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# THE LIMIT CYCLE BIFURCATIONS OF A WHIRLING PENDULUM WITH PIECEWISE SMOOTH PERTURBATIONS

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**Abstract** This paper deals with the problem of limit cycles for the whirling pendulum equation $\dot{x} = y,\ \dot{y} = \sin x(\cos x - r)$ under piecewise smooth perturbations of polynomials of $\cos x$, $\sin x$ and $y$ of degree $n$ with the switching line $x = 0$. The upper bounds of the number of limit cycles in both the oscillatory and the rotary regions are obtained using the Picard-Fuchs equations, which the generating functions of the associated first order Melnikov functions satisfy. Furthermore, the exact bound of a special case is given using the Chebyshev system. At the end, some numerical simulations are given to illustrate the existence of limit cycles.

**Key words** whirling pendulum; limit cycle; Melnikov function; Picard-Fuchs equation; Chebyshev system

**2020 MR Subject Classification** 34C08; 34C07

## 1 Introduction and Main Results

The piecewise smooth differential system has attracted the attention of many researchers interested in studying its limit cycles. One important reason for that is that a sudden behavior after a slow change is common in both natural and artificial systems, and that is usually described by the piecewise smooth mathematical model [25, 31]. Another interesting reason is that this problem can be seen as an extension to the piecewise smooth world of the Hilbert's 16-th problem, which is provided by Hilbert [6] in 1902. We recall that Hilbert's 16th problem asks for the maximum number of limit cycles of planar polynomial vector fields of degree $n$, $n \in \mathbb{N}^+$, and for their relative distributions on the plane. Later, this problem was posed again by Smale [30] in 1998. Although there are plenty of excellent articles corresponding to that, see, for instance, [8, 11, 22] and the references therein, the problem is still open.

Consider a piecewise smooth differential system with the form

$$\dot{x} = H_y(x, y) + \varepsilon f(x, y), \quad \dot{y} = H_x(x, y) + \varepsilon g(x, y), \tag{1.1}$$

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where $0<|\varepsilon|\ll 1$ is a small parameter, and

$$H(x,y)=\begin{cases}H^+(x,y), & x\geq 0,\\ H^-(x,y), & x<0,\end{cases}$$

$$f(x,y)=\begin{cases}f^+(x,y), & x\geq 0,\\ f^-(x,y), & x<0\end{cases}$$

and

$$g(x,y)=\begin{cases}g^+(x,y), & x\geq 0,\\ g^-(x,y), & x<0\end{cases}$$

with the functions $H^\pm$, $f^\pm$, $g^\pm$ being $C^\infty$ smooth.

To establish the first order Melnikov function, one must first make the following assumptions, as in [21]:

**Assumption (I)** There exist an interval $\Sigma$ and two points $A_1(h)=(0,a_1(h))$ and $A_2(h)=(0,a_2(h))$ such that, for $h\in\Sigma$,

$$H^+(A_1(h))=H^+(A_2(h))=h,\ H^-(A_1(h))=H^-(A_2(h)),\ a_1(h)>a_2(h).$$

**Assumption (II)** The equation $H^+(x,y)=h$, $x\geq 0$ defines an orbital arc $L_h^+$ starting from $A_1(h)$ and ending at $A_2(h)$; the equation $H^-(x,y)=H^-(A_2(h))$, $x\leq 0$ defines an orbital arc $L_h^-$ starting from $A_2(h)$ and ending at $A_1(h)$ such that the system $(1.1)|_{\varepsilon=0}$ has a family of clockwise oriented periodic orbits $L_h=L_h^+\cup L_h^-$, $h\in\Sigma$.

In order to study the problem of limit cycle bifurcations, the authors of [9, 17, 21] obtained the first order Melnikov function formula $M(h)$ of system (1.1) and the beautiful relationship between the limit cycles and the zeros of $M(h)$, as in the smooth case. We review these results here for the convenience of the reader.

**Theorem A** Under the assumptions (I) and (II), we have that

(i) the first order Melnikov function of system (1.1) has the following form

$$M(h)=\int_{L_h^+}g^+\mathrm{d}x-f^+\mathrm{d}y+\int_{L_h^-}g^-\mathrm{d}x-f^-\mathrm{d}y,\ h\in\Sigma;$$

(ii) if $M(h)$ has $k$ zeros in $h$ on the interval $\Sigma$, with each having an odd multiplicity, then system (1.1) has at least $k$ limit cycles bifurcating from the period annulus for $|\varepsilon|$ small;

(iii) if the function $M(h)$ has at most $k$ zeros in h on the interval $\Sigma$, taking the multiplicities into account, then there exist at most $k$ limit cycles of system (1.1) bifurcating from the period annulus up to the first order.

This theorem has many applications to Hopf, homoclinic and heteroclinic bifurcations for when $f^\pm(x,y)$ and $g^\pm(x,y)$ are polynomials of $x$ and $y$, see, [7, 18, 33–35, 37, 38], for instance. However, for when $f^\pm(x,y)$ and $g^\pm(x,y)$ are non-polynomials (e.g., trigonometric functions), there are, as far as we know, very few papers on this. For the smooth case, the analysis of pendulum-like equations appears in some of the literature. Examples include the perturbed whirling pendulum

$$\dot{x}=y,\quad \dot{y}=\sin x(\cos x-r)+\varepsilon y(\cos x+\alpha), \tag{1.2}$$

where $\alpha$ is a real parameter; that was considered by Lichardová in [19]. Via the Melnikov method and the Li and Zhang criterion [16], the author proved that, for $\varepsilon > 0$ small enough, this system has a unique limit cycle in a certain region of a two-dimensional space of parameters. In [20], Lichardová proved that the period function of system $(1.2)|_{\varepsilon=0}$ is either monotone or has exactly one critical point, by using the Picard-Fuchs equation. Another related problem is the study of the periodic solutions of the simple pendulum

$$\dot{x} = y, \quad \dot{y} = -\sin x.$$

The perturbed of this equation was studied by several authors; see [2, 4, 10, 12, 13, 27, 29, 36]. Gasull, Geyer and Mañosas [4] considered the pendulum-like equation

$$\dot{x} = y, \quad \dot{y} = -\sin x + \varepsilon \sum_{s=0}^{m} Q_{n,s}(x) y^s,$$

where $Q_{n,s}(x)$ are trigonometric polynomials of a degree of at most $n$, and $\varepsilon > 0$ is a small parameter. They gave upper bounds on the number of zeros of the associated first order Melnikov function in both the oscillatory and the rotary regions. These upper bounds are obtained by expressing the corresponding Abelian integrals in terms of polynomials and the complete elliptic functions of first and second kind. We refer the reader to the classical monograph [1, 26] for a complete survey of this problem.

In this paper, motivated by the above references, we will study the number of limit cycles bifurcating from the period annuli of the whirling pendulum when it is perturbed inside any polynomials of $\cos x$, $\sin x$ and $y$ of degree $n$ with the switching line $x = 0$. It is well-known that any polynomial of $\sin x$, $\cos x$ and $y$ of degree $n$ can be written as

$$\sum_{i+j=0}^{n} a_{i,j} y^j \cos^i x + \sum_{i+j=0}^{n-1} b_{i,j} y^j \cos^i x \sin x,$$

where $a_{i,j}$ and $b_{i,j}$ are real constants.

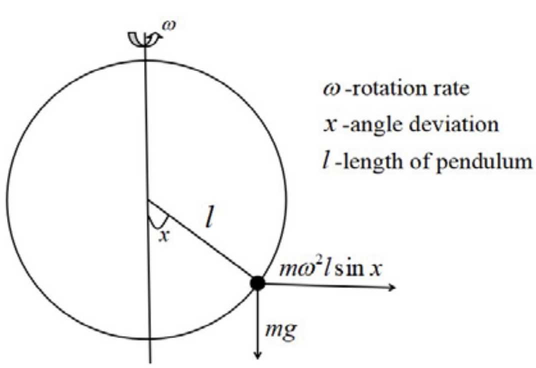


Figure 1 Whirling pendulum

The whirling pendulum is shown in Figure 1 above, it consists of a rigid frame that freely rotates about a vertical axis with constant rotation rate $\omega$ to which a planar pendulum with length $l$ and mass $m$ is attached, the pivot being on the vertical axis. If the angle deviation is denoted by $x$, then the centrifugal moment is $mw^2l^2 \sin x \cos x$, the gravity moment is $mgl \sin x$, and the moment of inertia is $ml^2$. Therefore, the motion of the whirling pendulum can be described by the equation (see [15, p272])

$$ml^2\ddot{x} - m\omega^2 l^2 \sin x \cos x + mgl \sin x = 0, \tag{1.3}$$

where $g$ is the gravity constant and the dot stands for the derivative with respect to the time $t$. Obviously, if there are forces to counteract gravity, then equation (1.3) is reduced to

$$ml^2\ddot{x} - m\omega^2 l^2 \sin x \cos x = 0.$$

Introducing a new variable $y = \dot{x}$ and then changing the variables $y \to \omega y, t \to t/\omega$ converts (1.3) to an equivalent planar system of first-order equations

$$\dot{x} = y, \quad \dot{y} = \sin x(\cos x - r), \tag{1.4}$$

where $r = \frac{g}{l\omega^2} \geq 0$ (when there are forces to counteract gravity, $r = 0$). This system is hamiltonian with the energy

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}\cos^2 x - r\cos x + r - \frac{1}{2}, \tag{1.5}$$

and its levels $H^{-1}(h) = L_h$ correspond to trajectories of system (1.4), where $h \in (h_m, +\infty)$ with

$$h_m = \begin{cases} -\dfrac{1}{2}(1-r)^2, & 0 \leq r < 1, \\ 0, & r \geq 1. \end{cases}$$

Depending on $r$, one can obtain three qualitatively different dynamics of system (1.4), and for all $r \geq 0$, the points $(\pm\pi, 0)$ in the phase plane are saddles (see Figures 2–4).

