$\}$, where

$$||f||_{\alpha,k}=\sup_{z\in B_n}(1-|z|^2)^\alpha|f(z)|\log^k\frac{\mathrm{e}}{1-|z|^2}.$$

For $p>0$ and a real number t, let

$$L^p(B_n,\mathrm{d}v_t)=\left\{f: ||f||_{p,t}=\left(\int_{B_n}|f(z)|^p\mathrm{d}v_t(z)\right)^{\frac{1}{p}}<\infty\right\},$$

where $\mathrm{d}v_t(z)=c_t(1-|z|^2)^t\mathrm{d}v(z)$, or $c_t=\frac{\Gamma(n+t+1)}{n!\Gamma(t+1)}$ when $t>-1$, or $c_t=1$ when $t\le -1$. Then

$$L^\infty(B_n)=\left\{f: ||f||_\infty=\operatorname{ess}\sup_{z\in B_n}|f(z)|<\infty\right\}.$$

For $p>0$ and real numbers t and k, let

$$L^p_{\log,k}(B_n,\mathrm{d}v_t)=\left\{f: ||f||_{p,t,\log,k}=\left(\int_{B_n}\left|f(z)\log^k\frac{\mathrm{e}}{1-|z|^2}\right|^p\mathrm{d}v_t(z)\right)^{\frac{1}{p}}<\infty\right\},$$

$$L^\infty_{\log,k}(B_n)=\left\{f: ||f||_{\infty,\log,k}=\operatorname{ess}\sup_{z\in B_n}|f(z)|\log^k\frac{\mathrm{e}}{1-|z|^2}<\infty\right\},$$

when $t>-1$, $L^\infty(B_n,\mathrm{d}v_t)=L^\infty(B_n)$ and $L^\infty_{\log,k}(B_n,\mathrm{d}v_t)=L^\infty_{\log,k}(B_n)$.

In 1974, Forelli and Rudin [8] introduced the following projection operator:

$$P_\tau f(z)=\int_{B_n}\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\tau}}\mathrm{d}v_\tau(w) \ \ (\tau>-1).$$

They proved that P_τ is a bounded operator from the Lebesgue space $L^p(B_n)$ to the Bergman space $A^p(B_n)$ if and only if $p(1+\tau)>1$ for $1\le p<\infty$. In 1979, Kolaski [9] considered P_τ from the weighted Lebesgue space $L^2(B_n,\mathrm{d}v_\alpha)$ to the weighted Bergman space $A^2_\alpha(B_n)$, and proved that P_τ is a bounded orthogonal projection if and only if $\tau=\alpha$ for $\alpha>-1$. In 1991, Zhu [10] studied more general Forelli-Rudin type operators $T_{\lambda,\tau}$ and $S_{\lambda,\tau}$ as

$$T_{\lambda,\tau}f(z)=(1-|z|^2)^\lambda\int_{B_n}\frac{(1-|w|^2)^\tau f(w)}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}\mathrm{d}v(w)$$

and

$$S_{\lambda,\tau}f(z)=(1-|z|^2)^\lambda\int_{B_n}\frac{(1-|w|^2)^\tau f(w)}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\mathrm{d}v(w) \ \ (z\in B_n),$$

where λ and τ are two real numbers. In 2006, Kures and Zhu [11] generalized the above two operators as

$$T_{\lambda,\tau,c}f(z)=(1-|z|^2)^\lambda\int_{B_n}\frac{(1-|w|^2)^\tau f(w)}{(1-\langle z,w\rangle)^c}\mathrm{d}v(w)$$

and

$$S_{\lambda,\tau,c}f(z)=(1-|z|^2)^\lambda\int_{B_n}\frac{(1-|w|^2)^\tau f(w)}{|1-\langle z,w\rangle|^c}\mathrm{d}v(w) \ \ (z\in B_n),$$

where λ, τ and c are three real numbers. Since Forelli-Rudin type operators are closely related to a large number of basic problems of function space theory and operator theory, many mathematicians are very interested in the boundedness of these operators between various function spaces. There is a lot of literature discussing the boundedness (see [8–27]). In [1] and [19], Zhu and Rudin gave the characterizations of the boundedness of P_τ from $L^p(B_n, \mathrm{d}v_\alpha)$ to $A^p_\alpha(B_n)$ for $p \geq 1$ and $\alpha > -1$. In [21] and [22], Zhao *et al* gave very beautiful results for the boundedness of $T_{\lambda,\tau,c}$ and $S_{\lambda,\tau,c}$ from $L^p(B_n, \mathrm{d}v_\alpha)$ to $L^q(B_n, \mathrm{d}v_\beta)$ for $1 \leq p, q \leq \infty$ and $\alpha, \beta > -1$. The general Hardy space $H^{p,q,s}(B_n)$ is a generalization of the Hardy space $H^p(B_n)$. Recently, we discussed the boundedness of $T_{\lambda,\tau}$ and $S_{\lambda,\tau}$ on its extension space $L^{p,q,s}(B_n)$ (see [27]). We know that $T_{\lambda,\tau,c}$ and $S_{\lambda,\tau,c}$ is the generalizations of $T_{\lambda,\tau}$ and $S_{\lambda,\tau}$. This mainly extends $n+1+\lambda+\tau$ to c, independently of λ and τ. Can $n+1+\lambda+\tau$ be generalized to another form? Or can the measure $(1-|w|^2)^\tau \mathrm{d}v(w)$ be generalized to another form? In this paper, we generalize the Forelli-Rudin type operators as follows:

$$T_{\lambda,\tau,k}f(z) = (1-|z|^2)^\lambda \int_{B_n} \frac{(1-|w|^2)^\tau f(w)}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{1-\langle z,w\rangle} \mathrm{d}v(w),$$
$$S_{\lambda,\tau,k}f(z) = (1-|z|^2)^\lambda \int_{B_n} \frac{(1-|w|^2)^\tau f(w)}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{|1-\langle z,w\rangle|} \mathrm{d}v(w),$$
$$Q_{\lambda,\tau,k}f(z) = (1-|z|^2)^\lambda \int_{B_n} \frac{(1-|w|^2)^\tau f(w)}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{1-|w|^2} \mathrm{d}v(w),$$
$$R_{\lambda,\tau,k}f(z) = (1-|z|^2)^\lambda \int_{B_n} \frac{(1-|w|^2)^\tau f(w)}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{1-|w|^2} \mathrm{d}v(w) \quad (z \in B_n),$$

there λ, τ and k are three real numbers. These generalized operators are often encountered in practical applications. In this paper, we first discuss the boundedness of $T_{\lambda,\tau,k}$, $S_{\lambda,\tau,k}$, $Q_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ on $L^1(B_n, \mathrm{d}v_t)$ or from $L^p_{\log,k}(B_n, \mathrm{d}v_t)$ to $L^p(B_n, \mathrm{d}v_t)$. Furthermore, we investigate these conditions such that $T_{\lambda,\tau,k}$, $S_{\lambda,\tau,k}$, $Q_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$ or from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$ in some cases. Our main results are the following:

Theorem 1.1 For $p \geq 1$, the following conditions are equivalent:

(1) $S_{\lambda,\tau,k}$ is bounded from $L^p_{\log,k}(B_n, \mathrm{d}v_t)$ to $L^p(B_n, \mathrm{d}v_t)$;

(2) $Q_{\lambda,\tau,k}$ is bounded from $L^p_{\log,k}(B_n, \mathrm{d}v_t)$ to $L^p(B_n, \mathrm{d}v_t)$;

(3) $R_{\lambda,\tau,k}$ is bounded from $L^p_{\log,k}(B_n, \mathrm{d}v_t)$ to $L^p(B_n, \mathrm{d}v_t)$;

(4) $-p\lambda < t+1 < p(\tau+1)$ ($t > -1$ when $p = 1$).

Theorem 1.2 For $t > -1$, the following conditions are equivalent:

(1) $T_{\lambda,\tau,k}$ is bounded on $L^1(B_n, \mathrm{d}v_t)$;

(2) $S_{\lambda,\tau,k}$ is bounded on $L^1(B_n, \mathrm{d}v_t)$;

(3) we have that either (i) $-\lambda < t+1 < \tau+1$ and $k \leq 0$, or (ii) $-\lambda < t+1 = \tau+1$ and $k < -1$.

Theorem 1.3 For $t > -1$, the following conditions are equivalent:

(1) $Q_{\lambda,\tau,k}$ is bounded on $L^1(B_n, \mathrm{d}v_t)$;

(2) $R_{\lambda,\tau,k}$ is bounded on $L^1(B_n, \mathrm{d}v_t)$;

(3) we have that either (i) $-\lambda < t+1 < \tau+1$ and $k \leq 0$, or (ii) $-\lambda < t+1 = \tau+1$ and $k \leq -1$.

Theorem 1.4 (1) If $p \geq 1$ and $0 \leq 2s < n$, then $S_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$ if and only if $-p\lambda < q+s < p(\tau+1)$.

(2) If $p \geq 1$ and $s \geq n$, then $S_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are bounded from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$ if and only if $-p\lambda < q+n < p(\tau+1)$.

(3) If $0 < p < 1$ and $s \geq n$, then $S_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are bounded from $\mathcal{H}^{p,q,s,k}(B_n)$ to $\mathcal{G}_{\frac{q+n}{p}}(B_n)$ if and only if $-p\lambda < q+n < p(\tau+1)$.

(4) If $p > 0$ and $s \geq n$, then $Q_{\lambda,\tau,k}$ is a bounded operator from $\mathcal{H}^{p,q,s,k}(B_n)$ to $\mathcal{H}^{p,q,s}(B_n)$ if and only if $-p\lambda < q+n < p(\tau+1)$.

In order to prove the above results, we need some key integral estimates. For a point in B_n, W. Rudin gave the following proposition in [19]:

Proposition A Let $t > -1$ and c be real. Then the integrals

$$I(z) = \int_{S_n} \frac{\mathrm{d}\sigma(\xi)}{|1-\langle \xi, z\rangle|^{n+c}}, \quad J(z) = \int_{B_n} \frac{(1-|w|^2)^t \mathrm{d}v(w)}{|1-\langle z, w\rangle|^{n+1+t+c}}$$

have the following asymptotic properties:

(1) $I(z) \asymp J(z) \asymp 1$ when $c < 0$;

(2) $I(z) \asymp J(z) \asymp \log \frac{\mathrm{e}}{1-|z|^2}$ when $c = 0$;

(3) $I(z) \asymp J(z) \asymp \frac{1}{(1-|z|^2)^c}$ when $c > 0$.

In terms of practical applications, these integrals are often encountered (for example, Zhou and Chen needed the case $k = 2$ in [28]). We also need some bidirectional estimates of these integrals in this paper:

$$G(w) = \int_{S_n} \frac{1}{|1-\langle \xi, w\rangle|^{n+c}} \left|\log \frac{\mathrm{e}}{1-\langle \xi, w\rangle}\right|^k \mathrm{d}\sigma(\xi),$$
$$H(w) = \int_{B_n} \frac{(1-|z|^2)^\delta}{|1-\langle z, w\rangle|^{n+1+\delta+c}} \log^k \frac{\mathrm{e}}{1-|z|^2} \mathrm{d}v(z)$$

and

$$F(w) = \int_{B_n} \frac{(1-|z|^2)^\delta}{|1-\langle z, w\rangle|^{n+1+\delta+c}} \left|\log \frac{\mathrm{e}}{1-\langle z, w\rangle}\right|^k \mathrm{d}v(z) \quad (w \in B_n).$$

Here $\delta > -1$, and c and k are real numbers.

There is here a natural problem. Do $G(w)$, $H(w)$ and $F(w)$ have bidirectional estimates? In this paper, we first discuss this problem, and give these bidirectional estimates for all of the cases in Proposition 3.1. Since k is an abstract real number, the original method of proof used method in Proposition A makes very difficult to estimate $F(w)$, $H(w)$ and $G(w)$. Therefore, we need to deal with the three integrals in a completely different way. For two points in B_n, we also need to estimate the integral

$$L_{w,\eta} = \int_{S_n} \frac{1}{|1-\langle \xi, w\rangle|^t \, |1-\langle \xi, \eta\rangle|^r} \left|\log \frac{\mathrm{e}}{1-\langle \xi, \eta\rangle}\right|^k \mathrm{d}\sigma(\xi) \quad (w, \eta \in B_n).$$

We give some bidirectional estimates in Proposition 3.2.

2 Some Lemmas

In order to prove our main results, we first give several lemmas.

Lemma 2.1 Let δ, c, k and k' be real numbers. Then integrals

$$I_1(\rho)=\int_0^1 \frac{(1-r)^{\delta}}{(1-r\rho)^{\delta+1+c}}\log^k \frac{\mathrm{e}(1-\rho r)}{1-\rho}\mathrm{d}r$$

and

$$I_2(\rho)=\int_0^1 \frac{(1-r)^{\delta}}{(1-\rho r)^{1+\delta+c}}\log^k \frac{\mathrm{e}}{1-\rho r}\log^{k'}\frac{\mathrm{e}}{1-r}\mathrm{d}r \quad (0\le \rho<1)$$

have the following bidirectional estimates:

(1)

$$I_1(\rho)\asymp\begin{cases}\log^k \dfrac{\mathrm{e}}{1-\rho}, & \delta>-1,\ c<0,\\ \dfrac{1}{(1-\rho)^c}, & \delta>-1,\ c>0,\\ \log^{k+1}\dfrac{\mathrm{e}}{1-\rho}, & \delta>-1,\ c=0,\ k>-1,\\ \log\log\dfrac{\mathrm{e}^2}{1-\rho}, & \delta>-1,\ c=0,\ k=-1,\\ 1, & \delta>-1,\ c=0,\ k<-1.\end{cases}$$

(2) $I_2(\rho)\asymp 1$ if one of the following conditions is satisfied: (i) $\delta>-1$, $c<0$; (ii) $\delta>-1$, $c=0$, $k+k'<-1$; (iii) $\delta=-1$, $c<0$, $k'<-1$; (iv) $\delta=-1$, $c=0$, $k+k'<-1$, $k'<-1$.

(3) $I_2(\rho)\asymp\frac{1}{(1-\rho)^c}\log^{k+k'}\frac{\mathrm{e}}{1-\rho}$ when $c>0$ and $\delta>-1$.

(4) $I_2(\rho)\asymp\frac{1}{(1-\rho)^c}\log^{k+k'+1}\frac{\mathrm{e}}{1-\rho}$ when $c>0$, $\delta=-1$ and $k'<-1$.

(5) $I_2(\rho)\asymp\log^{k+k'+1}\frac{\mathrm{e}}{1-\rho}$ if one of the following conditions is satisfied: (i) $\delta>-1$, $c=0$, $k+k'>-1$; (ii) $\delta=-1$, $c=0$, $k+k'>-1$, $k'<-1$.