**Case (I)** For $r \geq 1$ (i.e., for small rotation rate), the dynamics are the same as for that of a planar pendulum: there is a center $(0, 0)$, two saddles $(\pm\pi, 0)$, and two types of periodic orbits. For $h \in (0, 2r)$, the levels $L_h^0 = \{(x, y)|H(x, y) = h\}$ are ovals surrounding the origin. This corresponds to oscillations about the stable equilibrium $(0, 0)$. Meanwhile for $h \in (2r, +\infty)$, the corresponding levels have two connected components which are again ovals: one of these is contained in the region $y > 0$ denoted by $L_h^+$, corresponding to clockwise rotations of the pendulum, and the other one is contained in the region $y < 0$ denoted by $L_h^-$ corresponding to counterclockwise rotations; see Figure 2.

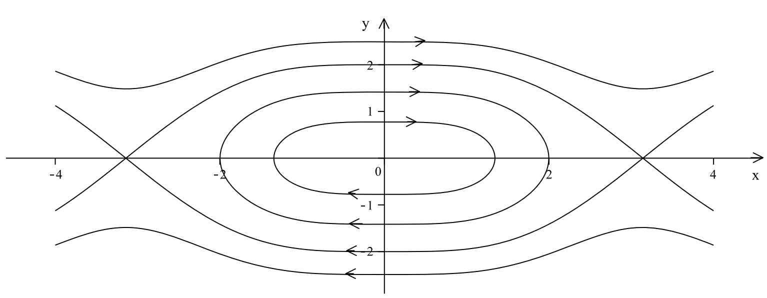


Figure 2 Phase portrait of system (1.4) with $r \geq 1$

**Case (II)** For $0 < r < 1$, i.e. if $\omega$ passes through the critical value $\sqrt{g/l}$, then $(0, 0)$ is a saddle point with two homoclinic loops (symmetric with respect to the $y$-axis). Inside each loop, there is a family of periodic solutions (deviated oscillations) $L_{h,+}^* = \{(x, y)|H(x, y) =$

$h, h \in (-\frac{1}{2}(1-r)^2, 0), x > 0\big\}$ and $L^*_{h,-} = \big\{(x,y)|H(x,y) = h, h \in (-\frac{1}{2}(1-r)^2, 0), x < 0\big\}$, which surround centers $(\arccos r, 0)$ and $(-\arccos r, 0)$, respectively; see Figure 3.

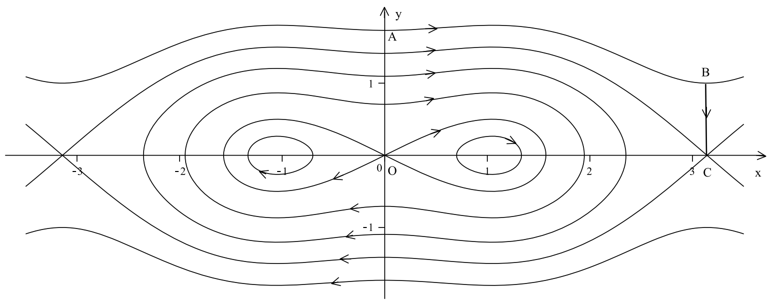


Figure 3 Phase portrait of system (1.4) with $0 < r < 1$

**Case (III)** For $r = 0$, that is to say, when there are forces to counteract gravity. $(0, 0)$ is also a saddle point with two heteroclinic loops (symmetric with respect to the $y$-axis). Inside each loop, there is a family of periodic solutions $L^*_{h,+} = \big\{(x,y)|H(x,y) = h, h \in (-\frac{1}{2}, 0), x > 0\big\}$ and $L^*_{h,-} = \big\{(x,y)|H(x,y) = h, h \in (-\frac{1}{2}, 0), x < 0\big\}$, which surround centers $(\frac{\pi}{2}, 0)$ and $(-\frac{\pi}{2}, 0)$, respectively; see Figure 4.

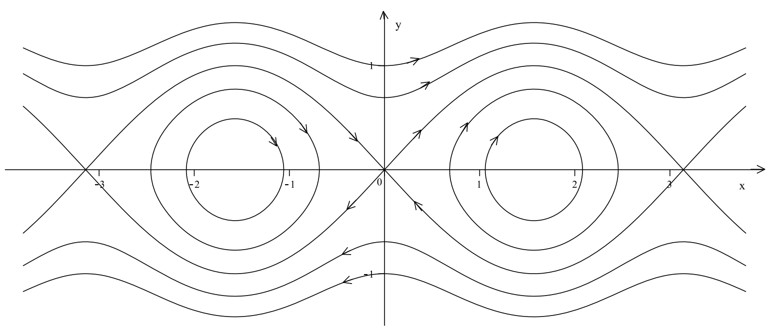


Figure 4 Phase portrait of system (1.4) with $r = 0$

In the sequel, we will take into consideration only the families $L^0_h$, $L^+_h$ and $L^*_{h,+}$, since, due to symmetry, the results for $L^-_h$ and $L^*_{h,-}$ are analogous. The superscripts 0, + and $*$ will denote which $L_h$-family is being used; for instance, $M^+(h)$ denotes a function $M(h)$ restricted to $L^+_h$.

Inspired by the non-smooth perturbation of the whirling pendulum, we will study the following perturbed equation of (1.4) with the switching line $x = 0$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon f^+(x,y) \\ \sin x(\cos x - r) + \varepsilon g^+(x,y) \end{pmatrix}, & x \geq 0, \\ \begin{pmatrix} y + \varepsilon f^-(x,y) \\ \sin x(\cos x - r) + \varepsilon g^-(x,y) \end{pmatrix}, & x < 0. \end{cases} \tag{1.6}$$

Here $0 < |\varepsilon| \ll 1$,

$$f^{\pm}(x,y) = \sum_{i+j=0}^{n} a^{\pm}_{i,j} y^j \cos^i x + \sum_{i+j=0}^{n-1} b^{\pm}_{i,j} y^j \cos^i x \sin x,$$
$$g^{\pm}(x,y) = \sum_{i+j=0}^{n} c^{\pm}_{i,j} y^j \cos^i x + \sum_{i+j=0}^{n-1} d^{\pm}_{i,j} y^j \cos^i x \sin x,$$

where $a^{\pm}_{i,j}, b^{\pm}_{i,j}, c^{\pm}_{i,j}$ and $d^{\pm}_{i,j}$ are real constants.

In order to simplify the notations, we first fix some statements. From now on we denote by $P_k(u)$ and $Q_k(u)$ the polynomials of a degree of at most $k$, and by $[x]$ the integer part for any real number $x$.

Without loss of generality, we will consider only the case $0 \le r < 1$ corresponding the phase portraits in Figure 3 and Figure 4, since, the case $r \ge 1$ is analogous and more digestible. For convenience, we first give the first order Melnikov functions of system (1.6). By the Poincaré-Pontryagin Theorem [3, 28] and Theorem A, the functions can be written as

$$\begin{aligned} M^+(h) = &\int_{L^+_{h,+}} g^+(x,y)\mathrm{d}x - f^+(x,y)\mathrm{d}y \\ &+ \int_{L^+_{h,-}} g^-(x,y)\mathrm{d}x - f^-(x,y)\mathrm{d}y,\ h \in (2r, +\infty), \end{aligned} \tag{1.7}$$

$$\begin{aligned} M^0(h) = &\int_{L^0_{h,+}} g^+(x,y)\mathrm{d}x - f^+(x,y)\mathrm{d}y \\ &+ \int_{L^0_{h,-}} g^-(x,y)\mathrm{d}x - f^-(x,y)\mathrm{d}y,\ h \in (0, 2r) \end{aligned} \tag{1.8}$$

and

$$M^*(h) = \oint_{L^*_{h,+}} g^+(x,y)\mathrm{d}x - f^+(x,y)\mathrm{d}y,\ h \in \big(-\frac{1}{2}(1-r)^2, 0\big), \tag{1.9}$$

where

$$\begin{aligned} L^+_{h,+} &= \{(x,y)|H(x,y) = h, h \in (2r,+\infty), x \ge 0, y > 0\}, \\ L^+_{h,-} &= \{(x,y)|H(x,y) = h, h \in (2r,+\infty), x < 0, y > 0\}, \\ L^0_{h,+} &= \{(x,y)|H(x,y) = h, h \in (0,2r), x \ge 0\}, \\ L^0_{h,-} &= \{(x,y)|H(x,y) = h, h \in (0,2r), x < 0\}, \\ L^*_{h,+} &= \big\{(x,y)|H(x,y) = h, h \in \big(-\frac{1}{2}(1-r)^2, 0\big), x \ge 0\big\}, \end{aligned}$$

and the number of isolated zeros of them, counting multiplicities, provides the upper bounds for the number of ovals of $H(x,y)$ that generate limit cycles of system (1.6) for $\varepsilon$ being close to zero, if they are not identically zero. The next three Theorems are our main results, and they hold only in one period annulus. The simultaneous bifurcation of limit cycles in more than one period annuli is a very intricate problem. This problem is of interest and deserves further work.

**Theorem 1.1** Consider the first order Melnikov functions of system (1.6) and $0 \le r < 1$. Then the following statements hold

(1) if $M^+(h)$ is not identically zero in $(2r, +\infty)$, then it has at most $137n + 15[\frac{n+1}{2}] + 482$ (resp. $14n + [\frac{n+1}{2}] + 30$) zeros counting in multiplicity in the interval $(2r, +\infty)$ for $0 < r < 1$ (resp. $r = 0$);

(2) if $M^0(h)$ is not identically zero in $(0,2r)$, then it has at most $55n+15[\frac{n+1}{2}]+173$ zeros counting in multiplicity in the interval $(0,2r)$ for $0<r<1$;

(3) if $M^*(h)$ is not identically zero in $\big(-\frac{1}{2}(1-r)^2,0\big)$, then it has at most $3n+25[\frac{n+1}{2}]+28$ (resp. $3n+[\frac{n+1}{2}]+4$) zeros counting in multiplicity in the interval $\big(-\frac{1}{2}(1-r)^2,0\big)$ for $0<r<1$ (resp. $r=0$).