(6) $I_2(\rho)\asymp\log\log\frac{e^2}{1-\rho}$ if one of the following conditions is satisfied: (i) $\delta>-1$, $c=0$, $k+k'=-1$; (ii) $\delta=-1$, $c=0$, $k+k'=-1$, $k'<-1$.

Proof If there exists a constant $0<\rho_0<1$ such that $0\le\rho\le\rho_0$, then these equivalents are obvious. Therefore, we may let ρ be sufficiently close to 1.

By changes of variables $x=(1-r)\rho/(1-\rho)$ and $y=1+x$, we have that

$$\begin{aligned}I_1(\rho)&=\frac{1}{(1-\rho)^c\rho^{\delta+1}}\int_0^{\frac{\rho}{1-\rho}}\frac{x^{\delta}}{(1+x)^{\delta+1+c}}\log^k \mathrm{e}(1+x)\mathrm{d}x\\ &\asymp\frac{1}{(1-\rho)^c}\left\{\int_0^1 x^{\delta}\mathrm{d}x+\int_2^{\frac{1}{1-\rho}}\frac{1}{y^{1+c}}\log^k \mathrm{e}y\mathrm{d}y\right\}.\end{aligned}$$

By a change of variables $x=(1-r)\rho/(1-r\rho)$, we have that

$$\begin{aligned}I_2(\rho)&=\frac{1}{(1-\rho)^c\rho^{\delta+1}}\int_0^{\rho}\frac{x^{\delta}(1-x)^{c-1}\log^k\frac{\mathrm{e}(1-x)}{1-\rho}}{\log^{-k'}\frac{\mathrm{e}\rho(1-x)}{x(1-\rho)}}\mathrm{d}x\\ &\asymp\frac{1}{(1-\rho)^c}\int_0^{\frac{1}{2}}\frac{x^{\delta}\log^k\frac{\mathrm{e}}{1-\rho}}{\log^{-k'}\frac{\mathrm{e}}{x(1-\rho)}}\mathrm{d}x+\int_{\frac{1}{2}}^{\rho}\frac{(1-x)^{c-1}}{(1-\rho)^c}\log^{k+k'}\frac{\mathrm{e}(1-x)}{1-\rho}\mathrm{d}x\\ &=\frac{\log^k\frac{\mathrm{e}}{1-\rho}}{(1-\rho)^{\delta+1+c}}\int_0^{\frac{1-\rho}{2}}y^{\delta}\log^{k'}\frac{\mathrm{e}}{y}\mathrm{d}y+\int_1^{\frac{1}{2(1-\rho)}}y^{c-1}\log^{k+k'}\mathrm{e}y\mathrm{d}y.\end{aligned}$$

If $\rho \to 1^-$, then we have the following results:

$$\int_1^{\frac{1}{2(1-\rho)}} y^{c-1} \log^{k+k'} \mathrm{e}y \mathrm{d}y \asymp 1 \quad \text{when} \quad c < 0;$$

$$\int_1^{\frac{1}{2(1-\rho)}} y^{c-1} \log^{k+k'} \mathrm{e}y \mathrm{d}y \asymp \frac{1}{(1-\rho)^c} \log^{k+k'} \frac{\mathrm{e}}{1-\rho} \quad \text{when} \quad c > 0;$$

$$\int_1^{\frac{1}{2(1-\rho)}} y^{-1} \log^{k+k'} \mathrm{e}y \mathrm{d}y \asymp \log^{k+k'+1} \frac{\mathrm{e}}{1-\rho} \quad \text{when} \quad k+k' > -1;$$

$$\int_1^{\frac{1}{2(1-\rho)}} y^{-1} \log^{-1} \mathrm{e}y \mathrm{d}y \asymp \log\log \frac{\mathrm{e}^2}{1-\rho};$$

$$\int_1^{\frac{1}{2(1-\rho)}} y^{-1} \log^{k+k'} \mathrm{e}y \mathrm{d}y \asymp 1 \quad \text{when} \quad k+k' < -1;$$

$$\int_2^{\frac{1}{1-\rho}} \frac{1}{y^{1+c}} \log^{k} \mathrm{e}y \mathrm{d}y \asymp (1-\rho)^c \log^k \frac{\mathrm{e}}{1-\rho} \quad \text{when} \quad c < 0;$$

$$\int_2^{\frac{1}{1-\rho}} \frac{1}{y^{1+c}} \log^{k} \mathrm{e}y \mathrm{d}y \asymp 1 \quad \text{when} \quad c > 0;$$

$$\int_0^{\frac{1-\rho}{2}} y^{\delta} \log^{k'} \frac{\mathrm{e}}{y} \mathrm{d}y \asymp (1-\rho)^{\delta+1} \log^{k'} \frac{\mathrm{e}}{1-\rho} \quad \text{when} \quad \delta > -1;$$

$$\int_0^{\frac{1-\rho}{2}} y^{-1} \log^{k'} \frac{\mathrm{e}}{y} \mathrm{d}y \asymp \log^{1+k'} \frac{\mathrm{e}}{1-\rho} \quad \text{when} \quad k' < -1.$$

Other cases are implied in the previous results. According to the different cases of δ, k and k', we can get these corresponding results. This proof is complete. □

Lemma 2.2 ([7]) For $r > 0$ and $t > 0$, let

$$I_{w,a} = \int_{S_n} \frac{\mathrm{d}\sigma(\xi)}{|1-\langle \xi, w\rangle|^t \, |1-\langle \xi, a\rangle|^r} \quad (w, a \in B_n).$$

Then

(1) $I_{w,a} \asymp \log \frac{\mathrm{e}}{|1-\langle w,a\rangle|}$ when $t + r = n$;

(2) $I_{w,a} \asymp \frac{1}{|1-\langle w,a\rangle|^{t+r-n}}$ when $t + r > n > \max\{r, n\}$.

These results come from Proposition 3.1 in [7].

Lemma 2.3 ([1]) The measures v and σ are related by

$$\int_{B_n} f(z)\mathrm{d}v(z) = 2n \int_0^1 r^{2n-1}\mathrm{d}r \int_{S_n} f(r\xi)\mathrm{d}\sigma(\xi).$$

This result comes from [1, Lemma 1.8].

Lemma 2.4 If $f \in \mathcal{H}^{p,q,s,k}(B_n)$, then

$$|f(z)| \lesssim \frac{||f||_{p,q,s,k} \log^{-k} \frac{\mathrm{e}}{1-|z|^2}}{(1-|z|^2)^{\frac{q+n}{p}}} \quad \text{for all } z \in B_n.$$

In particular, $\mathcal{H}^{p,q,s,k}(B_n) = \mathcal{H}^{\infty}_{\frac{q+n}{p},k}(B_n)$ and $||f||_{p,q,s,k} \asymp ||f||_{\frac{q+n}{p},k}$ when $s \geq n$.

Proof For any $f \in \mathcal{H}^{p,q,s,k}(B_n)$ and $z \in B_n$, it follows from the proof of Lemma 2.1 in [29] that $D(z, \log\sqrt{2}) \subset \frac{1+|z|}{2}B_n$. By (1.3), Lemma 2.20 in [1], Lemma 2.3, we have that

$$\left| f(z) \log^k \frac{\mathrm{e}}{1-|z|^2} \right|^p \lesssim \frac{1}{(1-|z|^2)^{n+1}} \int_{D(z,\log\sqrt{2})} \left| f(w) \log^k \frac{\mathrm{e}}{1-|w|^2} \right|^p \mathrm{d}v(w)$$

$$
\begin{aligned}
&\asymp \frac{1}{(1-|z|^2)^{q+n-1}} \int_{D(z,\log\sqrt{2})} \frac{\left|f(w)\log^k \frac{\mathrm{e}}{1-|w|^2}\right|^p (1-|\varphi_z(w)|^2)^s}{(1-|w|^2)^{-q}(1-|w|^2)^2} \mathrm{d}v(w) \\
&\lesssim \int_0^{\frac{1+|z|}{2}} \frac{(1-r^2)^{-2}\log^{pk}\frac{\mathrm{e}}{1-r^2}}{(1-|z|^2)^{q+n-1}} \left\{\int_{S_n} (1-r^2)^q |f(r\xi)|^p (1-|\varphi_z(r\xi)|^2)^s \mathrm{d}\sigma(\xi)\right\} \mathrm{d}r \\
&\le \frac{||f||^p_{p,q,s,k}}{(1-|z|^2)^{q+n-1}} \int_0^{\frac{1+|z|}{2}} \frac{\mathrm{d}r}{(1-r^2)^2} \asymp \frac{||f||^p_{p,q,s,k}}{(1-|z|^2)^{q+n}}.
\end{aligned}
$$

This means that $\mathcal{H}^{p,q,s,k}(B_n) \subseteq \mathcal{H}^\infty_{\frac{q+n}{p},k}(B_n)$ and $||f||_{\frac{q+n}{p},k} \lesssim ||f||_{p,q,s,k}$.

Moreover, if $s \ge n$ and $f \in \mathcal{H}^\infty_{\frac{q+n}{p},k}(B_n)$, it follows from Proposition A that

$$
\begin{aligned}
&\sup_{0\le r<1}\sup_{a\in B_n} (1-r^2)^q \int_{S_n} \left|f(r\xi)\log^k \frac{\mathrm{e}}{1-|r\xi|^2}\right|^p (1-|\varphi_a(r\xi)|^2)^s \mathrm{d}\sigma(\xi) \\
&\le ||f||^p_{\frac{q+n}{p},k} \sup_{0\le r<1}\sup_{a\in B_n} (1-r^2)^{s-n} \int_{S_n} \frac{(1-|a|^2)^s \mathrm{d}\sigma(\xi)}{|1-\langle a, r\xi\rangle|^{2s}} \lesssim ||f||^p_{\frac{q+n}{p},k}.
\end{aligned}
$$

This shows that $\mathcal{H}^\infty_{\frac{q+n}{p},k}(B_n) \subseteq \mathcal{H}^{p,q,s,k}(B_n)$ and $||f||_{p,q,s,k} \lesssim ||f||_{\frac{q+n}{p},k}$.

This proof is complete. □

Lemma 2.5 ([1]) There is a positive integer N such that, for any $0 < r \le 1$, one can find a sequence $\{a^k\} \subset B_n$ with $B_n = \bigcup\limits_{k=1}^{\infty} D(a^k, r)$, and for each point, $z \in B_n$ belongs to at most N of the sets $D(a^k, 4r)$.

This result comes from Theorem 2.23 in [1].

3 Main Results

We first prove two Propositions.

Proposition 3.1 Let c and k be real numbers, $\delta > -1$. Then the integrals

$$
G(w) = \int_{S_n} \frac{1}{|1-\langle \xi, w\rangle|^{n+c}} \left|\log \frac{\mathrm{e}}{1-\langle \xi, w\rangle}\right|^k \mathrm{d}\sigma(\xi),
$$

$$
H(w) = \int_{B_n} \frac{(1-|z|^2)^\delta}{|1-\langle z, w\rangle|^{n+1+\delta+c}} \log^k \frac{\mathrm{e}}{1-|z|^2} \mathrm{d}v(z)
$$

and

$$
F(w) = \int_{B_n} \frac{(1-|z|^2)^\delta}{|1-\langle z, w\rangle|^{n+1+\delta+c}} \left|\log \frac{\mathrm{e}}{1-\langle z, w\rangle}\right|^k \mathrm{d}v(z) \quad (w \in B_n)
$$

have the following bidirectional estimates:

(1) $G(w) \asymp H(w) \asymp F(w) \asymp 1$ when $c < 0$, or $c = 0$ and $k < -1$;

(2) $G(w) \asymp H(w) \asymp F(w) \asymp \frac{1}{(1-|w|^2)^c} \log^k \frac{\mathrm{e}}{1-|w|^2}$ when $c > 0$;

(3) $G(w) \asymp H(w) \asymp F(w) \asymp \log^{k+1} \frac{\mathrm{e}}{1-|w|^2}$ when $c = 0$ and $k > -1$;

(4) $G(w) \asymp H(w) \asymp F(w) \asymp \log\log \frac{\mathrm{e}^2}{1-|w|^2}$ when $c = 0$ and $k = -1$.

Proof If there exists a constant $0 < \rho_0 < 1$ such that $1 - |w|^2 \ge \rho_0$, then these bidirectional estimates are obvious. Therefore, we let $1 - |w|^2$ be sufficiently close to 0. It follows from (3.1) in [30] that we may get that

$$
G(w) \asymp \int_{S_n} \frac{1}{|1-\langle \xi, w\rangle|^{n+c}} \log^k \frac{\mathrm{e}}{|1-\langle \xi, w\rangle|} \mathrm{d}\sigma(\xi).
$$

When $c<0$, we may take $c<c'<0$ such that $\frac{1}{|1-\langle\xi,w\rangle|^{n+c}}\log^k\frac{\mathrm{e}}{|1-\langle\xi,w\rangle|}\lesssim\frac{1}{|1-\langle\xi,w\rangle|^{n+c'}}$ for all $\xi\in S_n$ and $w\in B_n$. It follows from the increasing property of the integral mean of the holomorphic function and Proposition A that

$$1\le G(w)\lesssim\int_{S_n}\frac{\mathrm{d}\sigma(\xi)}{|1-\langle\xi,w\rangle|^{n+c'}}\asymp 1.$$

By a change of variables $\xi=\varphi_w(\eta)$, (4.7) in [1], and (1.1)–(1.2), we have that

$$G(w)\asymp\frac{1}{(1-|w|^2)^c}\int_{S_n}\frac{1}{|1-\langle\eta,w\rangle|^{n-c}}\log^k\frac{\mathrm{e}|1-\langle\eta,w\rangle|}{1-|w|^2}\mathrm{d}\sigma(\eta)=\frac{J(w)}{(1-|w|^2)^c}.$$

Next, we consider $J(w)$ for $c\ge 0$.