**Theorem 1.2** If, in (1.6), $f^+(x,y)=f^-(x,y)$ and $g^+(x,y)=g^-(x,y)$ such that system (1.6) is smooth and $0\le r<1$, then the following statements hold:

(1) if $M^+(h)$ is not identically zero in $(2r,+\infty)$, then it has at most $3n+25[\frac{n+1}{2}]+28$ (resp. $3n+[\frac{n+1}{2}]+4$) zeros counting in multiplicity in the interval $(2r,+\infty)$ for $0<r<1$ (resp. $r=0$). The same result holds for $M^*(h)$ in the interval $\big(-\frac{1}{2}(1-r)^2,0\big)$;

(2) if $M^0(h)$ is not identically zero in $(0,2r)$, then it has at most $3n+3[\frac{n+1}{2}]+6$ zeros counting in multiplicity in the interval $(0,2r)$ for $0<r<1$.

The bounds given in Theorem 1.1 and Theorem 1.2 are not optimal. In the next theorems, we give optimal bounds for some particular smooth perturbations in the rotary region for $r=0$. In this case, there are forces to counteract gravity.

**Theorem 1.3** Consider the system

$$\begin{cases}\dot{x}=y,\\ \dot{y}=\sin x\cos x+\varepsilon\displaystyle\sum_{i=0}^{n}\big(a_iy^{2i}\cos^i x+b_iy^{2i}\cos^{i+1}x+c_iy^{2l+1}\cos^{2i}x\big),\end{cases}\tag{1.10}$$

or

$$\begin{cases}\dot{x}=y,\\ \dot{y}=\sin x\cos x+\varepsilon\displaystyle\sum_{i=0}^{n}\big(a_iy^{2i}\cos^i x+b_iy^{2i}\cos^{i+1}x+c_iy^{2l+1}\cos^{2i+1}x\big),\end{cases}\tag{1.11}$$

where $n,l\in\mathbb{N}$, $a_i,b_i,c_i\in\mathbb{R}$, and let $M^+(h)$ be its first order Melnikov function. Assume also that $M^+(h)$ is not identically zero. Then it has at most $2n+1$ zeros in $(0,+\infty)$, counting in multiplicity. This bound is optimal.

The techniques of the proofs of Theorems 1.1–1.3 mainly include use of the Melnikov function, the Picard-Fuchs equation, the Chebyshev criterion and the Gram determinant. We first obtain the algebraic structure of the first order Melnikov functions (see Lemma 2.3 and Lemma 2.5), which are more complicated than the Melnikov functions corresponding to the smooth case. Then we find that the corresponding generating functions of these satisfy some Picard-Fuchs equations (see Lemma 2.4). Finally, we give the upper bounds of the number of the zeros of the Melnikov functions by the Riccati equations and a derivation-division algorithm. For a special case, we get the exact bound by using the Chebyshev criterion and the Gram determinant, which is in fact similar to the proofs of [5]. It is worth noting that the Picard-Fuchs equation method can be applied to other situations regarding the investigation of limit cycles for differential systems under piecewise smooth non-polynomial perturbations.

The rest of the paper is organized as follows: in Section 2, we will give detailed expressions of the first order Melnikov functions, which can be expressed by some generating functions. The Picard-Fuchs equations of these generating functions are also derived. In Section 3 we prove Theorem 1.1, while Sections 4 and 5 address the proofs of Theorems 1.2 and 1.3, respectively. In Section 6, some numerical simulations are given to illustrate the existence of limit cycles.

## 2 Algebraic Structures of the First Order Melnikov Functions

For $i, j \in \mathbb{N}$, we denote that

$$\begin{aligned}
&I^+_{i,j}(h) = \int_{L^+_{h,+}} y^j \cos^i x \mathrm{d}x, \quad J^+_{i,j}(h) = \int_{L^+_{h,+}} y^j \cos^i x \sin x \mathrm{d}x,\\
&\bar{I}^+_{i,j}(h) = \int_{L^+_{h,-}} y^j \cos^i x \mathrm{d}x, \quad \bar{J}^+_{i,j}(h) = \int_{L^+_{h,-}} y^j \cos^i x \sin x \mathrm{d}x, \ \ h \in (2r, +\infty),\\
&I^0_{i,j}(h) = \int_{L^0_{h,+}} y^j \cos^i x \mathrm{d}x, \quad J^0_{i,j}(h) = \int_{L^0_{h,+}} y^j \cos^i x \sin x \mathrm{d}x,\\
&\bar{I}^0_{i,j}(h) = \int_{L^0_{h,-}} y^j \cos^i x \mathrm{d}x, \quad \bar{J}^0_{i,j}(h) = \int_{L^0_{h,-}} y^j \cos^i x \sin x \mathrm{d}x, \ \ h \in (0, 2r),\\
&I^*_{i,j}(h) = \int_{L^*_{h,+}} y^j \cos^i x \mathrm{d}x, \quad J^*_{i,j}(h) = \int_{L^*_{h,+}} y^j \cos^i x \sin x \mathrm{d}x, \ \ h \in \Big(-\frac{1}{2}(1-r)^2, 0\Big).
\end{aligned} \tag{2.1}$$

By a straightforward calculation, one has that

$$\begin{aligned}
&I^+_{i,j}(h) = \bar{I}^+_{i,j}(h), \quad J^+_{i,j}(h) = -\bar{J}^+_{i,j}(h),\\
&I^0_{i,j}(h) = \bar{I}_{i,j}(h), \quad J^0_{i,j}(h) = -\bar{J}_{i,j}(h).
\end{aligned} \tag{2.2}$$

**Lemma 2.1** If $0 \leq r < 1$, then the first order Melnikov functions $M^+(h)$, $M^0(h)$ and $M^*(h)$ can be written as

$$M^+(h) = \sum_{i+j=0}^{n+1} \lambda^+_{i,j} I^+_{i,j}(h) + \sum_{i+j=0}^{n} \mu^+_{i,j} J^+_{i,j}(h) + \sum_{j=0}^{n} \nu^+_j h^{\frac{j+1}{2}} + \sum_{j=0}^{n} \sigma^+_j (h-2r)^{\frac{j+1}{2}}, \tag{2.3}$$

$$M^0(h) = \sum_{i+j=0}^{n+1} \lambda^0_{i,j} I^0_{i,j}(h) + \sum_{i+j=0}^{n} \mu^0_{i,j} J^0_{i,j}(h) + \sum_{j=0}^{n} \nu^0_j h^{\frac{j+1}{2}} \tag{2.4}$$

and

$$M^*(h) = \sum_{i+j=0}^{n+1} \lambda^*_{i,j} I^*_{i,j}(h) + \sum_{i+j=0}^{n} \mu^*_{i,j} J^*_{i,j}(h). \tag{2.5}$$

Here $\lambda^+_{i,j}$, $\mu^+_{i,j}$, $\nu^+_j$, $\lambda^0_{i,j}$, $\mu^0_{i,j}$, $\nu^0_j$, $\sigma^+_j$, $\lambda^*_{i,j}$ and $\mu^*_{i,j}$ are arbitrary constants which can be expressed by the coefficients of $f^\pm(x,y)$ and $g^\pm(x,y)$.

**Proof** Without loss of generality, we only prove (2.3). The proofs of (2.4) and (2.5) follow in the same way. To see this, we need to add an auxiliary line $BC$, which is perpendicular to the $x$-axis at the saddle point $C(\pi, 0)$, and intersects $L^+_{h,+}$ at $B$; see Figure 3. Let $\Omega$ be the interior of $L^+_{h,+} \cup \overrightarrow{BC} \cup \overrightarrow{CO} \cup \overrightarrow{OA}$, and let $A$ be the point of intersection of $L^+_{h,+}$ with $y$-axis. By using Green's Theorem two times, one has that

$$\begin{aligned}
\int_{L^+_{h,+}} y^j \cos^i x \sin x \mathrm{d}y &= \oint_{L^+_{h,+} \cup \overrightarrow{BC} \cup \overrightarrow{CO} \cup \overrightarrow{OA}} y^j \cos^i x \sin x \mathrm{d}y\\
&= i \iint_\Omega y^j \cos^{i-1} x \mathrm{d}x\mathrm{d}y - (i+1) \iint_\Omega y^j \cos^{i+1} x \mathrm{d}x\mathrm{d}y,\\
\int_{L^+_{h,+}} y^j \cos^i x \mathrm{d}x &= \oint_{L^+_{h,+} \cup \overrightarrow{BC} \cup \overrightarrow{CO} \cup \overrightarrow{OA}} y^j \cos^i x \mathrm{d}x - \oint_{\overrightarrow{CO}} y^j \cos^i x \mathrm{d}x\\
&= j \iint_\Omega y^{j-1} \cos^i x \mathrm{d}x\mathrm{d}y - \oint_{\overrightarrow{CO}} y^j \cos^i x \mathrm{d}x.
\end{aligned}$$

Hence,

$$\int_{L_{h,+}^+} y^j \cos^i x \sin x \mathrm{d}y = \frac{i}{j+1} I_{i-1,j+1}^+(h) - \frac{i+1}{j+1} I_{i+1,j+1}^+(h),\ i \geq 0, j \geq 0. \tag{2.6}$$

In a similar way, we have that

$$\int_{L_{h,+}^+} y^j \cos^i x \mathrm{d}y = \frac{i}{j+1} J_{i-1,j+1}^+(h) - \frac{1}{j+1}(2h)^{\frac{j+1}{2}} + \frac{(-1)^i}{j+1}(2h-4r)^{\frac{j+1}{2}},\ i \geq 0, j \geq 0. \tag{2.7}$$

For the orbits $L_{h,-}^+$ in the left half plane, similarly to $L_{h,+}^+$, one can get, for $i \geq 0, j \geq 0$, that

$$\begin{aligned} &\int_{L_{h,-}^+} y^j \cos^i x \sin x \mathrm{d}y = \frac{i}{j+1} \bar{I}_{i-1,j+1}^+(h) - \frac{i+1}{j+1} \bar{I}_{i+1,j+1}^+(h), \\ &\int_{L_{h,-}^+} y^j \cos^i x \mathrm{d}y = \frac{i}{j+1} \bar{J}_{i-1,j+1}^+(h) + \frac{1}{j+1}(2h)^{\frac{j+1}{2}} - \frac{(-1)^i}{j+1}(2h-4r)^{\frac{j+1}{2}}. \end{aligned} \tag{2.8}$$

Therefore, by (1.7), (2.1) and (2.6)–(2.8), one has that

$$\begin{aligned} M^+(h) = &\sum_{i+j=0}^{n} c_{i,j}^+ I_{i,j}^+(h) + \sum_{i+j=0}^{n-1} d_{i,j}^+ J_{i,j}^+(h) - \sum_{i+j=0}^{n-1} b_{i,j}^+ \Big[\frac{i}{j+1} I_{i-1,j+1}^+(h) - \frac{i+1}{j+1} I_{i+1,j+1}^+(h)\Big] \\ &- \sum_{i+j=0}^{n} a_{i,j}^+ \Big[\frac{i}{j+1} J_{i-1,j+1}^+(h) - \frac{1}{j+1}(2h)^{\frac{j+1}{2}} + \frac{(-1)^i}{j+1}(2h-4r)^{\frac{j+1}{2}}\Big] \\ &+ \sum_{i+j=0}^{n} c_{i,j}^- \bar{I}_{i,j}^+(h) + \sum_{i+j=0}^{n-1} d_{i,j}^- \bar{J}_{i,j}^+(h) - \sum_{i+j=0}^{n-1} b_{i,j}^- \Big[\frac{i}{j+1} \bar{I}_{i-1,j+1}^+(h) - \frac{i+1}{j+1} \bar{I}_{i+1,j+1}^+(h)\Big] \\ &- \sum_{i+j=0}^{n} a_{i,j}^- \Big[\frac{i}{j+1} \bar{J}_{i-1,j+1}^+(h) + \frac{1}{j+1}(2h)^{\frac{j+1}{2}} - \frac{(-1)^i}{j+1}(2h-4r)^{\frac{j+1}{2}}\Big] \\ = &\sum_{i+j=0}^{n} a_{i,j} J_{i,j}^+(h) + \sum_{i+j=0}^{n+1} b_{i,j} I_{i,j}^+(h) + \sum_{i+j=0}^{n} \bar{a}_{i,j} \bar{J}_{i,j}^+(h) + \sum_{i+j=0}^{n+1} \bar{b}_{i,j} \bar{I}_{i,j}^+(h) \\ &+ \sum_{j=0}^{n} \nu_j^+ h^{\frac{j+1}{2}} + \sum_{j=0}^{n} \sigma_j^+ (h-2r)^{\frac{j+1}{2}}, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &a_{i,j} = d_{i,j}^+ - \frac{i+1}{j} a_{i+1,j-1}^+,\ b_{i,j} = c_{i,j}^+ - \frac{i+1}{j} b_{i+1,j-1}^+ + \frac{i}{j} b_{i-1,j-1}^+, \\ &\bar{a}_{i,j} = d_{i,j}^- - \frac{i+1}{j} a_{i+1,j-1}^-,\ \bar{b}_{i,j} = c_{i,j}^- - \frac{i+1}{j} b_{i+1,j-1}^- + \frac{i}{j} b_{i-1,j-1}^-, \\ &\nu_j^+ = \frac{2^{\frac{j+1}{2}}}{j+1} \sum_{i=0}^{n-j} (a_{i,j}^+ - a_{i,j}^-),\ \sigma_j^+ = \frac{2^{\frac{j+1}{2}}}{j+1} \sum_{i=0}^{n-j} (-1)^i (a_{i,j}^- - a_{i,j}^+). \end{aligned} \tag{2.10}$$

If the subscripts of $a_{i,j}^\pm$, $b_{i,j}^\pm$, $c_{i,j}^\pm$ and $d_{i,j}^\pm$ in (2.10) satisfy that $i < 0$ or $j < 0$, then they vanish. If $i+j > n$ (resp. $i+j > n-1$) in $a_{i,j}^\pm$ and $c_{i,j}^\pm$ (resp. $b_{i,j}^\pm$ and $d_{i,j}^\pm$) in (2.10), then $a_{i,j}^\pm = c_{i,j}^\pm = 0$ (resp. $b_{i,j}^\pm = d_{i,j}^\pm = 0$). Thus, inserting (2.2) into (2.9) gives (2.3) and

$$\lambda_{i,j}^+ = b_{i,j} + \bar{b}_{i,j}, \quad \mu_{i,j}^+ = a_{i,j} - \bar{a}_{i,j}.$$

This ends the proof. □

To understand the algebraic structure of the first order Melnikov functions $M^0(h)$ and $M^*(h)$, we need first to show that, for any $i, j \in \mathbb{N}$, $I_{i,2j}^0(h) = J_{i,2j}^0(h) = I_{i,2j}^*(h) = J_{i,2j}^*(h) = 0$.

Indeed, this is a direct consequence of the symmetry with respect to the $x$-axis of the integral paths and Green's Theorem.

**Lemma 2.2** For $0 \leq r < 1$, the following statements hold:

(1) letting $h \in (2r, +\infty)$, for $n \geq 3$,

$$\begin{aligned} &I^+_{i,2j+1}(h) = \alpha^+(h)I^+_{0,1}(h) + \beta^+(h)I^+_{1,1}(h) + \gamma^+(h)I^+_{2,1}(h), \quad i+2j+1=n,\\ &I^+_{i,2j}(h) = \xi^+(h)I^+_{0,0}(h) + \eta^+(h)I^+_{1,0}(h) + \zeta^+(h)I^+_{2,0}(h), \qquad i+2j=n \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &J^+_{i,2j+1}(h) = \delta^+(h)J^+_{0,1}(h) + h^{\frac{3}{2}}\varphi^+(h) + (h-2r)^{\frac{3}{2}}\psi^+(h), \quad i+2j+1=n,\\ &J^+_{i,2j}(h) = \phi^+(h), \qquad i+2j=n, \end{aligned} \tag{2.12}$$

where $\alpha^+(h)$, $\beta^+(h)$, $\gamma^+(h)$, $\xi^+(h)$, $\eta^+(h)$, $\zeta^+(h)$, $\delta^+(h)$, $\varphi^+(h)$, $\psi^+(h)$ and $\phi^+(h)$ are polynomials of $h$ with

$$\begin{aligned} &\deg\alpha^+(h), \deg\eta^+(h), \deg\delta^+(h) \leq [\frac{n-1}{2}],\ \deg\xi^+(h), \deg\phi^+(h) \leq [\frac{n}{2}],\\ &\deg\beta^+(h), \deg\zeta^+(h), \deg\varphi^+(h), \deg\psi^+(h) \leq [\frac{n-2}{2}],\ \deg\gamma^+(h) \leq [\frac{n-3}{2}]; \end{aligned}$$

(2) letting $h \in (0, 2r)$, for $i+2j+1 = n \geq 2$,

$$I^0_{i,2j+1}(h) = \alpha^0(h)I^0_{0,1}(h) + \beta^0(h)I^0_{1,1}(h) + \gamma^0(h)I^0_{2,1}(h), \tag{2.13}$$

and

$$J^0_{i,2j+1}(h) = \delta^0(h)J^0_{0,1}(h) + h^{\frac{3}{2}}\varphi^0(h), \tag{2.14}$$

where $\alpha^0(h)$, $\beta^0(h)$, $\gamma^0(h)$, $\delta^0(h)$ and $\varphi^0(h)$ are polynomials of $h$ with

$$\begin{aligned} &\deg\alpha^0(h) \leq [\frac{n-1}{2}], \quad \deg\beta^0(h), \deg\varphi^0 \leq [\frac{n-2}{2}],\\ &\deg\gamma^0(h) \leq [\frac{n-3}{2}], \quad \deg\delta^0(h) \leq [\frac{n-1}{2}]; \end{aligned}$$

(3) letting $h \in \big(-\frac{1}{2}(1-r)^2, 0\big)$, for $i+2j+1 = n \geq 2$,

$$I^*_{i,2j+1}(h) = \alpha^*(h)I^*_{0,1}(h) + \beta^*(h)I^*_{1,1}(h) + \gamma^*(h)I^*_{2,1}(h), \tag{2.15}$$

and

$$J^*_{i,2j+1}(h) = \delta^*(h)J^*_{0,1}(h), \tag{2.16}$$

where $\alpha^*(h)$, $\beta^*(h)$, $\gamma^*(h)$ and $\delta^*(h)$ are polynomials of $h$ with

$$\begin{aligned} &\deg\alpha^*(h) \leq [\frac{n-1}{2}], \quad \deg\beta^*(h) \leq [\frac{n-2}{2}],\\ &\deg\gamma^*(h) \leq [\frac{n-3}{2}], \quad \deg\delta^*(h) \leq [\frac{n-1}{2}]. \end{aligned}$$