When $n=1$, it follows from the rotation invariance of the integral that

$$\begin{aligned}J(w)&=\int_{-\pi}^{\pi}\frac{1}{|1-|w|\mathrm{e}^{\mathrm{i}\theta}|^{1-c}}\log^k\frac{\mathrm{e}|1-|w|\mathrm{e}^{\mathrm{i}\theta}|}{1-|w|^2}\ \frac{\mathrm{d}\theta}{2\pi}\\
&=\int_0^{\pi}\frac{1}{(1+|w|^2-2|w|\cos\theta)^{\frac{1-c}{2}}}\log^k\frac{\mathrm{e}^2(1+|w|^2-2|w|\cos\theta)}{(1-|w|^2)^2}\ \frac{\mathrm{d}\theta}{2^k\pi}\\
&=\frac{1}{2^k\pi}\int_{-1}^{1}\frac{(1-x^2)^{-\frac12}}{(1+|w|^2-2|w|x)^{\frac{1-c}{2}}}\log^k\frac{\mathrm{e}^2(1+|w|^2-2|w|x)}{(1-|w|^2)^2}\mathrm{d}x\\
&\asymp\log^k\frac{\mathrm{e}}{1-|w|^2}+\int_0^1\frac{(1-x)^{-\frac12}}{(1+|w|^2-2|w|x)^{\frac{1-c}{2}}}\log^k\frac{\mathrm{e}^2(1+|w|^2-2|w|x)}{(1-|w|^2)^2}\mathrm{d}x.\end{aligned}$$

Without losing generality, we let $|w|>1/2$. By a change of variables, $\rho=\frac{(1+|w|^2)(1-x)}{1+|w|^2-2|w|x}$, and we have that

$$\begin{aligned}J(w)&\asymp\log^k\frac{\mathrm{e}}{1-|w|^2}+(1-|w|)^c\int_0^1\frac{\log^k\frac{\mathrm{e}}{1-\frac{2|w|\rho}{1+|w|^2}}}{\rho^{\frac12}\big(1-\frac{2|w|\rho}{1+|w|^2}\big)^{1+\frac{c}{2}}}\mathrm{d}\rho\\
&\asymp\log^k\frac{\mathrm{e}}{1-|w|^2}+(1-|w|)^c\left\{1+\int_{\frac12}^1\frac{\log^k\frac{\mathrm{e}}{1-\frac{2|w|\rho}{1+|w|^2}}}{\big(1-\frac{2|w|\rho}{1+|w|^2}\big)^{1+\frac{c}{2}}}\mathrm{d}\rho\right\}\\
&\asymp\log^k\frac{\mathrm{e}}{1-|w|^2}+(1-|w|)^c\int_0^1\frac{\log^k\frac{\mathrm{e}}{1-\frac{2|w|\rho}{1+|w|^2}}}{\big(1-\frac{2|w|\rho}{1+|w|^2}\big)^{1+\frac{c}{2}}}\mathrm{d}\rho.\end{aligned}$$

It follows from Lemma 2.1 (case I_2 for $k'=0$) that we can get the estimates of $J(w)$ by different cases.

When $n>1$, it follows from (1.13) in [1] that

$$J(w)=(n-1)\int_D\frac{(1-|z|^2)^{n-2}}{|1-|w|z|^{n-c}}\log^k\frac{\mathrm{e}|1-|w|z|}{1-|w|^2}\mathrm{d}A(z).$$

If $c>0$ and $k\ge 0$, then it follows from Proposition A that

$$J(w)\lesssim\log^k\frac{\mathrm{e}}{1-|w|^2}\int_D\frac{(1-|z|^2)^{n-2}}{|1-|w|z|^{n-c}}\mathrm{d}A(z)\asymp\log^k\frac{\mathrm{e}}{1-|w|^2}.$$

If $c>0$ and $k<0$, then we take that $0<\varepsilon<c$. It is easy to obtain that

$$\sup_{1-|w|\le x\le 1+|w|}x^{\varepsilon}\log^k\frac{\mathrm{e}x}{1-|w|^2}\asymp\max\left\{(1-|w|^2)^{\varepsilon},\log^k\frac{\mathrm{e}}{1-|w|}\right\}\lesssim\log^k\frac{\mathrm{e}}{1-|w|^2}.$$

Therefore, it follows from Proposition A that

$$J(w) \lesssim \log^k \frac{\mathrm{e}}{1-|w|^2} \int_D \frac{(1-|z|^2)^{n-2}}{|1-|w|z|^{n-c+\varepsilon}} \mathrm{d}A(z) \asymp \log^k \frac{\mathrm{e}}{1-|w|^2}.$$

On the other hand, we have that

$$J(w) \gtrsim \int_{|z|\le \frac{1}{2}} \frac{(1-|z|^2)^{n-2}}{|1-|w|z|^{n-c}} \log^k \frac{\mathrm{e}|1-|w|z|}{1-|w|^2} \mathrm{d}A(z) \asymp \log^k \frac{\mathrm{e}}{1-|w|^2}.$$

This means that $J(w) \asymp \log^k \frac{\mathrm{e}}{1-|w|^2}$ when $c>0$. Therefore,

$$G(w) \asymp \frac{1}{(1-|w|^2)^c} \log^k \frac{\mathrm{e}}{1-|w|^2} \quad \text{when } c>0.$$

For any $1/2<\rho<1$ and any real number k, let $x=\frac{2\rho|w|(1-r)}{(1-\rho|w|)^2}$. Similar to the previous calculation, we can obtain that

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{1}{|1-\rho|w|\mathrm{e}^{\mathrm{i}\theta}|^n} \log^k \frac{\mathrm{e}|1-\rho|w|\mathrm{e}^{\mathrm{i}\theta}|}{1-|w|^2} \frac{\mathrm{d}\theta}{2\pi} \\
&\asymp \log^k \frac{\mathrm{e}}{1-|w|^2} + \int_0^1 \frac{1}{(1+\rho^2|w|^2-2\rho|w|r)^{\frac{n}{2}}} \log^k \left[\frac{\mathrm{e}^2(1+\rho^2|w|^2-2\rho|w|r)}{(1-|w|^2)^2}\right] \frac{\mathrm{d}r}{\sqrt{1-r}} \\
&\asymp \log^k \frac{\mathrm{e}}{1-|w|^2} + \frac{1}{(1-\rho|w|)^{n-1}} \int_0^{\frac{2\rho|w|}{(1-\rho|w|)^2}} \frac{\log^k \left[\frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2}(1+x)\right]}{x^{\frac{1}{2}}(1+x)^{\frac{n}{2}}} \mathrm{d}x.
\end{aligned}$$

It is clear that

$$\int_0^{\frac{8}{9}} \frac{1}{x^{\frac{1}{2}}(1+x)^{\frac{n}{2}}} \log^k \left[\frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2}(1+x)\right] \mathrm{d}x \asymp \log^k \frac{\mathrm{e}(1-\rho|w|)}{1-|w|}.$$

When $k \ge 0$, we have that

$$\log^k \left[\frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2}(1+x)\right] \asymp \log^k \frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2} + \log^k(1+x).$$

Therefore,

$$\begin{aligned}
&\int_{\frac{8}{9}}^{\frac{2\rho|w|}{(1-\rho|w|)^2}} \frac{1}{x^{\frac{1}{2}}(1+x)^{\frac{n}{2}}} \log^k \left[\frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2}(1+x)\right] \mathrm{d}x \\
&\lesssim \int_{\frac{8}{9}}^{\infty} \frac{1}{x^{\frac{n+1}{2}}} \left\{\log^k \frac{\mathrm{e}(1-\rho|w|)}{1-|w|^2} + \log^k(x+1)\right\} \mathrm{d}x \\
&\asymp \log^k \frac{\mathrm{e}(1-\rho|w|)}{1-|w|} + 1 \asymp \log^k \frac{\mathrm{e}(1-\rho|w|)}{1-|w|}.
\end{aligned}$$

When $k<0$, we have that

$$\log^k \left[\frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2}(1+x)\right] \le \log^k \frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2}.$$

Therefore,

$$\begin{aligned}
&\int_{\frac{8}{9}}^{\frac{2\rho|w|}{(1-\rho|w|)^2}} \frac{1}{x^{\frac{1}{2}}(1+x)^{\frac{n}{2}}} \log^k \left[\frac{\mathrm{e}^2(1-\rho|w|)^2}{(1-|w|^2)^2}(1+x)\right] \mathrm{d}x \\
&\lesssim \log^k \frac{\mathrm{e}(1-\rho|w|)}{1-|w|^2} \int_{\frac{8}{9}}^{\infty} \frac{1}{x^{\frac{n+1}{2}}} \mathrm{d}x \asymp \log^k \frac{\mathrm{e}(1-\rho|w|)}{1-|w|}.
\end{aligned}$$

This means that

$$\int_{-\pi}^{\pi}\frac{1}{|1-\rho|w|e^{i\theta}|^{n}}\log^{k}\frac{e|1-\rho|w|e^{i\theta}|}{1-|w|}\frac{d\theta}{2\pi}\asymp\log^{k}\frac{e}{1-|w|^{2}}+\frac{1}{(1-\rho|w|)^{n-1}}\log^{k}\frac{e(1-\rho|w|)}{1-|w|}.$$

If $c=0$, then it follows from the polar coordinate and the above result that

$$J(w)\asymp\log^{k}\frac{e}{1-|w|^{2}}\int_{0}^{1}(1-\rho)^{n-2}d\rho+\int_{0}^{1}\frac{(1-\rho)^{n-2}}{(1-\rho|w|)^{n-1}}\log^{k}\frac{e(1-\rho|w|)}{1-|w|}d\rho.$$

By Lemma 2.1 (case I_1), we can get the estimates of $J(w)$ for all of the cases.

Finally, we consider $H(w)$ and $F(w)$.

First, according to the increasing property of the integral mean of the holomorphic function, it can be obtained that

$$H(w)\geq 2n\int_{0}^{1}r^{2n-1}(1-r^{2})^{\delta}\log^{k}\frac{e}{1-r^{2}}dr\asymp 1,$$

$$F(w)\geq 2n\int_{0}^{1}r^{2n-1}(1-r^{2})^{\delta}dr\asymp 1.$$

When $c<0$, and let $c<c'<0$ and $0<\varepsilon<\min\{-c,\ \delta+1\}$. By (3.1) in [30] and Proposition A, we have that

$$1\lesssim H(w)\lesssim\int_{B_n}\frac{(1-|z|^{2})^{\delta-\varepsilon}}{|1-\langle z,w\rangle|^{n+1+\delta+c}}dv(z)\asymp 1$$

and

$$\begin{aligned}1\lesssim F(w)&\asymp\int_{B_n}\frac{(1-|z|^{2})^{\delta}}{|1-\langle z,w\rangle|^{n+1+\delta+c}}\log^{k}\frac{e}{|1-\langle z,w\rangle|}dv(z)\\&\lesssim\int_{B_n}\frac{(1-|z|^{2})^{\delta}}{|1-\langle z,w\rangle|^{n+1+\delta+c'}}dv(z)\asymp 1.\end{aligned}$$

When $c\geq 0$, by (3.1) in [30], Lemma 2.3 and the estimate of $G(w)$, we have that

$$H(w)\asymp\int_{0}^{1}\frac{(1-\rho)^{\delta}}{(1-\rho|w|)^{\delta+1+c}}\log^{k}\frac{e}{1-\rho}d\rho$$

and

$$F(w)\asymp\int_{0}^{1}\frac{(1-\rho)^{\delta}}{(1-\rho|w|)^{\delta+1+c}}\log^{k}\frac{e}{1-\rho|w|}d\rho.$$

It follows from Lemma 2.1 (case I_2 for $k=0$ or $k'=0$) that we can get the estimates of $H(w)$ and $F(w)$ in different cases.

The proof is complete. □

Proposition 3.2 For real number r,t,k, let

$$L_{w,\eta}=\int_{S_n}\frac{1}{|1-\langle\xi,w\rangle|^{t}\,|1-\langle\xi,\eta\rangle|^{r}}\left|\log\frac{e}{1-\langle\xi,\eta\rangle}\right|^{k}d\sigma(\xi)\quad(w,\eta\in B_n).$$

Then we have the following estimates:

(1) $$L_{w,\eta}\asymp\frac{1}{(1-|\eta|^{2})^{r-n}|1-\langle w,\eta\rangle|^{t}}\log^{k}\frac{e}{1-|\eta|^{2}}\quad\text{when}\quad r>n>t\geq 0;$$

(2) $$\begin{aligned}L_{w,\eta}\asymp&\frac{1}{|1-\langle w,\eta\rangle|^{r}}\log^{k}\frac{e}{|1-\langle w,\eta\rangle|}\log\frac{e}{|1-\langle w,\varphi_w(\eta)\rangle|}\\&+\frac{1}{(1-|\eta|^{2})^{r-n}|1-\langle w,\eta\rangle|^{n}}\log^{k}\frac{e}{1-|\eta|^{2}}\quad\text{when}\quad r>n=t;\end{aligned}$$

$$(3)\quad L_{w,\eta} \asymp \frac{1}{(1-|w|^2)^{t-n}|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|} + \frac{1}{(1-|\eta|^2)^{r-n}|1-\langle w,\eta\rangle|^t}\log^k\frac{\mathrm{e}}{1-|\eta|^2} \quad \text{when } r>n \text{ and } t>n.$$

Proof Without losing generality, let $1-|w|^2$ and $|1-\langle w,\eta\rangle|$ be sufficiently close to 0 such that they meet the needs of all of the relevant proof processes.

It follows from (3.1) in [30] that

$$L_{w,\eta} \asymp \int_{S_n} \frac{1}{|1-\langle \xi,w\rangle|^t\,|1-\langle \xi,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle \xi,\eta\rangle|}\mathrm{d}\sigma(\xi).$$

If $t=0$, then it follows from Proposition 3.1 that the result of (1) is true.

In that follows, we let $t>0$ and let $d(z,u)=|\langle z-u,z\rangle|+|\langle u-z,u\rangle|$ $(z,u\in\overline{B_n})$. By [31], there exists a constant $c_d>0$ such that $d(z,u)\le c_d\{d(z,a)+d(a,u)\}$ $(z,u,a\in\overline{B_n})$.

For $w,\eta\in B_n$, we consider a partition of S_n and get that

$$\Omega_1=\left\{\xi\in S_n: d(\xi,w)\le\frac{d(w,\eta)}{2c_d}\right\};\quad \Omega_2=\left\{\xi\in S_n: d(\xi,\eta)\le\frac{d(w,\eta)}{2c_d}\right\};$$

$$\Omega_3=\left\{\xi\in S_n: \frac{d(w,\eta)}{2c_d}<d(\xi,w)\le d(\xi,\eta)\right\};$$

$$\Omega_4=\left\{\xi\in S_n: \frac{d(w,\eta)}{2c_d}<d(\xi,\eta)\le d(\xi,w)\right\}.$$

Then $S_n=\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4$, where Ω_j and Ω_k $(j\ne k)$ are mutually disjoint. By Lemma 3.3 in [31], we have that $|1-\langle\xi,\eta\rangle|\gtrsim|1-\langle w,\eta\rangle|$ when $\xi\in\Omega_1\cup\Omega_3$, and $|1-\langle\xi,w\rangle|\gtrsim|1-\langle w,\eta\rangle|$ when $\xi\in\Omega_2\cup\Omega_4$.