**Proof** (1) It follows from

$$\frac{1}{2}y^2 + \frac{1}{2}\cos^2 x - r\cos x + r - \frac{1}{2} = h \tag{2.17}$$

that

$$y\frac{\partial y}{\partial x} - \cos x\sin x + r\sin x = 0. \tag{2.18}$$

Multiplying both sides of (2.18) by $y^j\cos^{i-3}x\sin x\mathrm{d}x$ and integrating along $L^+_{h,+}$, in view of (2.6), one gets that

$$I^+_{i,j}(h)=rI^+_{i-1,j}(h)+I^+_{i-2,j}(h)-rI^+_{i-3,j}(h)+\frac{i-2}{j+2}I^+_{i-2,j+2}(h)-\frac{i-3}{j+2}I^+_{i-4,j+2}(h). \quad (2.19)$$

Analogously, multiplying both sides of (2.17) by $y^{j-2}\cos^i x\mathrm{d}x$ and integrating along $L^+_{h,+}$, yields that

$$I^+_{i,j}(h)=(2h-2r+1)I^+_{i,j-2}(h)+2rI^+_{i+1,j-2}(h)-I^+_{i+2,j-2}(h). \quad (2.20)$$

We want the sum of the subscripts of the elements on the right side of equation (2.19) to be less than $i+j$. To this end, eliminating $I_{i-2,j+2}(h)$ in (2.19) using (2.20) gives that

$$\begin{aligned}I^+_{i,j}(h)=\frac{1}{i+j}\big[&\big(2(i-2)(h-r)+i+j\big)I^+_{i-2,j}(h)+(2i+j-2)rI^+_{i-1,j}(h)\\&-(j+2)rI^+_{i-3,j}(h)-(i-3)I^+_{i-4,j+2}(h)\big].\end{aligned} \quad (2.21)$$

Now we prove the first equality of (2.11) by induction on $n$, using (2.20) and (2.21). In fact, in view of (2.20) and (2.21), one has that

$$\begin{cases}I^+_{0,3}(h)=(2h-2r+1)I^+_{0,1}(h)+2rI^+_{1,1}(h)-I^+_{2,1}(h),\\ I^+_{1,3}(h)=(2h-2r+1)I^+_{1,1}(h)+2rI^+_{2,1}(h)-I^+_{3,1}(h),\\ I^+_{3,1}(h)=-\dfrac{3}{4}rI^+_{0,1}(h)+\dfrac{1}{2}(h-r+2)I^+_{1,1}(h)+\dfrac{5}{4}rI^+_{2,1}(h),\end{cases} \quad (2.22)$$

which proves the result for $n=3,4$. Now assume that the first equality of (2.11) holds for all $i+j\le n-1$, and that $n$ is an even number (if $n$ is an odd number, the proof is similar). Then, taking that $(i,j)=(1,n-1),(3,n-3),\cdots,(n-3,3)$ in (2.20) and that $(i,j)=(n-1,1)$ in (2.21), respectively, one obtains that

$$\mathbf{A}\begin{pmatrix}I^+_{1,n-1}(h)\\ I^+_{3,n-3}(h\\ \cdots\\ I^+_{n-3,3}(h)\\ I^+_{n-1,1}(h)\end{pmatrix}=\begin{pmatrix}(2h-2r+1)I^+_{1,n-3}(h)+2rI^+_{2,n-3}(h)\\ (2h-2r+1)I^+_{3,n-5}(h)+2rI^+_{4,n-5}(h)\\ \cdots\\ (2h-2r+1)I^+_{n-3,1}(h)+2rI^+_{n-2,1}(h)\\ \frac{1}{n}\big[(2n-3)I^+_{n-2,1}(h)+((n-3)(2h-2r+1)+3)I^+_{n-3,1}(h)\\ -3rI^+_{n-4,1}(h)-(n-4)I^+_{n-5,3}(h)\big]\end{pmatrix}, \quad (2.23)$$

where

$$\mathbf{A}=\begin{pmatrix}1&1&0&\cdots&0&0\\0&1&1&\cdots&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&1&1\\0&0&0&\cdots&0&1\end{pmatrix}.$$

According to (2.23), one has, for $(i,j)=(1,n-1),(3,n-3),\cdots,(n-3,3)$, that

$$\begin{aligned}I^+_{i,j}(h)=(2h-2r+1)&\Big[\alpha^{(n-2)}(h)I^+_{0,1}(h)+\beta^{(n-2)}(h)I^+_{1,1}(h)+\gamma^{(n-2)}(h)I^+_{2,1}(h)\Big]\\&+\alpha^{(n-1)}(h)I^+_{0,1}(h)+\beta^{(n-1)}(h)I^+_{1,1}(h)+\gamma^{(n-1)}(h)I^+_{2,1}(h)\end{aligned}$$

$$:= \alpha^{(n)}(h)I_{0,1}^{+}(h) + \beta^{(n)}(h)I_{1,1}^{+}(h) + \gamma^{(n)}(h)I_{2,1}^{+}(h),$$

where $\alpha^{(n-s)}(h)$, $\beta^{(n-s)}(h)$ and $\gamma^{(n-s)}(h)$ are polynomials of $h$ satisfying that

$$\deg\alpha^{(n-s)}(h) \leq [\frac{n-1-s}{2}],\ \deg\beta^{(n-s)}(h) \leq [\frac{n-2-s}{2}],$$
$$\deg\gamma^{(n-s)}(h) \leq [\frac{n-3-s}{2}],\ s=1,2.$$

Therefore, by using the induction hypothesis, on gets that

$$\deg\alpha^{(n)}(h) \leq [\frac{n-1}{2}],\ \deg\beta^{(n)}(h) \leq [\frac{n-2}{2}],\ \deg\gamma^{(n)}(h) \leq [\frac{n-3}{2}].$$

If $(i,j)=(n-1,1)$, we can prove the first equality in (2.11) in a similar way. The proof of the second equality in (2.11) follows in the same way.

To show the statement (2.12), in view of (2.7), we proceed by multiplying (2.18) $y^j\cos^{i-1}x\mathrm{d}x$ and integrating along $L_{h,+}^{+}$, which gives that

$$J_{i,j}^{+}(h) = rJ_{i-1,j}^{+}(h) + \frac{i-1}{j+2}J_{i-2,j+2}^{+}(h) - \frac{1}{j+2}(2h)^{\frac{j+2}{2}} - \frac{(-1)^i}{j+2}(2h-4r)^{\frac{j+2}{2}}. \tag{2.24}$$

Multiplying both sides of (2.17) by $y^{j-2}\cos^i x\sin x\mathrm{d}x$ implies that

$$J_{i,j}^{+}(h) = (2h-2r+1)J_{i,j-2}^{+}(h) + 2rJ_{i+1,j-2}^{+}(h) - J_{i+2,j-2}^{+}(h). \tag{2.25}$$

Elementary manipulation reduces (2.24) and (2.25) to

$$\begin{aligned} J_{i,j}^{+}(h) = \frac{1}{i+j+1}\Big[&(i-1)(2h-2r+1)J_{i-2,j}^{+}(h) + (2i+j)rJ_{i-1,j}^{+}(h) \\ &-(2h)^{\frac{j+2}{2}} - (-1)^i(2h-4r)^{\frac{j+2}{2}}\Big] \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} J_{i,j}^{+}(h) = \frac{1}{i+j+1}\Big[&j(2h-2r+1)J_{i,j-2}^{+}(h) + jrJ_{i+1,j-2}^{+}(h) \\ &+(2h)^{\frac{j}{2}} + (-1)^i(2h-4r)^{\frac{j}{2}}\Big]. \end{aligned} \tag{2.27}$$

Then, the first equality in (2.12) follows, by induction, using the above two equalities. For the sake of brevity we only give the details of the second equality of (2.12). In fact, a simple computation shows that

$$\begin{aligned} J_{i,2j}^{+}(h) &= \int_0^\pi y^{2j}\cos^i x\sin x\mathrm{d}x \\ &= \int_0^\pi \cos^i x(2h-2r+1-\cos^2 x+2r\cos x)^j\sin x\mathrm{d}x \\ &= \sum_{k=0}^{j} C_j^k(2h)^{j-k}\int_0^\pi \cos^i x(1-2r-\cos^2 x+2r\cos x)^j\sin x\mathrm{d}x, \end{aligned}$$

which gives the desired result for $i+2j=n$.

(2) Similarly to (2.20), (2.21), (2.26) and (2.27), one can get that

$$\begin{aligned} I_{i,j}^{0}(h) &= (2h-2r+1)I_{i,j-2}^{0}(h) + 2rI_{i+1,j-2}^{0}(h) - I_{i+2,j-2}^{0}(h), \\ I_{i,j}^{0}(h) &= \frac{1}{i+j}\big[\big(2(i-2)(h-r)+i+j\big)I_{i-2,j}^{0}(h) + (2i+j-2)rI_{i-1,j}^{0}(h) \\ &\quad - (j+2)rI_{i-3,j}^{0}(h) - (i-3)I_{i-4,j+2}^{0}(h)\big] \end{aligned} \tag{2.28}$$

and

$$J^0_{i,j}(h) = \frac{1}{i+j+1}\Big[(i-1)(2h-2r+1)J^0_{i-2,j}(h) + (2i+j)rJ^0_{i-1,j}(h) - \big(1-(-1)^j\big)(2h)^{\frac{j+2}{2}}\Big],$$

$$J^0_{i,j}(h) = \frac{1}{i+j+1}\Big[j(2h-2r+1)J^0_{i,j-2}(h) + jrJ^0_{i+1,j-2}(h) + \big(1-(-1)^j\big)(2h)^{\frac{j}{2}}\Big]. \tag{2.29}$$

The proofs of (2.13) and (2.14) are conducted along the lines of the proofs of (2.11) and (2.12), using (2.28) and (2.29).