If $r>n$, then it follows from Proposition 3.1 that

$$\begin{aligned} L_1 &= \int_{\Omega_2\cup\Omega_4}\frac{1}{|1-\langle \xi,w\rangle|^t\,|1-\langle \xi,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle \xi,\eta\rangle|}\mathrm{d}\sigma(\xi)\\ &\lesssim \frac{1}{|1-\langle w,\eta\rangle|^t}\int_{S_n}\frac{1}{|1-\langle \xi,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle \xi,\eta\rangle|}\mathrm{d}\sigma(\xi)\\ &\asymp \frac{1}{(1-|\eta|^2)^{r-n}|1-\langle w,\eta\rangle|^t}\log^k\frac{\mathrm{e}}{1-|\eta|^2}. \end{aligned}\tag{3.1}$$

If $t>n$, then it follows from Proposition A that

$$\begin{aligned} L_2 &= \int_{\Omega_1\cup\Omega_3}\frac{1}{|1-\langle \xi,w\rangle|^t\,|1-\langle \xi,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle \xi,\eta\rangle|}\mathrm{d}\sigma(\xi)\\ &\lesssim \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_{S_n}\frac{1}{|1-\langle \xi,w\rangle|^t}\mathrm{d}\sigma(\xi)\\ &\asymp \frac{1}{(1-|w|^2)^{t-n}|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}. \end{aligned}\tag{3.2}$$

We take $r-n<\varepsilon<r$ such that $0<r-\varepsilon<n$. By a change of variables $\xi=\varphi_w(\zeta)$, (4.7) in [1], (1.1)–(1.2), Lemma 2.2, if $t=n$, then we have that

$$\begin{aligned} L_2 &\lesssim \frac{1}{|1-\langle w,\eta\rangle|^\varepsilon}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_{S_n}\frac{1}{|1-\langle \xi,w\rangle|^n|1-\langle \xi,\eta\rangle|^{r-\varepsilon}}\mathrm{d}\sigma(\xi)\\ &= \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_{S_n}\frac{\mathrm{d}\sigma(\zeta)}{|1-\langle \zeta,\varphi_w(\eta)\rangle|^{r-\varepsilon}|1-\langle \zeta,w\rangle|^{n-(r-\varepsilon)}}\\ &\asymp \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\log\frac{\mathrm{e}}{|1-\langle w,\varphi_w(\eta)\rangle|}. \end{aligned}\tag{3.3}$$

When $t<n<r$, we take that $r-n<\varepsilon<t+r-n$. It follows from Lemma 2.2 that

$$\begin{aligned} L_2 &\lesssim \frac{1}{|1-\langle w,\eta\rangle|^{\varepsilon}}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_{S_n}\frac{\mathrm{d}\sigma(\xi)}{|1-\langle \xi,w\rangle|^t|1-\langle\xi,\eta\rangle|^{r-\varepsilon}} \\ &\asymp \frac{1}{|1-\langle w,\eta\rangle|^{t+r-n}}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}. \end{aligned} \tag{3.4}$$

By (3.1)–(3.4), these " $\lesssim$ " parts are true. However, we need to notice that

$$\frac{1}{|1-\langle w,\eta\rangle|^{t+r-n}}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\lesssim\frac{1}{(1-|\eta|^2)^{r-n}|1-\langle w,\eta\rangle|^t}\log^k\frac{\mathrm{e}}{1-|\eta|^2}.$$

It follows form Lemma 2.2 in [30] that " $\gtrsim$ " parts of (1) and (3) are true. It remains to prove the " $\gtrsim$ " part of (2).

Let $|w|>1/2$ and $|\varphi_w(\eta)|>1/2$. By the unitary invariance of integral on S_n, we may let $\varphi_w(\eta)=(|\varphi_w(\eta)|,0,0,\cdots,0)$ and $w=(\lambda_1,\lambda_2,0,\cdots,0)$, where $\lambda_2\ge 0$ and $|\lambda_1|^2+\lambda_2^2=|w|^2$ (a similar treatment can be found in [32]). Let $\Omega=\{u\in\overline{B_n}:2|1-\langle|\varphi_w(\eta)|u,\lambda_1e_1\rangle|\ge |1-\langle|\varphi_w(\eta)|u,\varphi_w(\eta)\rangle|\}$.

When $u\in\Omega$ and $k\ge 0$, we have that

$$\begin{aligned} &\left(\frac{|1-\langle|\varphi_w(\eta)|u,\lambda_1e_1\rangle|}{|1-\langle|\varphi_w(\eta)|u,\varphi_w(\eta)\rangle|}\right)^{r-n}\log^k\frac{\mathrm{e}|1-\langle|\varphi_w(\eta)|u,\lambda_1e_1\rangle|}{|1-\langle w,\eta\rangle||1-\langle|\varphi_w(\eta)|u,\varphi_w(\eta)\rangle|} \\ \ge{}& \frac{1}{2^{r-n}}\log^k\frac{\mathrm{e}}{2|1-\langle w,\eta\rangle|}\asymp\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}. \end{aligned} \tag{3.5}$$

When $u\in\Omega$ and $k<0$, let $M=\sup\limits_{0<x\le 2}x^{\frac{r-n}{-k}}\log\frac{2}{x}$. We have that

$$\begin{aligned} &\left(\frac{|1-\langle|\varphi_w(\eta)|u,\lambda_1e_1\rangle|}{|1-\langle|\varphi_w(\eta)|u,\varphi_w(\eta)\rangle|}\right)^{r-n}\log^k\frac{\mathrm{e}|1-\langle|\varphi_w(\eta)|u,\lambda_1e_1\rangle|}{|1-\langle w,\eta\rangle||1-\langle|\varphi_w(\eta)|u,\varphi_w(\eta)\rangle|} \\ \ge{}& \left\{\left(\frac{|1-\langle|\varphi_w(\eta)|u,\varphi_w(\eta)\rangle|}{|1-\langle|\varphi_w(\eta)|u,\lambda_1e_1\rangle|}\right)^{\frac{r-n}{-k}}\log\frac{\mathrm{e}}{2|1-\langle w,\eta\rangle|}+M\right\}^k \\ \ge{}& \left\{2^{\frac{r-n}{-k}}\log\frac{\mathrm{e}}{2|1-\langle w,\eta\rangle|}+M\right\}^k\asymp\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}. \end{aligned} \tag{3.6}$$

For any $0\le\rho<1$, we consider the function

$$f(z)=\frac{\{1-(\overline{\lambda_1}\rho z+\lambda_2\sqrt{1-\rho^2}\mathrm{e}^{\mathrm{i}\theta})\}^{r-2}}{(1-|\varphi_w(\eta)|\rho z)^r}\log^k\frac{\mathrm{e}\{1-(\overline{\lambda_1}\rho z+\lambda_2\sqrt{1-\rho^2}\mathrm{e}^{\mathrm{i}\theta})\}}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|\rho z)}.$$

Then f is an analytical function on $\overline{D}$. It follows from the increasing of integral mean of analytic function that

$$\int_{-\pi}^{\pi}|f(\mathrm{e}^{\mathrm{i}\varphi})|\mathrm{d}\varphi\ge\int_{-\pi}^{\pi}|f(|\varphi_w(\eta)|\mathrm{e}^{\mathrm{i}\varphi})|\mathrm{d}\varphi.$$

By the polar coordinate formula, we may get that

$$\begin{aligned} &\int_D\left|\frac{\{1-(\overline{\lambda_1}\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}^{r-2}}{(1-|\varphi_w(\eta)|\zeta_1)^r}\log^k\frac{\mathrm{e}\{1-(\overline{\lambda_1}\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|\zeta_1)}\right|\mathrm{d}v(\zeta_1) \\ \ge{}&\int_D\left|\frac{\{1-(\overline{\lambda_1}|\varphi_w(\eta)|\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}^{r-2}}{(1-|\varphi_w(\eta)|^2\zeta_1)^r}\right. \\ &\left.\times\log^k\frac{\mathrm{e}\{1-(\overline{\lambda_1}|\varphi_w(\eta)|\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|^2\zeta_1)}\right|\mathrm{d}v(\zeta_1). \end{aligned} \tag{3.7}$$

(i) When $n=1$, let $w=\lambda_1=|w|\mathrm{e}^{\mathrm{i}\alpha}$ and $T=\{\theta\in[-\pi,\pi]:\mathrm{e}^{\mathrm{i}\theta}\in\Omega\}$. After calculation, we have that $\Omega=\{z\in\overline{D}:|z-z_0|\geq R\}$, where

$$z_0=\frac{4|w|\mathrm{e}^{\mathrm{i}\alpha}-|\varphi_w(\eta)|}{(4|w|^2-|\varphi_w(\eta)|^2)|\varphi_w(\eta)|},\quad R=\frac{2\sqrt{|w|^2-2|w||\varphi_w(\eta)|\cos\alpha+|\varphi_w(\eta)|^2}}{(4|w|^2-|\varphi_w(\eta)|^2)|\varphi_w(\eta)|}.$$

For any $0\leq x\leq 1$, we may obtain that

$$\begin{aligned}&4|1-|\varphi_w(\eta)||w|\mathrm{e}^{-\mathrm{i}\alpha}x|^2-(1-|\varphi_w(\eta)|^2x)^2\\ \geq\ &(3-2|w||\varphi_w(\eta)|x-|\varphi_w(\eta)|^2x)\{(|w|-|\varphi_w(\eta)|)^2x+1-|w|^2x\}>0.\end{aligned}$$

This means that the interval on the real axis is $[0,1]\subset\Omega$. Therefore, we have at least one of the sets $\{z:z\in\overline{D}\text{ and }0\leq\arg z\leq\pi/2\}$ or $\{z:z\in\overline{D}\text{ and }-\pi/2\leq\arg z\leq 0\}$, included in Ω. We may let $\{z:z\in\overline{D}\text{ and }0\leq\arg z\leq\pi/2\}\subset\Omega$. This shows that $[0,\frac{\pi}{2}]\subset T$. By a change of variables $\xi=\varphi_w(\zeta)$, (4.7) in [1], (1.1)–(1.2), increasing the integral mean of the analytic function, and (3.5)–(3.6), we have that

$$\begin{aligned}L_{w,\eta}&=\frac{1}{|1-\langle w,\eta\rangle|^r}\int_{S_1}\left|\frac{(1-\langle\zeta,w\rangle)^{r-1}\log^k\frac{\mathrm{e}(1-\langle\zeta,w\rangle)}{(1-\langle w,\eta\rangle)(1-\langle\zeta,\varphi_w(\eta)\rangle)}}{(1-\langle\zeta,\varphi_w(\eta)\rangle)^r}\right|\mathrm{d}\sigma(\zeta)\\
&\geq\frac{1}{|1-\langle w,\eta\rangle|^r}\int_{S_1}\left|\frac{(1-\langle|\varphi_w(\eta)|\zeta,w\rangle)^{r-1}\log^k\frac{\mathrm{e}(1-\langle|\varphi_w(\eta)|\zeta,w\rangle)}{(1-\langle w,\eta\rangle)(1-\langle|\varphi_w(\eta)|\zeta,\varphi_w(\eta)\rangle)}}{(1-\langle|\varphi_w(\eta)|\zeta,\varphi_w(\eta)\rangle)^r}\right|\mathrm{d}\sigma(\zeta)\\
&\gtrsim\frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_T\frac{\mathrm{d}\theta}{|1-|\varphi_w(\eta)|^2\mathrm{e}^{\mathrm{i}\theta}|}\\
&\geq\frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{|1-|\varphi_w(\eta)|^2\mathrm{e}^{\mathrm{i}\theta}|}\\
&\asymp\frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\log\frac{\mathrm{e}}{1-|\varphi_w(\eta)|^2}\\
&\gtrsim\frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\log\frac{\mathrm{e}}{|1-\langle w,\varphi_w(\eta)\rangle|}.\end{aligned}$$

(ii) When $n=2$, by increasing the integral mean of the analytic function, Lemma 1.10 in [1], (3.5)–(3.7), we have that

$$\begin{aligned}L_{w,\eta}&=\frac{1}{|1-\langle w,\eta\rangle|^r}\int_{S_2}\left|\frac{(1-\langle\zeta,w\rangle)^{r-2}\log^k\frac{\mathrm{e}(1-\langle\zeta,w\rangle)}{(1-\langle w,\eta\rangle)(1-\langle\zeta,\varphi_w(\eta)\rangle)}}{(1-\langle\zeta,\varphi_w(\eta)\rangle)^r}\right|\mathrm{d}\sigma(\zeta)\\
&=\frac{1}{|1-\langle w,\eta\rangle|^r}\int_D\frac{1}{2\pi}\int_{-\pi}^{\pi}|\frac{\{1-(\overline{\lambda_1}\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}^{r-2}}{(1-|\varphi_w(\eta)|\zeta_1)^r}\\
&\quad\times\log^k\frac{\mathrm{e}\{1-(\overline{\lambda_1}\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|\zeta_1)}|\mathrm{d}\theta\mathrm{d}v(\zeta_1)\\
&\geq\frac{1}{|1-\langle w,\eta\rangle|^r}\int_D\frac{1}{2\pi}\int_{-\pi}^{\pi}|\frac{\{1-(\overline{\lambda_1}|\varphi_w(\eta)|\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}^{r-2}}{(1-|\varphi_w(\eta)|^2\zeta_1)^r}\\
&\quad\times\log^k\frac{\mathrm{e}\{1-(\overline{\lambda_1}|\varphi_w(\eta)|\zeta_1+\lambda_2\sqrt{1-|\zeta_1|^2}\mathrm{e}^{\mathrm{i}\theta})\}}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|^2\zeta_1)}|\mathrm{d}\theta\mathrm{d}v(\zeta_1)\\
&\geq\int_D\frac{|1-|\varphi_w(\eta)|\overline{\lambda_1}\zeta_1|^{r-2}\left|\log^k\frac{\mathrm{e}(1-|\varphi_w(\eta)|\overline{\lambda_1}\zeta_1)}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|^2\zeta_1)}\right|}{|1-\langle w,\eta\rangle|^r|1-|\varphi_w(\eta)|^2\zeta_1|^r}\mathrm{d}v(\zeta_1)\end{aligned}$$

$$
\begin{aligned}
&\asymp \int_0^1 \int_{-\pi}^{\pi} \frac{|1-|\varphi_w(\eta)|\overline{\lambda_1}\rho \mathrm{e}^{\mathrm{i}\theta}|^{r-2}\log^k \frac{\mathrm{e}|1-|\varphi_w(\eta)|\overline{\lambda_1}\rho\mathrm{e}^{\mathrm{i}\theta}|}{|1-\langle w,\eta\rangle||1-|\varphi_w(\eta)|^2\rho\mathrm{e}^{\mathrm{i}\theta}|}}{|1-\langle w,\eta\rangle|^r|1-|\varphi_w(\eta)|^2\rho\mathrm{e}^{\mathrm{i}\theta}|^r}\mathrm{d}\theta\mathrm{d}\rho\\
&\gtrsim \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_0^1\int_T \frac{1}{|1-|\varphi_w(\eta)|^2\rho\mathrm{e}^{\mathrm{i}\theta}|^2}\mathrm{d}\theta\mathrm{d}\rho\\
&\gtrsim \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_0^1\frac{\mathrm{d}\rho}{1-|\varphi_w(\eta)|^2\rho}\\
&\asymp \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\log\frac{\mathrm{e}}{1-|\varphi_w(\eta)|^2}.
\end{aligned}
$$

(iii) When $n>2$, by Lemmas 1.8–1.9 in [1], increasing the integral mean of the analytic function, (3.5)–(3.6), Lemma 2.1 (case I_2 for $k=k'=0$), we have that

$$
\begin{aligned}
L_{w,\eta} &= \frac{1}{|1-\langle w,\eta\rangle|^r}\int_{S_n}\left|\frac{(1-\langle\zeta,w\rangle)^{r-n}\log^k\frac{\mathrm{e}(1-\langle\zeta,w\rangle)}{(1-\langle w,\eta\rangle)(1-\langle\zeta,\varphi_w(\eta)\rangle)}}{(1-\langle\zeta,\varphi_w(\eta)\rangle)^r}\right|\mathrm{d}\sigma(\zeta)\\
&= \frac{(n-1)(n-2)}{2}\int_{|u_1|^2+|u_2|^2<1}\frac{(1-|u_1|^2-|u_2|^2)^{n-3}}{|1-\langle w,\eta\rangle|^r}\\
&\quad\times\left|\frac{(1-\overline{\lambda_1}u_1-\lambda_2u_2)^{r-n}}{(1-|\varphi_w(\eta)|u_1)^r}\log^k\frac{\mathrm{e}(1-\overline{\lambda_1}u_1-\lambda_2u_2)}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|u_1)}\right|\mathrm{d}v(u_1,u_2)\\
&= \frac{(n-1)(n-2)}{2}\int_D\int_0^{\sqrt{1-|u_1|^2}}\frac{\rho(1-|u_1|^2-\rho^2)^{n-3}}{|1-\langle w,\eta\rangle|^r}\\
&\quad\times\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{(1-\overline{\lambda_1}u_1-\lambda_2\rho\mathrm{e}^{\mathrm{i}\theta})^{r-n}\log^k\frac{\mathrm{e}(1-\overline{\lambda_1}u_1-\lambda_2\rho\mathrm{e}^{\mathrm{i}\theta})}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|u_1)}}{(1-|\varphi_w(\eta)|u_1)^r}\right|\mathrm{d}\theta\right\}\mathrm{d}\rho\mathrm{d}v(u_1)\\
&\ge (n-1)(n-2)\int_D\int_0^{\sqrt{1-|u_1|^2}}\frac{\rho(1-|u_1|^2-\rho^2)^{n-3}}{|1-\langle w,\eta\rangle|^r}\\
&\quad\times\left|\frac{(1-\overline{\lambda_1}u_1)^{r-n}\log^k\frac{\mathrm{e}(1-\overline{\lambda_1}u_1)}{(1-\langle w,\eta\rangle)(1-|\varphi_w(\eta)|u_1)}}{(1-|\varphi_w(\eta)|u_1)^r}\right|\mathrm{d}\rho\mathrm{d}v(u_1)\\
&\asymp \int_D\frac{(1-|u_1|^2)^{n-2}|1-\overline{\lambda_1}u_1|^{r-n}}{|1-\langle w,\eta\rangle|^r|1-|\varphi_w(\eta)|u_1|^r}\log^k\frac{\mathrm{e}|1-\overline{\lambda_1}u_1|}{|1-\langle w,\eta\rangle||1-|\varphi_w(\eta)|u_1|}\mathrm{d}v(u_1)\\
&\gtrsim \int_0^1\frac{(1-\rho)^{n-2}}{|1-\langle w,\eta\rangle|^r}\int_{-\pi}^{\pi}\frac{|1-\overline{\lambda_1}\rho|\varphi_w(\eta)|\mathrm{e}^{\mathrm{i}\theta}|^{r-n}\log^k\frac{\mathrm{e}|1-\overline{\lambda_1}\rho|\varphi_w(\eta)|\mathrm{e}^{\mathrm{i}\theta}|}{|1-\langle w,\eta\rangle||1-|\varphi_w(\eta)|^2\rho\mathrm{e}^{\mathrm{i}\theta}|}}{|1-|\varphi_w(\eta)|^2\rho\mathrm{e}^{\mathrm{i}\theta}|^r}\mathrm{d}\theta\mathrm{d}\rho\\
&\gtrsim \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_0^1\int_T\frac{(1-\rho)^{n-2}}{|1-|\varphi_w(\eta)|^2\rho\mathrm{e}^{\mathrm{i}\theta}|^n}\mathrm{d}\theta\mathrm{d}\rho\\
&\asymp \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\int_0^1\frac{(1-\rho)^{n-2}}{(1-|\varphi_w(\eta)|^2\rho)^{n-1}}\mathrm{d}\rho\\
&\asymp \frac{1}{|1-\langle w,\eta\rangle|^r}\log^k\frac{\mathrm{e}}{|1-\langle w,\eta\rangle|}\log\frac{\mathrm{e}}{1-|\varphi_w(\eta)|^2}.
\end{aligned}
$$

Therefore, the " $\ge$ " part of (2) is true. The proof is complete. □

Next, we consider the boundedness of the generalized Forelli-Rudin type operators from $L^p_{\log,k}(B_n,\mathrm{d}v_t)$ to $L^p(B_n,\mathrm{d}v_t)$.

The proof of Theorem 1.1 (1) ⇒ (4)

We choose α such that $p\alpha+t>-1$, $\tau+\alpha>-1$ and $\alpha>\lambda$.

Take that $f(z)=(1-|z|^2)^{\alpha}\quad(z\in B_n)$. Then

$$\|f\|_{p,t,\log,k}^p=\int_{B_n}\left|f(z)\log^k\frac{\mathrm{e}}{1-|z|^2}\right|^p\mathrm{d}v_t(z)=c_t\int_{B_n}(1-|z|^2)^{p\alpha+t}\log^{pk}\frac{\mathrm{e}}{1-|z|^2}\mathrm{d}v(z)\asymp 1.$$

This means that $f\in L^p_{\log,k}(B_n,\mathrm{d}v_t)$. It follows from Proposition 3.1 that

$$\begin{aligned}S_{\lambda,\tau,k}f(z)&=(1-|z|^2)^{\lambda}\int_{B_n}\frac{(1-|w|^2)^{\tau+\alpha}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(w)\\&\asymp(1-|z|^2)^{\lambda}\quad(z\in B_n).\end{aligned}$$

The boundedness of $S_{\lambda,\tau,k}$ from $L^p_{\log,k}(B_n,\mathrm{d}v_t)$ to $L^p(B_n,\mathrm{d}v_t)$ means that the function $(1-|z|^2)^{\lambda}$ belongs to $L^p(B_n,\mathrm{d}v_t)$. Therefore, we get that $p\lambda+t>-1$.

Furthermore, it follows from $S_{\lambda,\tau,k}:L^p_{\log,k}(B_n,\mathrm{d}v_t)\to L^p(B_n,\mathrm{d}v_t)$ that

$$S^*_{\lambda,\tau,k}:(L^p(B_n,\mathrm{d}v_t))^*=L^{p'}(B_n,\mathrm{d}v_t)\to(L^p_{\log,k}(B_n,\mathrm{d}v_t))^*=L^{p'}_{\log,-k}(B_n,\mathrm{d}v_t),$$

where $1/p+1/p'=1$. By $\langle f,S_{\lambda,\tau,k}g\rangle=\langle S^*_{\lambda,\tau,k}f,g\rangle$ $(f\in L^{p'}(B_n,\mathrm{d}v_t),\ g\in L^p_{\log,k}(B_n,\mathrm{d}v_t))$, we may get the conjugate operator

$$S^*_{\lambda,\tau,k}f(w)=(1-|w|^2)^{\tau-t}\int_{B_n}\frac{(1-|z|^2)^{\lambda+t}f(z)}{|1-\langle w,z\rangle|^{n+1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{|1-\langle w,z\rangle|}\mathrm{d}v(z)\quad(w\in B_n).$$

When $p>1$, if we choose $\beta>\max\{-(1+t)/p',\ -1-\lambda-t,\ \tau-t\}$, then $g(z)=(1-|z|^2)^{\beta}\in L^{p'}(B_n,\mathrm{d}v_t)$. It follows from Proposition 3.1 that

$$\begin{aligned}S^*_{\lambda,\tau,k}g(z)&=(1-|z|^2)^{\tau-t}\int_{B_n}\frac{(1-|w|^2)^{\lambda+\beta+t}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(w)\\&\asymp(1-|z|^2)^{\tau-t}\quad(z\in B_n).\end{aligned}$$

The boundedness of $S^*_{\lambda,\tau,k}$ from $L^{p'}(B_n,\mathrm{d}v_t)$ to $L^{p'}_{\log,-k}(B_n,\mathrm{d}v_t)$ means that the function $(1-|z|^2)^{\tau-t}$ belongs to $L^{p'}_{\log,-k}(B_n,\mathrm{d}v_t)$. This implies that $t+1<p(\tau+1)$, or $t+1=p(\tau+1)$ and $k>1/p'$.

If $t+1=p(\tau+1)$ and $k>1/p'$, then we take that

$$h(z)=(1-|z|^2)^{-\frac{1+t}{p'}}\log^{-1-\frac{1}{p'}}\frac{\mathrm{e}}{1-|z|^2}\quad(z\in B_n).$$

Then $h\in L^{p'}(B_n,\mathrm{d}v_t)$. This means that $S^*_{\lambda,\tau,k}h\in L^{p'}_{\log,-k}(B_n,\mathrm{d}v_t)$. On the other hand, the conditions $-p\lambda<t+1=p(\tau+1)$ mean that $\lambda+\tau+1>0$ and $\lambda+t-(1+t)/p'>-1$. By Proposition 3.1, $k>1/p'$ and Lemma 2.1 (case I_2 for $k'=-1-1/p'$), we get that

$$\begin{aligned}S^*_{\lambda,\tau,k}h(z)&=(1-|z|^2)^{-\frac{1+t}{p'}}\int_{B_n}\frac{(1-|w|^2)^{\lambda+t-\frac{1+t}{p'}}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\frac{\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}}{\log^{1+\frac{1}{p'}}\frac{\mathrm{e}}{1-|w|^2}}\mathrm{d}vy(w)\\&\asymp(1-|z|^2)^{-\frac{1+t}{p'}}\int_0^1\frac{(1-\rho)^{\lambda+t-\frac{1+t}{p'}}}{(1-\rho|z|)^{1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{1-\rho|z|}\log^{-1-\frac{1}{p'}}\frac{\mathrm{e}}{1-\rho}\mathrm{d}\rho\\&\asymp(1-|z|^2)^{-\frac{1+t}{p'}}\log^{k-\frac{1}{p'}}\frac{\mathrm{e}}{1-|z|^2}.\end{aligned}$$

This shows that

$$\int_{B_n}\left|S^*_{\lambda,\tau,k}h(z)\log^{-k}\frac{\mathrm{e}}{1-|z|^2}\right|^{p'}\mathrm{d}v_t(z)\asymp\int_{B_n}\frac{1}{1-|z|^2}\log^{-1}\frac{\mathrm{e}}{1-|z|^2}\mathrm{d}v(z)=\infty.$$

This contradiction means that the cases $t+1=p(\tau+1)$ and $k>1/p'$ are impossible.

When $p=1$, it follows from Proposition 3.1 that

$$S^*_{\lambda,\tau,k}1=(1-|z|^2)^{\tau-t}\int_{B_n}\frac{(1-|w|^2)^{\lambda+t}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(w)$$
$$\asymp\begin{cases}\log^k\dfrac{\mathrm{e}}{1-|z|^2}, & \tau>t,\\ \log^{k+1}\dfrac{\mathrm{e}}{1-|z|^2}, & \tau=t \text{ and } k>-1,\\ \log\log\dfrac{\mathrm{e}^2}{1-|z|^2}, & \tau=t \text{ and } k=-1,\\ 1, & \tau=t \text{ and } k<-1,\\ (1-|z|^2)^{\tau-t}, & \tau<t.\end{cases}$$

It is clear that there must be $\tau>t$ when $S^*_{\lambda,\tau,k}1\in L^\infty_{\log,-k}(B_n)$.

Therefore, we obtain that $-p\lambda<t+1<p(\tau+1)$ for all $p\geq 1$.

(2) $\Rightarrow$ (4)

This proof is easier than the proof of (1) $\Rightarrow$ (4). Notice that

$$Q^*_{\lambda,\tau,k}f(w)=\frac{(1-|w|^2)^{\tau-t}}{\log^{-k}\frac{\mathrm{e}}{1-|w|^2}}\int_{B_n}\frac{(1-|z|^2)^{\lambda+t}f(z)}{(1-\langle w,z\rangle)^{n+1+\lambda+\tau}}\mathrm{d}v(z)\quad(w\in B_n).$$

We omit the proof process.

(4) $\Rightarrow$ (1)

When $p=1$, the conditions $-\lambda<t+1<\tau+1$ mean that $\lambda+t>-1$ and $\tau-t>0$. By Fubini's Theorem and Proposition 3.1, we have that

$$\begin{aligned}||S_{\lambda,\tau,k}f||_{1,t}&\leq\int_{B_n}S_{\lambda,\tau,k}|f|(z)\mathrm{d}v_t(z)\\ &=c_t\int_{B_n}|f(w)|(1-|w|^2)^\tau\left\{\int_{B_n}\frac{(1-|z|^2)^{\lambda+t}\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\mathrm{d}v(z)\right\}\mathrm{d}v(w)\\ &\asymp\int_{B_n}|f(w)|\log^k\frac{\mathrm{e}}{1-|w|^2}\mathrm{d}v_t(w)=||f||_{1,t,\log,k}.\end{aligned}$$

When $p>1$, let $1/p+1/p'=1$. If $-p\lambda<t+1<p(\tau+1)$, then we may choose $\lambda+\tau-p\lambda<\tau_1<p(\tau+1)-1$ such that

$$\left(\tau-\frac{\tau_1}{p}\right)p'>-1,\ (n+1+\lambda+\tau)-\left(\tau-\frac{\tau_1}{p}\right)p'-n-1>0.$$

For any $f\in L^p_{\log,k}(B_n,\mathrm{d}v_t)$, Hölder's inequality and Proposition A show that

$$\begin{aligned}\{S_{\lambda,\tau,k}|f|(z)\}^p&\leq(1-|z|^2)^{p\lambda}\left\{\int_{B_n}\frac{(1-|w|^2)^{(\tau-\frac{\tau_1}{p})p'}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\ \mathrm{d}v(w)\right\}^{\frac{p}{p'}}\\ &\quad\times\int_{B_n}\frac{|f(w)|^p(1-|w|^2)^{\tau_1}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\log^{pk}\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(w)\\ &\asymp(1-|z|^2)^{\lambda+\tau-\tau_1}\int_{B_n}\frac{|f(w)|^p(1-|w|^2)^{\tau_1}\log^{pk}\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}.\end{aligned}\tag{3.8}$$

If $-p\lambda < t+1 < p(\tau+1)$, then we may also choose $t < \tau_1 < t+\lambda+\tau+1$. By (3.8) and Proposition 3.1(2), we have that

$$\begin{aligned}||S_{\lambda,\tau,k}f||_{p,t}^p &\le \int_{B_n}\{S_{\lambda,\tau,k}|f|(z)\}^p\mathrm{d}v_t(z)\\ &\lesssim \int_{B_n}|f(w)|^p(1-|w|^2)^{\tau_1}\left\{\int_{B_n}\frac{(1-|z|^2)^{\lambda+\tau-\tau_1+t}\log^{pk}\frac{\mathrm{e}}{|1-\langle z,w\rangle|}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\mathrm{d}v(z)\right\}\mathrm{d}v(w)\\ &\asymp \int_{B_n}|f(w)|^p\log^{pk}\frac{\mathrm{e}}{1-|w|^2}\mathrm{d}v_t(w) = ||f||_{p,t,\log,k}^p.\end{aligned}$$

This shows that $S_{\lambda,\tau,k}$ is bounded from $L^p_{\log,k}(B_n,\mathrm{d}v_t)$ to $L^p(B_n,\mathrm{d}v_t)$ for all $p\ge 1$.