The proof of statment (3) is much easier, and follows by using the same arguments, so we omit it for the sake of brevity. The proof of Lemma 2.2 is complete. □

The following lemma gives the algebraic structure of the first Melnikov functions $M^+(h)$, $M^0(h)$ and $M^*(h)$ in (1.7)–(1.9), which play a crucial role in the study of the number of limit cycles of equation (1.6):

**Lemma 2.3** For $0 \le r < 1$, $M^+(h)$, $M^0(h)$ and $M^*(h)$ can be expressed as

$$\begin{aligned} M^+(h) = {} & P^+_{2[\frac{n}{2}]+2}(\sqrt{h}) + P^+_{[\frac{n}{2}]}(h)\sqrt{h-2r} + P^+_{[\frac{n+1}{2}]}(h)\Big(\arctan\frac{r+1}{\sqrt{2h-4r}} - \arctan\frac{r-1}{\sqrt{2h}}\Big) \\ & + P^+_{[\frac{n}{2}]}(h)I^+_{0,1}(h) + P^+_{[\frac{n-1}{2}]}(h)I^+_{1,1}(h) + P^+_{[\frac{n-2}{2}]}(h)I^+_{2,1}(h),\ h \in (2r, +\infty), \end{aligned} \tag{2.30}$$

$$\begin{aligned} M^0(h) = {} & P^0_{2[\frac{n}{2}]+2}(\sqrt{h}) + P^0_{[\frac{n+1}{2}]}(h)\arctan\frac{r-1}{\sqrt{2h}} + P^0_{[\frac{n}{2}]}(h)I^0_{0,1}(h) \\ & + P^0_{[\frac{n-1}{2}]}(h)I^0_{1,1}(h) + P^0_{[\frac{n-2}{2}]}(h)I^0_{2,1}(h),\ h \in (0, 2r) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} M^*(h) = {} & P^*_{[\frac{n+1}{2}]}(h) + P^*_{[\frac{n}{2}]}(h)I^*_{0,1}(h) + P^*_{[\frac{n-1}{2}]}(h)I^*_{1,1}(h) \\ & + P^*_{[\frac{n-2}{2}]}(h)I^*_{2,1}(h),\ h \in \Big(-\frac{1}{2}(1-r)^2, 0\Big), \end{aligned} \tag{2.32}$$

where $P^+_k(h)$, $P^0_k(h)$ and $P^*_k(h)$ are polynomials of a degree of at most $k$ and $P^+_{[\frac{n-2}{2}]}(h) = P^0_{[\frac{n-2}{2}]}(h) = P^*_{[\frac{n-2}{2}]}(h) = 0$ for $n = 1$.

**Proof** By a straightforward calculation, one has that

$$\begin{aligned} I^+_{0,0}(h) &= \int_0^\pi \mathrm{d}x = \pi, \quad I^+_{1,0}(h) = \int_0^\pi \cos x\mathrm{d}x = 0, \quad I^+_{2,0}(h) = \int_0^\pi \cos^2 x\mathrm{d}x = \frac{\pi}{2}, \\ J^+_{0,1}(h) &= \int_0^\pi \sqrt{2h-2r+1-\cos^2 x + 2r\cos x}\sin x\mathrm{d}x \\ &= \frac{1}{2}(2h+r^2-2r+1)\Big[\arctan\frac{r+1}{\sqrt{2h-4r}} - \arctan\frac{r-1}{\sqrt{2h}}\Big] \\ &\quad + \frac{\sqrt{2}}{2}(r+1)\sqrt{h-2r} - \frac{\sqrt{2}}{2}(r-1)\sqrt{h}. \end{aligned} \tag{2.33}$$

If $n \ge 3$, then inserting (2.11), (2.12) and (2.33) into (2.3) gives (2.30). If $1 \le n \le 2$, (2.30) follows from directly (2.3), (2.20), (2.26) and (2.33). (2.31) and (2.32) can be proven similarly. This ends the proof. □

**Lemma 2.4** If $0 \le r < 1$, then the vector functions $V_1(h) = \big(I^+_{0,1}(h), I^+_{1,1}(h), I^+_{2,1}(h)\big)^T$, $V_2(h) = \big(I^0_{0,1}(h), I^0_{1,1}(h), I^0_{2,1}(h)\big)^T$ and $V_3(h) = \big(I^*_{0,1}(h), I^*_{1,1}(h), I^*_{2,1}(h)\big)^T$ satisfy the Picard-Fuchs equations

$$(Bh + C)V_i' = V_i,\ i = 1, 2, 3, \tag{2.34}$$

respectively, where

$$B=\begin{pmatrix}2&0&0\\0&1&0\\\frac{2}{3}&\frac{r}{3}&\frac{2}{3}\end{pmatrix},\quad C=\begin{pmatrix}1-2r&2r&-1\\\frac{r}{2}&-r&\frac{r}{2}\\-\frac{1}{6}r^2-\frac{2}{3}r+\frac{1}{3}&-\frac{r^2}{3}+\frac{4r}{3}&\frac{1}{2}r^2-\frac{2}{3}r-\frac{1}{3}\end{pmatrix}.$$

**Proof** According to (1.5), one has that

$$\frac{1}{2}y^2+\frac{1}{2}\cos^2 x-r\cos x+r-\frac{1}{2}=h. \tag{2.35}$$

Differentiating the above equation with respect to $h$ gives $\frac{\partial y}{\partial h}=\frac{1}{y}$, which implies that

$$I'_{i,j}(h)=j\int_{L^0_{h,+}}y^{j-2}\cos^i x\mathrm{d}x. \tag{2.36}$$

Hence,

$$I_{i,j}(h)=\frac{1}{j+2}I'_{i,j+2}(h). \tag{2.37}$$

Multiplying both sides of (2.36) by $h$, one gets that

$$\begin{aligned}hI'_{i,j}(h)&=j\int_{L^0_{h+}}y^{j-2}\cos^i x\big(\frac{1}{2}y^2+\frac{1}{2}\cos^2 x-r\cos x+r-\frac{1}{2}\big)\mathrm{d}x\\&=\frac{j}{2(j+2)}I'_{i,j+2}(h)+\frac{1}{2}I'_{i+2,j}(h)-rI'_{i+1,j}(h)+(r-\frac{1}{2})I'_{i,j}(h).\end{aligned} \tag{2.38}$$

Thus, by (2.37) and (2.38), we have that

$$I_{i,j}(h)=\frac{1}{j}\big[(2h-2r+1)I'_{i,j}(h)+2rI'_{i+1,j}(h)-I'_{i+2,j}(h)\big], \tag{2.39}$$

which yields that

$$\begin{cases}I_{0,1}(h)=(2h-2r+1)I'_{0,1}(h)+2rI'_{1,1}(h)-I'_{2,1}(h),\\I_{1,1}(h)=(2h-2r+1)I'_{1,1}(h)+2rI'_{2,1}(h)-I'_{3,1}(h),\\I_{2,1}(h)=(2h-2r+1)I'_{2,1}(h)+2rI'_{3,1}(h)-I'_{4,1}(h),\end{cases}$$

which, together with (2.21), yields (2.34) for $i=1$. Similarly, we can prove (2.34) for $i=2,3$. This completes the proof. □

By using the above lemma, the forms of the first order Melnikov functions $M^+(h)$ and $M^*(h)$ for $r=0$ are simpler, they are given in the following lemma:

**Lemma 2.5** For $r=0$,

$$\begin{aligned}M^+(h)=&P^+_{2[\frac{n}{2}]+2}(\sqrt{h})+P^+_{[\frac{n+1}{2}]}(h)\arctan\frac{1}{\sqrt{2h}}\\&+P^+_{[\frac{n}{2}]}(h)I^+_{0,1}(h)+P^+_{[\frac{n-2}{2}]}(h)I^+_{2,1}(h),\ h\in(0,+\infty),\end{aligned}$$

and

$$M^*(h)=P^*_{[\frac{n+1}{2}]}(h)+P^*_{[\frac{n}{2}]}(h)I^*_{0,1}(h)+P^*_{[\frac{n-2}{2}]}(h)I^*_{2,1}(h),\ h\in\big(-\frac{1}{2},0\big),$$

where $P^+_k(h)$ and $P^*_k(h)$ are polynomials of degree at most $k$.

**Proof** In view of $r = 0$ and the second equation in (2.34) for $i = 1$, one has that $I^+_{1,1}(h) = h\frac{\mathrm{d}I^+_{1,1}(h)}{\mathrm{d}h}$, which yields that $I^+_{1,1}(h) = c_0 h$, where $c_0$ is a constant. Substituting the above equality into (2.30) gives the desired result regarding $M^+(h)$. The other equality can be proven similarly. This ends the proof. □

According to Theorem 1 in [20], we have, for $r > 0$, that

$$\frac{\mathrm{d}^2 I^+_{0,1}(h)}{\mathrm{d}h^2} < 0,\ h \in (2r, +\infty); \quad \frac{\mathrm{d}^2 I^*_{0,1}(h)}{\mathrm{d}h^2} > 0,\ h \in \big(-\frac{1}{2}(1-r)^2, 0\big),$$

and that $\frac{\mathrm{d}^2 I^0_{0,1}(h)}{\mathrm{d}h^2} > 0$ in $h \in (0, 2r)$ for $r \geq 4$ and that $\frac{\mathrm{d}^2 I^0_{0,1}(h)}{\mathrm{d}h^2}$ has exactly one zero in $(0, 2r)$ for $0 < r < 4$, denoted by $h_0$. If $r = 0$, then it is easy to check that $I^+_{0,1}(h) \neq 0$ in $h \in (0, +\infty)$ and that $I^*_{0,1}(h) \neq 0$ in $(-\frac{1}{2}, 0)$. Thus, let

$$\begin{aligned}
\varpi_1(h) &= \frac{\mathrm{d}^2 I^+_{1,1}(h)}{\mathrm{d}h^2}\Big/\frac{\mathrm{d}^2 I^+_{0,1}(h)}{\mathrm{d}h^2},\ h \in (2r, +\infty),\\
\varpi_2(h) &= \frac{\mathrm{d}^2 I^0_{1,1}(h)}{\mathrm{d}h^2}\Big/\frac{\mathrm{d}^2 I^0_{0,1}(h)}{\mathrm{d}h^2},\ h \in (0, h_0) \cup (h_0, 2r),\\
\varpi_3(h) &= \frac{\mathrm{d}^2 I^*_{1,1}(h)}{\mathrm{d}h^2}\Big/\frac{\mathrm{d}^2 I^*_{0,1}(h)}{\mathrm{d}h^2},\ h \in \big(-\frac{1}{2}(1-r)^2, 0\big),\\
\varrho_1(h) &= \frac{I^+_{2,1}(h)}{I^+_{0,1}(h)},\ h \in (0, +\infty),\\
\varrho_2(h) &= \frac{I^*_{2,1}(h)}{I^*_{0,1}(h)},\ h \in \big(-\frac{1}{2}, 0\big).
\end{aligned} \tag{2.40}$$