Similarly, we may prove that (4) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2)

Let $R_{\lambda,\tau,k}$ be a bounded operator from $L^p_{\log,k}(B_n,\mathrm{d}v_t)$ to $L^p(B_n,\mathrm{d}v_t)$. For any $f\in L^p_{\log,k}(B_n,\mathrm{d}v_t)$, we have that $||R_{\lambda,\tau,k}f||_{p,t}\le ||R_{\lambda,\tau,k}||.||f||_{p,t,\log,k}$. Therefore,

$$||Q_{\lambda,\tau,k}f||_{p,t}\le ||R_{\lambda,\tau,k}|f|||_{p,t}\le ||R_{\lambda,\tau,k}||.|||f|||_{p,t,\log,k} = ||R_{\lambda,\tau,k}||.||f||_{p,t,\log,k}.$$

This means that $Q_{\lambda,\tau,k}$ is a bounded operator from $L^p_{\log,k}(B_n,\mathrm{d}v_t)$ to $L^p(B_n,\mathrm{d}v_t)$.

This proof is complete. □

The proof of Theorem 1.2 (1) $\Rightarrow$ (3)

When $\alpha+t>-1$ and $\tau+\alpha>-1$, we have that $f(z)=(1-|z|^2)^\alpha\in L^1(B_n,\mathrm{d}v_t)$. The symmetry of B_n shows that

$$\begin{aligned}T_{\lambda,\tau,k}f(z) &= (1-|z|^2)^\lambda\int_{B_n}\frac{(1-|w|^2)^{\tau+\alpha}}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{1-\langle z,w\rangle}\mathrm{d}v(w)\\ &= \frac{(1-|z|^2)^\lambda}{c_{\tau+\alpha}}\quad (z\in B_n).\end{aligned}$$

The boundedness of $T_{\lambda,\tau,k}$ on $L^1(B_n,\mathrm{d}v_t)$ means that the function $(1-|z|^2)^\lambda$ belongs to $L^1(B_n,\mathrm{d}v_t)$. Thus, we get that $\lambda+t>-1$. It is easy to calculate that the conjugate operator of $T_{\lambda,\tau,k}$ on $L^1(B_n,\mathrm{d}v_t)$ is

$$T^*_{\lambda,\tau,k}f(z) = (1-|z|^2)^{\tau-t}\int_{B_n}\frac{(1-|w|^2)^{\lambda+t}f(w)}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{1-\langle z,w\rangle}\mathrm{d}v(w)\quad (z\in B_n).$$

It follows from $T^*_{\lambda,\tau,k}1=\frac{1}{c_{\lambda+t}}(1-|z|^2)^{\tau-t}\in L^\infty(B_n)$ that $\tau\ge t$.

For any $z\in B_n$, we take that

$$g_z(w) = \frac{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\log^{-k}\frac{\mathrm{e}}{1-\langle z,w\rangle}\quad (w\in B_n).$$

Then $g_z\in L^\infty(B_n)$ and $||g_z||_\infty\asymp 1$. This means that $||T^*_{\lambda,\tau,k}g_z||_\infty\lesssim ||T^*_{\lambda,\tau,k}||$. It follows from Proposition 3.1 that

$$\begin{aligned}||T^*_{\lambda,\tau,k}|| &\gtrsim \sup_{w\in B_n}|T^*_{\lambda,\tau,k}g_z(w)|\ge |T^*_{\lambda,\tau,k}g_z(z)|\\ &= \int_{B_n}\frac{(1-|z|^2)^{\tau-t}(1-|w|^2)^{\lambda+t}\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\mathrm{d}v(w)\end{aligned}$$

for all $z\in B_n$ if and only if $\tau>t$ and $k\le 0$, or $\tau=t$ and $k<-1$.

(3) $\Rightarrow$ (2)

When $-\lambda < t+1$, we have that $\lambda + t > -1$. Let $\tau > t$ and $k \le 0$, or $\tau = t$ and $k < -1$. For any $f \in L^1(B_n, \mathrm{d}v_t)$, it follows from Proposition 3.1 that

$$\begin{aligned}||S_{\lambda,\tau,k}f||_{1,t} &\lesssim \int_{B_n} S_{\lambda,\tau,k}|f|(z)(1-|z|^2)^t \mathrm{d}v(z)\\ &= \int_{B_n}(1-|w|^2)^\tau |f(w)| \left\{\int_{B_n} \frac{(1-|z|^2)^{\lambda+t}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(z)\right\}\mathrm{d}v(w)\\ &\lesssim \int_{B_n}|f(w)|\mathrm{d}v_t(w) = ||f||_{1,t}.\end{aligned}$$

(2) $\Rightarrow$ (1)

This proof is the same as that of Theorem 1.1. □

The proof of Theorem 1.3 (1) $\Rightarrow$ (3)

This proof of $\lambda + t > -1$ is the same as that of Theorem 1.2. It is easy to calculate that the conjugate operator of $Q_{\lambda,\tau,k}$ on $L^1(B_n, \mathrm{d}v_t)$ is

$$Q^*_{\lambda,\tau,k}f(z) = (1-|z|^2)^{\tau-t}\log^k \frac{\mathrm{e}}{1-|z|^2}\int_{B_n}\frac{(1-|w|^2)^{\lambda+t}f(w)}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}\mathrm{d}v(w) \quad (z\in B_n).$$

For any $z \in B_n$, we take that

$$g_z(w) = \frac{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}} \quad (w \in B_n).$$

Then $g_z \in L^\infty(B_n)$ and $||g_z||_\infty = 1$. It follows from Proposition A that

$$||Q^*_{\lambda,\tau,k}|| \gtrsim |Q^*_{\lambda,\tau,k}g_z(z)| = \int_{B_n}\frac{(1-|w|^2)^{\lambda+t}(1-|z|^2)^{\tau-t}\log^k\frac{\mathrm{e}}{1-|z|^2}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\mathrm{d}v(w)$$

for all $z \in B_n$ if and only if $\tau > t$ and $k \le 0$, or $\tau = t$ and $k \le -1$.

(3) $\Rightarrow$ (2)

When $-\lambda < t+1$, we have that $\lambda + t > -1$. Let $\tau > t$ and $k \le 0$, or $\tau = t$ and $k \le -1$. For any $f \in L^1(B_n, \mathrm{d}v_t)$, it follows from Proposition A that

$$\begin{aligned}||R_{\lambda,\tau,k}f||_{1,t} &\lesssim \int_{B_n} R_{\lambda,\tau,k}|f|(z)(1-|z|^2)^t \mathrm{d}v(z)\\ &= \int_{B_n}(1-|w|^2)^\tau |f(w)|\log^k\frac{\mathrm{e}}{1-|w|^2} \left\{\int_{B_n} \frac{(1-|z|^2)^{\lambda+t}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\mathrm{d}v(z)\right\}\mathrm{d}v(w)\\ &\lesssim \int_{B_n}|f(w)|\mathrm{d}v_t(w) = ||f||_{1,t}.\end{aligned}$$

(2) $\Rightarrow$ (1)

This proof is the same as that of Theorem 1.1. This proof is complete. □

Finally, we consider the boundedness of the generalized Forelli-Rudin type operators from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

Proposition 3.3 (1) When $p \ge 1$ and $0 \le 2s < n$, if $-p\lambda < q+s < p(\tau+1)$, then $T_{\lambda,\tau,k}$ ($Q_{\lambda,\tau,k}$) and $S_{\lambda,\tau,k}$ ($R_{\lambda,\tau,k}$) are bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

(2) When $p \ge 1$ and $n \le 2s < 2n$, if $-p\lambda < q+s < q+n < p(\tau+1)$ and $\lambda+\tau+1 > (n-s)\mathrm{sgn}\{\max(p-1,0)\}$, then $T_{\lambda,\tau,k}$ ($Q_{\lambda,\tau,k}$) and $S_{\lambda,\tau,k}$ ($R_{\lambda,\tau,k}$) are bounded from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, where sgn is the symbol function.

(3) When $0 < p < 1$ and $0 \le s < n$, if $-p\lambda < q + s < q + n < p(\tau + 1 + n) - n$, then $T_{\lambda,\tau,k}$ ($Q_{\lambda,\tau,k}$) and $S_{\lambda,\tau,k}$ ($R_{\lambda,\tau,k}$) are bounded from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

(4) When $p > 0$ and $s \ge n$, if $-p\lambda < q + n < p(\tau + 1)$, then $T_{\lambda,\tau,k}$ ($Q_{\lambda,\tau,k}$) and $S_{\lambda,\tau,k}$ ($R_{\lambda,\tau,k}$) are bounded from $\mathcal{H}^{p,q,s,k}(B_n)$ to $\mathcal{G}_{\frac{q+n}{p}}(B_n)$.

Proof For any $f \in L^{p,q,s,k}(B_n)$ or $f \in \mathcal{H}^{p,q,s,k}(B_n)$, we only need to discuss the boundedness of $S_{\lambda,\tau,k}|f|$.

(1) Case $p = 1$.

For any $0 \le \rho < 1$ and $a \in B_n$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\int_{B_n} \frac{(1-|u|^2)^\tau |f(u)|}{|1-\langle \rho a, \rho u\rangle|^{2s}(1-\rho^2|u|^2)^{1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{1-\rho^2|u|^2} \mathrm{d}v(u) \\
&= 2n \int_0^1 \frac{t^{2n-1}(1-t^2)^\tau}{(1-\rho^2t^2)^{1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{1-\rho^2t^2} \left\{ \int_{S_n} \frac{|f(t\xi)|\mathrm{d}\sigma(\xi)}{|1-\rho^2\langle a, t\xi\rangle|^{2s}} \right\} \mathrm{d}t \\
&= 2n \int_0^1 \frac{t^{2n-1}(1-t^2)^{\tau-q-s}}{(1-\rho^2t^2)^{1+\lambda+\tau}(1-\rho^4|a|^2)^s} \log^k \frac{\mathrm{e}}{1-\rho^2t^2} \log^{-k} \frac{\mathrm{e}}{1-t^2} \\
&\quad \times \left\{ (1-t^2)^q \int_{S_n} \left| f(t\xi) \log^k \frac{\mathrm{e}}{1-|t\xi|^2} \right| (1-|\varphi_{\rho^2 a}(t\xi)|^2)^s \mathrm{d}\sigma(\xi) \right\} \mathrm{d}t \\
&\lesssim \int_0^1 \frac{||f||_{1,q,s,k}(1-t)^{\tau-q-s}}{(1-\rho t)^{1+\lambda+\tau}(1-\rho^4|a|^2)^s} \log^k \frac{\mathrm{e}}{1-\rho t} \log^{-k} \frac{\mathrm{e}}{1-t} \mathrm{d}t. \qquad (3.9)
\end{aligned}$$

The conditions $-\lambda < q + s < \tau + 1$ show that $\tau - q - s > -1$, $\lambda + \tau + 1 + n > n$ and $q + s + \lambda > 0$. For any $a \in B_n$, by Fubini's theorem, Proposition 3.2(1), (3.9) and Lemma 2.1 (case I_2 for $k' = -k$), we have that

$$\begin{aligned}
&(1-\rho^2)^q \int_{S_n} S_{\lambda,\tau,k}|f|(\rho\xi)(1-|\varphi_a(\rho\xi)|^2)^s \mathrm{d}\sigma(\xi) \\
&= \int_{B_n} \frac{|f(u)|}{(1-|u|^2)^{-\tau}} \left\{ \int_{S_n} \frac{(1-\rho^2)^{q+s+\lambda}(1-|a|^2)^s \log^k \frac{\mathrm{e}}{|1-\langle \rho\xi, u\rangle|} \mathrm{d}\sigma(\xi)}{|1-\langle \rho\xi, u\rangle|^{n+1+\lambda+\tau}|1-\langle \rho\xi, a\rangle|^{2s}} \right\} \mathrm{d}v(u) \\
&\asymp \int_{B_n} \frac{(1-\rho^2)^{q+s+\lambda}(1-|a|^2)^s(1-|u|^2)^\tau |f(u)|}{|1-\langle \rho a, \rho u\rangle|^{2s}(1-\rho^2|u|^2)^{1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{1-\rho^2|u|^2} \mathrm{d}v(u) \\
&\lesssim \frac{(1-\rho)^{q+s+\lambda}(1-|a|^2)^s}{(1-\rho^4|a|^2)^s} ||f||_{1,q,s,k} \int_0^1 \frac{(1-t)^{\tau-q-s}}{(1-\rho t)^{1+\lambda+\tau}} \log^k \frac{\mathrm{e}}{1-\rho t} \log^{-k} \frac{\mathrm{e}}{1-t} \mathrm{d}t \\
&\lesssim ||f||_{1,q,s,k}.
\end{aligned}$$

Case $p > 1$.

It follows from $-p\lambda < q + s < p(\tau + 1)$ that we may choose $\max\{\lambda + \tau - p\lambda,\ q + s - 1\} < \tau_1 < \min\{p(\tau + 1) - 1,\ \lambda + \tau + q + s\}$ such that $\lambda + \tau - p\lambda < \tau_1 < p(\tau + 1) - 1$, $1 + \lambda + \tau > 0$, $\tau_1 - q - s > -1$ and $(1 + \lambda + \tau) - (\tau_1 - q - s) - 1 > 0$. By (3.8), Proposition 3.2(1), Lemma 2.3 and Lemma 2.1 (the case I_2 for $k' = -pk$), we have that

$$\begin{aligned}
&(1-\rho^2)^q \int_{S_n} \{S_{\lambda,\tau,k}|f|(\rho\xi)\}^p (1-|\varphi_a(\rho\xi)|^2)^s \mathrm{d}\sigma(\xi) \\
&\lesssim \int_{B_n} (1-|w|^2)^{\tau_1} |f(w)|^p \left(\int_{S_n} \frac{(1-\rho^2)^{q+s+\lambda+\tau-\tau_1}(1-|a|^2)^s \log^{pk} \frac{\mathrm{e}}{|1-\langle \xi, \rho w\rangle|} \mathrm{d}\sigma(\xi)}{|1-\langle \xi, \rho w\rangle|^{n+1+\lambda+\tau}|1-\langle \xi, \rho a\rangle|^{2s}} \right) \mathrm{d}v(w) \\
&\asymp \int_{B_n} \frac{(1-\rho^2)^{q+s+\lambda+\tau-\tau_1}(1-|a|^2)^s(1-|w|^2)^{\tau_1}|f(w)|^p \log^{pk} \frac{\mathrm{e}}{1-\rho^2|w|^2}}{(1-\rho^2|w|^2)^{1+\lambda+\tau}|1-\langle w, \rho^2 a\rangle|^{2s}} \mathrm{d}v(w)
\end{aligned}$$

$$\lesssim \frac{||f||^p_{p,q,s,k}(1-\rho^2)^{q+s+\lambda+\tau-\tau_1}(1-|a|^2)^s}{(1-\rho^4|a|^2)^s}\int_0^1 \frac{(1-t)^{\tau_1-q-s}\log^{pk}\frac{\mathrm{e}}{1-\rho t}}{(1-\rho t)^{1+\lambda+\tau}\log^{pk}\frac{\mathrm{e}}{1-t}}\mathrm{d}t$$
$$\lesssim ||f||^p_{p,q,s,k}.$$

This means that $T_{\lambda,\tau,k}$ and $S_{\lambda,\tau,k}$ are bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

Similarly, we may prove that $Q_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are two bounded operators from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

(2) When $2s=n$ and $p=1$, it follows from $-\lambda<q+s<q+n<\tau+1$ that we choose $0<\varepsilon<\min\{1,s,1+\lambda+\tau,\lambda+q+n\}$. By Proposition 3.2(2), Lemma 2.3,

$$\sup_{0<x\le 1} x^\varepsilon \log\frac{\mathrm{e}}{x}=\frac{\mathrm{e}^{\varepsilon-1}}{\varepsilon},$$

Lemma 2.3, Proposition 3.1 and Lemma 2.1 (case I_2 for $k'=-k$), we have that

$$\begin{aligned}
&(1-\rho^2)^q\int_{S_n} S_{\lambda,\tau,k}|f|(\rho\xi)(1-|\varphi_a(\rho\xi)|^2)^s\mathrm{d}\sigma(\xi)\\
&\asymp \int_{B_n}\frac{(1-\rho^2)^{q+s+\lambda}(1-|a|^2)^s(1-|u|^2)^\tau|f(u)|}{|1-\langle\rho a,\rho u\rangle|^{2s}(1-\rho^2|u|^2)^{1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{1-\rho^2|u|^2}\mathrm{d}v(u)\\
&\quad+\int_{B_n}\frac{(1-|a|^2)^s(1-|u|^2)^\tau|f(u)|\log^k\frac{\mathrm{e}}{|1-\langle\rho u,\rho a\rangle|}}{(1-\rho^2)^{-q-s-\lambda}|1-\langle\rho u,\rho a\rangle|^{n+1+\lambda+\tau}}\log\frac{\mathrm{e}}{|1-\langle\rho a,\varphi_{\rho a}(\rho u)\rangle|}\mathrm{d}v(u)\\
&\lesssim ||f||_{1,q,s,k}+\int_{B_n}\frac{(1-|a|^2)^{s-\varepsilon}|f(u)|(1-|u|^2)^\tau\log^k\frac{\mathrm{e}}{|1-\langle\rho u,\rho a\rangle|}}{(1-\rho^2)^{-q-s-\lambda}|1-\langle\rho u,\rho a\rangle|^{n+1+\lambda+\tau-\varepsilon}}\mathrm{d}v(u)\\
&\lesssim ||f||_{1,q,s,k}+\int_{B_n}\frac{||f||_{1,q,s,k}(1-|a|^2)^{s-\varepsilon}(1-|u|^2)^{\tau-q-n}\log^k\frac{\mathrm{e}}{|1-\langle u,\rho^2a\rangle|}}{(1-\rho^2)^{-q-s-\lambda}|1-\langle u,\rho^2 a\rangle|^{n+1+\lambda+\tau-\varepsilon}\log^k\frac{\mathrm{e}}{1-|u|^2}}\mathrm{d}v(u)\\
&\asymp ||f||_{1,q,s,k}+\int_0^1\frac{||f||_{1,q,s,k}(1-|a|^2)^{s-\varepsilon}(1-t)^{\tau-q-n}\log^k\frac{\mathrm{e}}{1-t\rho^2|a|}}{(1-\rho^2)^{-q-s-\lambda}(1-t\rho^2|a|)^{1+\lambda+\tau-\varepsilon}\log^k\frac{\mathrm{e}}{1-t}}\mathrm{d}t\\
&\asymp ||f||_{1,q,s,k}+\frac{(1-|a|^2)^{s-\varepsilon}(1-\rho^2)^{q+s+\lambda}||f||_{1,q,s,k}}{(1-\rho^2|a|)^{\lambda+q+n-\varepsilon}}\lesssim ||f||_{1,q,s,k}.
\end{aligned}$$

When $2s>n$ and $p=1$, it follows from $-\lambda<q+s<q+n<\tau+1$ that $\tau-q-s>-1$, $q+s+\lambda>0$, $\tau-q-n>-1$ and $q+n+\lambda>0$ hold. By Fubini's theorem, Proposition 3.2(3), Lemmas 2.3–2.4, Proposition 3.1 and Lemma 2.1 (case I_2 for $k'=-k$), we get that

$$\begin{aligned}
&(1-\rho^2)^q\int_{S_n} S_{\lambda,\tau,k}|f|(\rho\xi)(1-|\varphi_a(\rho\xi)|^2)^s\ \mathrm{d}\sigma(\xi)\\
&\asymp \int_{B_n}\frac{(1-\rho^2)^{q+s+\lambda}(1-|a|^2)^s(1-|u|^2)^\tau|f(u)|}{|1-\langle\rho a,\rho u\rangle|^{2s}(1-\rho^2|u|^2)^{1+\lambda+\tau}}\log^k\frac{\mathrm{e}}{1-\rho^2|u|^2}\mathrm{d}v(u)\\
&\quad+\int_{B_n}\frac{(1-\rho^2)^{q+s+\lambda}(1-|a|^2)^s(1-|u|^2)^\tau|f(u)|}{|1-\langle\rho a,\rho u\rangle|^{n+1+\lambda+\tau}(1-\rho^2|a|^2)^{2s-n}}\log^k\frac{\mathrm{e}}{|1-\langle u,\rho^2a\rangle|}\mathrm{d}v(u)\\
&\lesssim ||f||_{1,q,s,k}+\int_{B_n}\frac{||f||_{1,q,s,k}(1-\rho^2)^{q+s+\lambda}(1-|u|^2)^{\tau-q-n}\log^k\frac{\mathrm{e}}{|1-\langle u,\rho^2a\rangle|}\mathrm{d}v(u)}{(1-|a|^2)^{-s}|1-\langle\rho a,\rho u\rangle|^{n+1+\lambda+\tau}(1-\rho^2|a|^2)^{2s-n}\log^k\frac{\mathrm{e}}{1-|u|^2}}\\
&\asymp ||f||_{1,q,s,k}+\int_0^1\frac{||f||_{1,q,s,k}(1-\rho^2)^{q+s+\lambda}(1-t)^{\tau-q-n}\log^k\frac{\mathrm{e}}{1-\rho^2|a|t}\mathrm{d}t}{(1-\rho^2|a|^2)^{2s-n}(1-|a|^2)^{-s}(1-\rho^2t|a|)^{1+\lambda+\tau}\log^k\frac{\mathrm{e}}{1-t}}\\
&\asymp ||f||_{1,q,s,k}+\frac{||f||_{1,q,s,k}(1-\rho^2)^{q+s+\lambda}(1-|a|^2)^s}{(1-\rho^2|a|^2)^{2s-n}(1-\rho^2|a|)^{q+n+\lambda}}\lesssim ||f||_{1,q,s,k}.
\end{aligned}$$

This means that $T_{\lambda,\tau,k}$ and $S_{\lambda,\tau,k}$ are bounded from $\mathcal{H}^{1,q,s,k}(B_n)$ to $L^{1,q,s}(B_n)$.

Case $p>1$.

The conditions $-p\lambda < q+s < q+n < p(\tau+1)$ and $\lambda+\tau+1 > n-s$ show that we may choose $\max\{\lambda+\tau-p\lambda,\ q+n-1\} < \tau_1 < \min\{p(\tau+1)-1,\ \lambda+\tau+q+s\}$ such that $\lambda+\tau-p\lambda < \tau_1 < p(\tau+1)-1$ and $-(\lambda+\tau-\tau_1) < q+s < q+n < \tau_1+1$. By (3.8) and the previous proof process for $p=1$, we can get that $T_{\lambda,\tau,k}$ and $S_{\lambda,\tau,k}$ are bounded from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$. We omit the details of the process.

Similarly, we may prove that $Q_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are two bounded operators from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

(3) Let $p(\tau+n+1)-n-1=\tau'$ and $0<r\le 1$. By (1.3), Lemma 2.20 and (2.20) in [1], and Lemma 2.5, we have that

$$\begin{aligned}
\{S_{\lambda,\tau,k}|f|(z)\}^p &\le \left\{\sum_{j=1}^{\infty}\int_{D(a^j,r)}\frac{(1-|w|^2)^{\tau}|f(w)|\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(w)}{(1-|z|^2)^{-\lambda}|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\right\}^p\\
&\lesssim \left\{(1-|z|^2)^{\lambda}\sum_{j=1}^{\infty}\frac{(1-|a^j|^2)^{\tau+n+1}\log^k\frac{\mathrm{e}}{|1-\langle z,a^j\rangle|}}{|1-\langle z,a^j\rangle|^{n+1+\lambda+\tau}}\sup_{w\in D(a^j,r)}|f(w)|\right\}^p\\
&\le (1-|z|^2)^{p\lambda}\sum_{j=1}^{\infty}\frac{(1-|a^j|^2)^{p(\tau+n+1)}\log^{pk}\frac{\mathrm{e}}{|1-\langle z,a^j\rangle|}}{|1-\langle z,a^j\rangle|^{p(n+1+\lambda+\tau)}}\sup_{w\in D(a^j,r)}|f(w)|^p\\
&\lesssim (1-|z|^2)^{p\lambda}\sum_{j=1}^{\infty}\frac{(1-|a^j|^2)^{\tau'}\log^{pk}\frac{\mathrm{e}}{|1-\langle z,a^j\rangle|}}{|1-\langle z,a^j\rangle|^{n+1+p\lambda+\tau'}}\sup_{w\in D(a^j,r)}\int_{D(w,r)}|f(u)|^p\mathrm{d}v(u)\\
&\le (1-|z|^2)^{p\lambda}\sum_{j=1}^{\infty}\frac{(1-|a^j|^2)^{\tau'}\log^{pk}\frac{\mathrm{e}}{|1-\langle z,a^j\rangle|}}{|1-\langle z,a^j\rangle|^{n+1+p\lambda+\tau'}}\int_{D(a^j,2r)}|f(u)|^p\mathrm{d}v(u)\\
&\lesssim (1-|z|^2)^{p\lambda}\sum_{j=1}^{\infty}\int_{D(a^j,4r)}\frac{(1-|u|^2)^{\tau'}|f(u)|^p\log^{pk}\frac{\mathrm{e}}{|1-\langle z,u\rangle|}}{|1-\langle z,u\rangle|^{n+1+p\lambda+\tau'}}\mathrm{d}v(u)\\
&\le N(1-|z|^2)^{p\lambda}\int_{B_n}\frac{(1-|u|^2)^{\tau'}|f(u)|^p}{|1-\langle z,u\rangle|^{n+1+p\lambda+\tau'}}\log^{pk}\frac{\mathrm{e}}{|1-\langle z,u\rangle|}\mathrm{d}v(u). \qquad (3.10)
\end{aligned}$$

The conditions $-p\lambda < q+s < q+n < p(\tau+1+n)-n$ mean that $-p\lambda < q+s < q+n < \tau'+1$. By (3.10) and the previous proof process for $p=1$, we can get that $T_{\lambda,\tau,k}$ and $S_{\lambda,\tau,k}$ are bounded from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

Similarly, we may prove that $Q_{\lambda,\tau,k}$ and $R_{\lambda,\tau,k}$ are two bounded operators from $\mathcal{H}^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

(4) If $s\ge n$, then it follows from Lemma 2.4 that $\mathcal{H}^{p,q,s,k}(B_n)=\mathcal{H}^{\infty}_{\frac{q+n}{p},k}(B_n)$. When $-p\lambda < q+n < p(\tau+1)$, by Lemma 2.4, Proposition 3.1 and Lemma 2.1 (the case I_2 for $k'=-k$), we have that

$$\begin{aligned}
\big\|S_{\lambda,\tau,k}|f|\big\|_{\frac{q+n}{p}} &= \sup_{z\in B_n}(1-|z|^2)^{\frac{q+n}{p}}|S_{\lambda,\tau,k}|f|(z)|\\
&\lesssim ||f||_{\frac{q+n}{p},k}\sup_{z\in B_n}(1-|z|^2)^{\frac{q+n}{p}+\lambda}\int_{B_n}\frac{(1-|w|^2)^{\tau-\frac{q+n}{p}}\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}\log^k\frac{\mathrm{e}}{1-|w|^2}}\\
&\asymp ||f||_{\frac{q+n}{p},k} \quad\Rightarrow\quad T_{\lambda,\tau,k}f,\ S_{\lambda,\tau,k}f\in\mathcal{G}_{\frac{q+n}{p}}(B_n).
\end{aligned}$$

Similarly, we may prove that $Q_{\lambda,\tau,k}f,\ R_{\lambda,\tau,k}f\in\mathcal{G}_{\frac{q+n}{p}}(B_n)$. The proof is complete. □

Next, we consider the necessary conditions.

Proposition 3.4 (1) If $S_{\lambda,\tau,k}$ ($R_{\lambda,\tau,k}$) is a bounded operator from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then $-p\lambda<q+\min(s,n)<p(\tau+1)$.

(2) If $T_{\lambda,\tau,k}$ is a bounded operator from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then (i) $-p\lambda\le q+s<p(\tau+1)$, or $-p\lambda\le q+s=p(\tau+1)$ and $k>1$ when $0\le s<n$; (ii) $-p\lambda<q+n<p(\tau+1)$, or $-p\lambda<q+n=p(\tau+1)$ and $k>1$ when $s\ge n$.

(3) If $Q_{\lambda,\tau,k}$ is a bounded operator from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then (i) $-p\lambda\le q+s<p(\tau+1)$ when $0\le s<n$; (ii) $-p\lambda<q+n<p(\tau+1)$ when $s\ge n$.