**Lemma 2.6** (1) If $0 < r < 1$, then $\varpi_i(h)$ satisfies the Riccati equations

$$G(h)\varpi_i'(h) = \kappa_2(h)\varpi_i^2(h) + \kappa_1(h)\varpi_i(h) + \kappa_0(h),\ i = 1, 2, 3, \tag{2.41}$$

where

$$\begin{aligned}
G(h) &= \frac{2}{3}h(h-2r)(2h + r^2 - 2r + 1),\\
\kappa_0(h) &= rh + r^2(r-1),\\
\kappa_1(h) &= \frac{4}{3}(2r^2 - 1)h - \frac{2}{3}r(r-1)(r+2),\\
\kappa_2(h) &= -\frac{2}{3r}h^2 - \frac{1}{3}(5r-4)h - \frac{r}{3}(r-1)(r-4).
\end{aligned}$$

(2) If $r = 0$, then $\varrho_i(h)$ satisfies the Riccati equations

$$G_0(h)\varrho_i'(h) = -\varrho_i^2(h) + \frac{4}{3}(h+1)\varrho_i(h) - \frac{2}{3}h - \frac{1}{3},\ i = 1, 2, \tag{2.42}$$

where $G_0(h) = \frac{2}{3}h(2h+1)$.

**Proof** (1) From (2.34), one has that

$$(Bh + C)V_1''(h) = (E - B)V_1'(h),$$

which gives that

$$\frac{\mathrm{d}^2 I^+_{2,1}(h)}{\mathrm{d}h^2} = -\frac{\mathrm{d}^2 I^+_{0,1}(h)}{\mathrm{d}h^2} - \frac{2}{r}(h-r)\frac{\mathrm{d}^2 I^+_{1,1}(h)}{\mathrm{d}h^2}, \tag{2.43}$$

and

$$G(h)V_1'''(h) = (Bh + C)^*(E - 2B)V_1''(h), \tag{2.44}$$

where $E$ is a $3\times 3$ identity matrix, and $(Bh+C)^*$ is the adjoint matrix of $Bh+C$. Then the statement (2.41) for $i=1$ follows from (2.43) and (2.44). The other two cases can be shown similarly.

(2) Noting that $r=0$ and (2.34), one gets that

$$\begin{aligned}I_{0,1}^{+}(h)&=(2h+1)\frac{\mathrm{d}I_{0,1}^{+}(h)}{\mathrm{d}h}-\frac{\mathrm{d}I_{2,1}^{+}(h)}{\mathrm{d}h},\\ I_{2,1}^{+}(h)&=\frac{1}{3}(2h+1)\frac{\mathrm{d}I_{0,1}^{+}(h)}{\mathrm{d}h}+\frac{1}{3}(2h-1)\frac{\mathrm{d}I_{2,1}^{+}(h)}{\mathrm{d}h}.\end{aligned}$$

Thus, by the above two equations, one can obtain the statement (2.42) for $i=1$. The proof of the case $i=2$ follows by using the same arguments. The proof is complete. $\square$

## 3 Proof of Theorem 1.1

In the sequel we will use the notation $\#\{h\in(\rho_1,\rho_2)\,|\,\phi(h)=0\}$ to indicate the number of zeros of the function $\phi(h)$ in the interval $(\rho_1,\rho_2)$, taking into account their multiplicities.

(1) In what follows, we drop the superscript $+$ in $I^+(h)$, $P_l^+(h)$ and $Q_l^+(h)$ in (2.30), for the sake of lighter notation. Suppose that $0<r<1$, $n\geq 3$ and

$$\Sigma=(2r,+\infty)\backslash\{h\in(2r,+\infty)|P_{[\frac{n+1}{2}]}(h)=0\}.$$

Then, for $h\in\Sigma$, in view of (2.30) and (2.34), one gets that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}h}\Big(\frac{M^{+}(h)}{P_{[\frac{n+1}{2}]}(h)}\Big)=-\frac{(1+r)\sqrt{h}+(1-r)\sqrt{h-2r}}{\sqrt{2h(h-2r)}(2h+r^2-2r+1)}\\ &\quad+\frac{\mathrm{d}}{\mathrm{d}h}\Big[\frac{P_{2[\frac{n}{2}]+2}(\sqrt{h})+\sqrt{h-2r}P_{[\frac{n}{2}]}(h)+P_{[\frac{n}{2}]}(h)I_{0,1}(h)+P_{[\frac{n-1}{2}]}(h)I_{1,1}(h)+P_{[\frac{n-2}{2}]}(h)I_{2,1}(h)}{P_{[\frac{n+1}{2}]}(h)}\Big]\\ &=\frac{1}{h(h-2r)(2h+r^2-2r+1)P^2_{[\frac{n+1}{2}]}(h)}\Big\{P_{2n+6}(\sqrt{h})+\sqrt{h-2r}P_{n+2}(h)\\ &\quad+\frac{1}{G(h)}\Big[P_{n+5}(h)I_{0,1}(h)+P_{2[\frac{n+1}{2}]+4}(h)I_{1,1}(h)+P_{n+4}(h)I_{2,1}(h)\Big]\Big\}\\ &:=\frac{M_1^{+}(h)}{h(h-2r)(2h+r^2-2r+1)P^2_{[\frac{n+1}{2}]}(h)}.\end{aligned}\tag{3.1}$$

Letting $u=\sqrt{h}$, $h\in(2r,+\infty)$, $M_1^+(h)$ in the above equality can be written as

$$\begin{aligned}M_2^{+}(u)&=P_{2n+6}(u)+\sqrt{h-2r}P_{n+2}(h)\\ &\quad+\frac{1}{G(h)}\Big[P_{n+5}(h)I_{0,1}(h)+P_{2[\frac{n+1}{2}]+4}(h)I_{1,1}(h)+P_{n+4}(h)I_{2,1}(h)\Big].\end{aligned}\tag{3.2}$$

Hence, $M_2^+(u)$ and $M_1^+(h)$ have the same number of zeros for $u\in(\sqrt{2r},+\infty)$ and $h\in(2r,+\infty)$. We want to eliminate $P_{2n+6}(u)$ in $M_2^+(u)$. To this end, we differentiate $M_2^+(u)$ in the above expression with respect to $u$ using Lemma 2.4. A simple computation shows that

$$\begin{aligned}\frac{\mathrm{d}M_2^{+}(u)}{\mathrm{d}u}&=P_{2n+5}(u)+u(h-2r)^{-\frac{1}{2}}P_{n+2}(h)\\ &\quad+\frac{u}{G^2(h)}\Big[P_{n+7}(h)I_{0,1}(h)+P_{2[\frac{n+1}{2}]+6}(h)I_{1,1}(h)+P_{n+6}(h)I_{2,1}(h)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2M_2^+(u)}{\mathrm{d}u^2}&=P_{2n+4}(u)+(h-2r)^{-\frac{3}{2}}P_{n+3}(h)\\&\quad+\frac{1}{G^3(h)}\Big[P_{n+10}(h)I_{0,1}(h)+P_{2[\frac{n+1}{2}]+9}(h)I_{1,1}(h)+P_{n+9}(h)I_{2,1}(h)\Big],\end{aligned}$$

In general, we have, for $1\le k\le 2n+7$, that

$$\begin{aligned}\frac{\mathrm{d}^kM_2^+(u)}{\mathrm{d}u^k}&=P_{2n+6-k}(u)+\frac{\big(1+(-1)^{k+1}\big)u+1+(-1)^k}{2}\Big[(h-2r)^{-k+\frac{1}{2}}P_{n+2+[\frac{k}{2}]}(h)\\&\quad+\frac{1}{G^{k+1}(h)}\Big(P_{n+6+5[\frac{k}{2}]+(-1)^{k+1}}(h)I_{0,1}(h)+P_{2[\frac{n+1}{2}]+5+5[\frac{k}{2}]+(-1)^{k+1}}(h)I_{1,1}(h)\\&\quad+P_{n+5+5[\frac{k}{2}]+(-1)^{k+1}}(h)I_{2,1}(h)\Big)\Big],\end{aligned}$$

Let $k=2n+7$ in the above expression. Namely, we differentiate $M_2^+(u)$ in (3.2) $2n+7$ times, and the polynomial $P_{2n+6}(u)$ vanishes. Then we have that

$$\begin{aligned}\frac{\mathrm{d}^{2n+7}M_2^+(u)}{\mathrm{d}u^{2n+7}}&=u\Big[(h-2r)^{-2n-\frac{13}{2}}P_{2n+5}(h)+\frac{1}{G^{2n+8}(h)}\Big(P_{6n+22}(h)I_{0,1}(h)\\&\quad+P_{5n+2[\frac{n+1}{2}]+21}(h)I_{1,1}(h)+P_{6n+21}(h)I_{2,1}(h)\Big)\Big]\\&=u(h-2r)^{-2n-\frac{13}{2}}\Big[P_{2n+5}(h)+\frac{\sqrt{h-2r}}{G^{2n+8}(h)}\Big(P_{8n+28}(h)I_{0,1}(h)\\&\quad+P_{7n+2[\frac{n+1}{2}]+27}(h)I_{1,1}(h)+P_{8n+27}(h)I_{2,1}(h)\Big)\Big]\\&:=\frac{u}{(h-2r)^{2n+\frac{13}{2}}}M_3^+(h).\end{aligned}\tag{3.3}$$