Proof (1) Let $f(z)=(1-|z|^2)^\alpha\log^{-k}\frac{\mathrm{e}}{1-|z|^2}$. It follows from Proposition A that

$$||f||^p_{p,q,s,k}=\sup_{0\le\rho<1}\sup_{a\in B_n}\int_{S_n}\frac{(1-\rho^2)^{q+s+p\alpha}(1-|a|^2)^s}{|1-\langle a,\rho\xi\rangle|^{2s}}\mathrm{d}\sigma(\xi)<\infty$$

if and only if $q+\min(s,n)+p\alpha\ge0$. In particular,

$$f(z)=(1-|z|^2)^{-\frac{q+\min(s,n)}{p}}\log^{-k}\frac{\mathrm{e}}{1-|z|^2}\in L^{p,q,s,k}(B_n).$$

If $S_{\lambda,\tau,k}$ is bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then we may take that $z_0\in\{z\in\mathbb{C}^n:|z|\le1/2\}$ such that

$$\infty>|S_{\lambda,\tau,k}f(z_0)|\asymp\int_{B_n}(1-|w|^2)^{\tau-\frac{q+\min(s,n)}{p}}\log^{-k}\frac{\mathrm{e}}{1-|w|^2}\mathrm{d}v(w).$$

This means that $\tau-\frac{q+\min(s,n)}{p}>-1$, or $\tau-\frac{q+\min(s,n)}{p}=-1$ and $k>1$.

On the other hand, it is clear that $S_{\lambda,\tau,k}f(z)\gtrsim(1-|z|^2)^\lambda$. It follows from $S_{\lambda,\tau,k}f\in L^{p,q,s}(B_n)$ that we have that $q+\min(s,n)+p\lambda\ge0$. This shows that $-p\lambda\le q+\min(s,n)<p(\tau+1)$, or $-p\lambda\le q+\min(s,n)=p(\tau+1)$ and $k>1$.

Let $z\in B_n$. When $1+\lambda+\tau>0$, by Lemma 2.3, Proposition 3.1 and Lemma 2.1 (case I_2 for $k'=-k$), we have that

$$\begin{aligned}S_{\lambda,\tau,k}f(z)&=(1-|z|^2)^\lambda\int_{B_n}\frac{(1-|w|^2)^{\tau-\frac{q+\min(s,n)}{p}}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\frac{\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}}{\log^k\frac{\mathrm{e}}{1-|w|^2}}\mathrm{d}v(w)\\&\asymp(1-|z|^2)^\lambda\int_0^1\frac{(1-\rho)^{\tau-\frac{q+\min(s,n)}{p}}\log^k\frac{\mathrm{e}}{1-\rho|z|}}{(1-\rho|z|)^{1+\lambda+\tau}\log^k\frac{\mathrm{e}}{1-\rho}}\mathrm{d}\rho\asymp g(z),\quad\text{where}\end{aligned}$$

$$g(z)=\begin{cases}(1-|z|^2)^{-\frac{q+\min(s,n)}{p}}, & -\lambda<\dfrac{q+\min(s,n)}{p}<\tau+1,\\(1-|z|^2)^{-\frac{q+\min(s,n)}{p}}\log\dfrac{\mathrm{e}}{1-|z|^2}, & -\lambda<\dfrac{q+\min(s,n)}{p}=\tau+1\text{ and }k>1,\\(1-|z|^2)^{-\frac{q+\min(s,n)}{p}}\log\dfrac{\mathrm{e}}{1-|z|^2}, & -\lambda=\dfrac{q+\min(s,n)}{p}<\tau+1.\end{cases}$$

If $S_{\lambda,\tau,k}$ is a bounded operator from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then we have that $S_{\lambda,\tau,k}f\in L^{p,q,s}(B_n)\ \Leftrightarrow\ g\in L^{p,q,s}(B_n)$. By calculation, g belongs to $L^{p,q,s}(B_n)$ if and only if $-p\lambda<q+\min(s,n)<p(\tau+1)$.

If $1+\lambda+\tau=0$, then there must be $-p\lambda=q+\min(s,n)=p(\tau+1)$ and $k>1$. By Proposition 3.1 and Lemma 2.1 (case I_2 for $k'=-k$ and $\delta=-1$), we have that

$$\begin{aligned}S_{\lambda,\tau,k}f(z)&=(1-|z|^2)^{\lambda}\int_{B_n}\frac{(1-|w|^2)^{-1}}{|1-\langle z,w\rangle|^n}\frac{\log^k\frac{\mathrm{e}}{|1-\langle z,w\rangle|}}{\log^k\frac{\mathrm{e}}{1-|w|^2}}\mathrm{d}v(w)\\&\asymp(1-|z|^2)^{\lambda}\int_0^1\frac{(1-\rho)^{-1}}{\log^k\frac{\mathrm{e}}{1-\rho}}\log^{k+1}\frac{\mathrm{e}}{1-\rho|z|}\mathrm{d}\rho\\&\asymp(1-|z|^2)^{\lambda}\log^2\frac{\mathrm{e}}{1-|z|^2}=h(z).\end{aligned}$$

It is easy to prove that h does not belong to $L^{p,q,s}(B_n)$.

Similarly, it is easier to prove that $-p\lambda<q+\min(s,n)<p(\tau+1)$ when $R_{\lambda,\tau,k}$ is bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$.

(2) We have proven that $f(z)=(1-|z|^2)^{-\frac{q+\min(s,n)}{p}}\log^{-k}\frac{\mathrm{e}}{1-|z|^2}\in L^{p,q,s,k}(B_n)$. If $T_{\lambda,\tau,k}$ is bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then we have that $T_{\lambda,\tau,k}f(0)<\infty$. This shows that $\tau-\frac{q+\min(s,n)}{p}>-1$, or $\tau-\frac{q+\min(s,n)}{p}=-1$ and $k>1$.

This symmetry of B_n shows that there exists a constant c such that

$$\begin{aligned}T_{\lambda,\tau,k}f(z)&=(1-|z|^2)^{\lambda}\int_{B_n}\frac{(1-|w|^2)^{\tau-\frac{q+\min(s,n)}{p}}}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}\frac{\log^k\frac{\mathrm{e}}{1-\langle z,w\rangle}}{\log^k\frac{\mathrm{e}}{1-|w|^2}}\mathrm{d}v(w)\\&=c(1-|z|^2)^{\lambda}\ \text{ when }\tau-\frac{q+\min(s,n)}{p}>-1,\text{ or }\tau-\frac{q+\min(s,n)}{p}=-1\text{ and }k>1.\end{aligned}$$

If $T_{\lambda,\tau,k}$ is a bounded operator from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then $T_{\lambda,\tau,k}f\in L^{p,q,s}(B_n)$. This means that $-p\lambda\le q+\min(s,n)<p(\tau+1)$, or $-p\lambda\le q+\min(s,n)=p(\tau+1)$ and $k>1$.

For any $z\in B_n$ and λ with $q+\min(s,n)+p\lambda\ge 0$, we take that

$$F_z(w)=\frac{(1-|w|^2)^{\lambda}(1-\langle z,w\rangle)^{n+1+\lambda+\tau}|\log\frac{\mathrm{e}}{1-\langle w,z\rangle}|^k}{|1-\langle w,z\rangle|^{n+1+\lambda+\tau}\log^k\frac{\mathrm{e}}{1-\langle w,z\rangle}}\log^{-k}\frac{\mathrm{e}}{1-|w|^2}\quad(w\in B_n).$$

By $|F_z(w)|=(1-|w|^2)^{\lambda}\log^{-k}\frac{\mathrm{e}}{1-|w|^2}$, we have that $F_z\in L^{p,q,s,k}(B_n)$ and $||F_z||_{p,q,s,k}\asymp 1$. If $T_{\lambda,\tau,k}$ is bounded from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$, then $T_{\lambda,\tau,k}F_z\in\mathcal{H}^{p,q,s}(B_n)$. In fact, $T_{\lambda,\tau,k}F_z(w)=(1-|w|^2)^{\lambda}g_z(w)$ $(w\in B_n)$, where $g_z\in H(B_n)$. By the boundedness of $T_{\lambda,\tau,k}$ from $L^{p,q,s,k}(B_n)$ to $L^{p,q,s}(B_n)$ and Lemma 2.3, we have that

$$\begin{aligned}||T_{\lambda,\tau,k}||&\gtrsim||T_{\lambda,\tau,k}||.||F_z||_{p,q,s,k}\ge||T_{\lambda,\tau,k}F_z||_{p,q,s}\gtrsim(1-|z|^2)^{\frac{q+n}{p}}|(T_{\lambda,\tau,k}F_z)(z)|\\&\asymp(1-|z|^2)^{\frac{q+n}{p}+\lambda}\int_{B_n}\frac{(1-|w|^2)^{\tau+\lambda}\log^k\frac{\mathrm{e}}{|1-\langle w,z\rangle|}}{|1-\langle w,z\rangle|^{n+1+\lambda+\tau}\log^k\frac{\mathrm{e}}{1-|w|^2}}\mathrm{d}v(w).\end{aligned}$$

This implies that $\lambda+\tau>-1$, or $\lambda+\tau=-1$ and $k>1$. When $\lambda+\tau>-1$, it follows from Proposition 3.1 and Lemma 2.1 (case I_2 for $k'=-k$) that

$$(1-|z|^2)^{\frac{q+n}{p}+\lambda}\log\frac{\mathrm{e}}{1-|z|^2}\lesssim||T_{\lambda,\tau,k}||$$

for all $z\in B_n$. Therefore, there must be $q+n+p\lambda>0$.

When $\lambda+\tau=-1$ and $k>1$, it follows from Proposition 3.1 and Lemma 2.1 (the case I_2 for $k'=-k$ and $\delta=-1$) that

$$\int_{B_n}\frac{(1-|w|^2)^{-1}\log^k\frac{\mathrm{e}}{|1-\langle w,z\rangle|}}{|1-\langle w,z\rangle|^n\log^k\frac{\mathrm{e}}{1-|w|^2}}\mathrm{d}v(w)\asymp\int_0^1\frac{\log^{k+1}\frac{\mathrm{e}}{1-|z|\rho}}{(1-\rho)\log^k\frac{\mathrm{e}}{1-\rho}}\mathrm{d}\rho\asymp\log^2\frac{\mathrm{e}}{1-|z|^2}.$$

This shows that

$$(1-|z|^2)^{\frac{q+n}{p}+\lambda}\log^2\frac{\mathrm{e}}{1-|z|^2}\lesssim ||T_{\lambda,\tau,k}||$$

for all $z\in B_n$. Therefore, there must be $q+n+p\lambda>0$.

(3) For $f(z)=(1-|z|^2)^{-\frac{q+\min(s,n)}{p}}\log^{-k}\frac{\mathrm{e}}{1-|z|^2}\in L^{p,q,s,k}(B_n)$, we take that $z_0\in\{z\in\mathbb{C}^n: |z|\le 1/2\}$ such that

$$\infty>|Q_{\lambda,\tau,k}f(z_0)|\asymp\int_{B_n}(1-|w|^2)^{\tau-\frac{q+\min(s,n)}{p}}\mathrm{d}v(w).$$

This means that $q+\min(s,n)<p(\tau+1)$.

At the same time, we have that

$$Q_{\lambda,\tau,k}f(z)=(1-|z|^2)^{\lambda}\int_{B_n}\frac{(1-|w|^2)^{\tau-\frac{q+\min(s,n)}{p}}}{(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}\mathrm{d}v(w)=\frac{(1-|z|^2)^{\lambda}}{c_{\tau-\frac{q+\min(s,n)}{p}}}.$$

It follows from $Q_{\lambda,\tau,k}f\in L^{p,q,s}(B_n)$ that $q+\min(s,n)+p\lambda\ge 0$.

For any $z\in B_n$ and λ with $q+\min(s,n)+p\lambda\ge 0$, we take that

$$G_z(w)=\frac{(1-|w|^2)^{\lambda}(1-\langle z,w\rangle)^{n+1+\lambda+\tau}}{|1-\langle w,z\rangle|^{n+1+\lambda+\tau}}\log^{-k}\frac{\mathrm{e}}{1-|w|^2}\quad(w\in B_n).$$

Then $G_z\in L^{p,q,s,k}(B_n)$ and $||G_z||_{p,q,s,k}\lesssim 1$. By $Q_{\lambda,\tau,k}G_z(0)<\infty$, we have that $\lambda+\tau>-1$. Therefore, it follows from Proposition A that

$$Q_{\lambda,\tau,k}G_z(z)=(1-|z|^2)^{\lambda}\int_{B_n}\frac{(1-|w|^2)^{\tau+\lambda}}{|1-\langle z,w\rangle|^{n+1+\lambda+\tau}}\mathrm{d}v(w)\asymp(1-|z|^2)^{\lambda}\log\frac{\mathrm{e}}{1-|z|^2}.$$

Since $Q_{\lambda,\tau,k}G_z\in\mathcal{H}^{p,q,s}(B_n)$, it follows from Lemma 2.4 (case $k=0$) that

$$\begin{aligned}||Q_{\lambda,\tau,k}||&\gtrsim||Q_{\lambda,\tau,k}||.||G_z||_{p,q,s,k}\ge||Q_{\lambda,\tau,k}G_z||_{\frac{q+n}{p}}\gtrsim(1-|z|^2)^{\frac{q+n}{p}}Q_{\lambda,\tau,k}G_z(z)\\&\asymp(1-|z|^2)^{\frac{q+n}{p}+\lambda}\log\frac{\mathrm{e}}{1-|z|^2}\quad\text{for all } z\in B_n.\end{aligned}$$

This means that $q+n+p\lambda>0$. The proof is complete. □

Note When $s\ge n$, it follows from the test function in the proof of Proposition 3.4 that $-p\lambda<q+n<p(\tau+1)$ if $S_{\lambda,\tau,k}$ ($R_{\lambda,\tau,k}$) is bounded from $\mathcal{H}^{\infty}_{\frac{q+n}{p},k}(B_n)$ to $\mathcal{G}_{\frac{q+n}{p}}(B_n)$, or $Q_{\lambda,\tau,k}$ is bounded from $\mathcal{H}^{\infty}_{\frac{q+n}{p},k}(B_n)$ to $\mathcal{H}^{\infty}_{\frac{q+n}{p}}(B_n)$.

The proof of Theorem 1.4 By Propositions 3.3–3.4 and the above note, (1), (2), (3) and (4) are true. In (3) and (4), we need to notice that $\mathcal{H}^{p,q,s,k}(B_n)=\mathcal{H}^{\infty}_{\frac{q+n}{p},k}(B_n)$ and $\mathcal{H}^{p,q,s}(B_n)=\mathcal{H}^{\infty}_{\frac{q+n}{p}}(B_n)$ when $s\ge n$. In addition, $Q_{\lambda,\tau,k}f\in\mathcal{H}^{p,q,s}(B_n)$ when $f\in\mathcal{H}^{p,q,s,k}(B_n)$. The proof is complete. □

Conflict of Interest The authors declare no conflict of interest.

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