Similarly, we eliminate polynomial $P_{2n+5}(h)$ by differentiating $M_3^+(h)$ in (3.3) $2n+6$ times using Lemma 2.4. Then, we have that

$$\begin{aligned}\frac{\mathrm{d}^{2n+6}M_3^+(h)}{\mathrm{d}h^{2n+6}}&=\frac{1}{(h-2r)^{2n+\frac{11}{2}}G^{4n+14}(h)}\Big[P_{14n+46}(h)I_{0,1}(h)\\&\quad+P_{13n+2[\frac{n+1}{2}]+45}(h)I_{1,1}(h)+P_{14n+45}(h)I_{2,1}(h)\Big]\\&:=\frac{M_4^+(h)}{(h-2r)^{2n+\frac{11}{2}}G^{4n+14}(h)}.\end{aligned}\tag{3.4}$$

Therefore, it suffices to get the upper bound of the number of zeros of $M_4^+(h)$ in $(2r,+\infty)$. To this end, we now express $M_4^+(h)$ in terms of $I''_{0,1}(h)$, $I''_{1,1}(h)$ and $I'_{1,1}(h)$. We begin by working with the Picard-Fuchs equation in (2.34). Differentiating (2.34) for $i=1$ with respect to $h$ gives that

$$(Bh+C)V_1''(h)=(E-B)V_1'(h),\tag{3.5}$$

which implies that

$$\begin{aligned}I'_{0,1}(h)&=I''_{2,1}(h)-(2h-2r+1)I''_{0,1}(h)-2rI''_{1,1}(h),\\I''_{2,1}(h)&=-I''_{0,1}(h)-\frac{2}{r}(h-r)I''_{1,1}(h),\\I'_{2,1}(h)&=rI'_{1,1}(h)-(2h+\frac{r^2}{2}-2r+1)I''_{0,1}(h)+(hr-r^2)I''_{1,1}(h)\\&\quad+(2h+\frac{3r^2}{2}-2r+1)I''_{2,1}(h),\end{aligned}\tag{3.6}$$

where we have used the first equality in the third equality of the above expression. Then, by (2.44) and the second equality of (3.6), one obtains that

$$\begin{aligned} I'''_{0,1}(h) &= \frac{1}{G(h)}\Big[-\Big(2h^2+(r-4)rh-r^2(r-1)\Big)I''_{0,1}(h) \\ &\quad +\Big(\frac{2}{3r}h^2+\frac{1}{3}(5r-4)h+\frac{r}{3}(r-1)(r-4)\Big)I''_{1,1}(h)\Big], \\ I'''_{1,1}(h) &= \frac{1}{G(h)}\Big[\Big(rh+r^2(r-1)\Big)I''_{0,1}(h) \\ &\quad -\Big(2h^2-\frac{1}{3}(5r^2+12r-4)h-\frac{r}{3}(r-1)(r-4)\Big)I''_{1,1}(h)\Big]. \end{aligned} \tag{3.7}$$

Using (2.34), $M_4^+(h)$ becomes

$$M_4^+(h)=P_{14n+47}(h)I'_{0,1}(h)+P_{13n+2[\frac{n+1}{2}]+46}(h)I'_{1,1}(h)+P_{14n+46}(h)I'_{2,1}(h).$$

Differentiating $M_4^+(h)$ with respect to $h$ and using (3.6) yields that

$$\begin{aligned} \frac{\mathrm{d}M_4^+(h)}{\mathrm{d}h} &= P'_{14n+47}(h)I'_{0,1}(h)+P'_{13n+2[\frac{n+1}{2}]+46}(h)I'_{1,1}(h)+P'_{14n+46}(h)I'_{2,1}(h) \\ &\quad +P_{14n+47}(h)I''_{0,1}(h)+P_{13n+2[\frac{n+1}{2}]+46}(h)I''_{1,1}(h)+P_{14n+56}(h)I''_{2,1}(h) \\ &= P_{14n+47}(h)I''_{0,1}(h)+Q_{14n+47}(h)I''_{1,1}(h)+P_{13n+2[\frac{n+1}{2}]+45}(h)I'_{1,1}(h). \end{aligned} \tag{3.8}$$

In view of (3.7), one has that

$$\begin{aligned} \frac{\mathrm{d}^2M_4^+(h)}{\mathrm{d}h^2} &= P'_{14n+47}(h)I''_{0,1}(h)+Q'_{14n+47}(h)I''_{1,1}(h)+P'_{13n+2[\frac{n+1}{2}]+45}(h)I'_{1,1}(h) \\ &\quad +P_{14n+47}(h)I'''_{0,1}(h)+Q_{14n+47}(h)I'''_{1,1}(h)+P_{13n+2[\frac{n+1}{2}]+45}(h)I''_{1,1}(h) \\ &= \frac{1}{G(h)}\Big[P_{14n+49}(h)I''_{0,1}(h)+Q_{14n+49}(h)I''_{1,1}(h)\Big]+P_{13n+2[\frac{n+1}{2}]+44}(h)I'_{1,1}(h). \end{aligned}$$

Thus, differentiating (3.8) $13n+2[\frac{n+1}{2}]+46$ times using (3.7) implies that

$$\begin{aligned} \frac{\mathrm{d}^mM_4^+(h)}{\mathrm{d}h^m} &= \frac{1}{G^{m-1}(h)}\Big[P_{40n+4[\frac{n+1}{2}]+139}(h)I''_{0,1}(h)+Q_{40n+4[\frac{n+1}{2}]+139}(h)I''_{1,1}(h)\Big] \\ &:= \frac{1}{G^{m-1}(h)}M_5^+(h), \end{aligned} \tag{3.9}$$

where $m=13n+2[\frac{n+1}{2}]+47$. We need to study the number of zeros of $M_5^+(h)$ on $(2r,+\infty)$. To see this, let

$$\chi_1(h)=\frac{M_5^+(h)}{I''_{0,1}(h)}(I''_{0,1}(h)\neq 0).$$

Then, by (2.41), one gets that

$$G(h)P_{40n+4[\frac{n+1}{2}]+139}(h)\chi'_1(h)=\kappa_2(h)\chi_1^2(h)+F_1(h)\chi_1(h)+F_0(h), \tag{3.10}$$

where $F_0(h)$ and $F_1(h)$ are polynomials of $h$ with

$$\deg F_0(h)\le 80n+8[\frac{n+1}{2}]+280.$$

By Lemma 4.4 in [39], we have, for $h\in(0,2r)$, that

$$\begin{aligned} \#\{\chi_1(h)=0\} &\le \#\{F_0(h)=0\}+\#\{P_{40n+4[\frac{n+1}{2}]+139}(h)=0\}+1, \\ &\le 120n+12[\frac{n+1}{2}]+420. \end{aligned} \tag{3.11}$$

Suppose that $h_0$ in (2.40) is a zero point of $M^+(h)$. Therefore, applying Rolle's Theorem, and in view of (3.1), (3.3), (3.4) and (3.9), we have that

$$\begin{aligned}\#\{h\in(2r,+\infty)|M^+(h)=0\} &\le \#\{h\in(2r,+\infty)|M_1^+(h)=0\}+[\frac{n+1}{2}]+2\\ &\le \#\{h\in(2r,+\infty)|M_3^+(h)=0\}+2n+[\frac{n+1}{2}]+9\\ &\le \#\{h\in(2r,+\infty)|M_4^+(h)=0\}+4n+[\frac{n+1}{2}]+15\\ &\le \#\{h\in(2r,+\infty)|M_5^+(h)=0\}+17n+3[\frac{n+1}{2}]+62\\ &\le 137n+15[\frac{n+1}{2}]+482. \end{aligned} \tag{3.12}$$

Similarly, one can check that (3.12) is also true for $n=1,2$. If $r=0$, we can prove statement (1) along the lines of the above proof by using Lemma 2.5 and (2.42) in a similar fashion.

The proofs of the statements (2) and (3) are much easier, and follow by using the same arguments, which we omit. This ends the proof of Theorem 1.1.

## 4 Proof of Theorem 1.2

Without loss of generality, we only prove the case $0<r<1$. If, in (1.6), $f^+(x,y)=f^-(x,y)$ and $g^+(x,y)=g^-(x,y)$ such that system (1.6) is smooth, then system (1.6) reads as

$$\begin{cases}\dot{x}=y+\varepsilon f^+(x,y),\\ \dot{y}=\sin x(\cos x-r)+\varepsilon g^+(x,y).\end{cases} \tag{4.1}$$

It is well known that the first order Melnikov functions $M^+(h)$, $M^0(h)$ of system (4.1) have the form

$$\begin{aligned}M^+(h)&=\oint_{L_h^+} g^+(x,y)\mathrm{d}x-f^+(x,y)\mathrm{d}y,\ \ h\in(2r,+\infty),\\ M^0(h)&=\oint_{L_h^0} g^+(x,y)\mathrm{d}x-f^+(x,y)\mathrm{d}y,\ \ h\in(0,2r),\end{aligned}$$

where $L_h^+=L_{h,+}^+\cup L_{h,-}^+$, $L_h^0=L_{h,+}^0\cup L_{h,-}^0$, and $M^*(h)$ is defined as (1.9). Thus we only consider $M^+(h)$ and $M^0(h)$ in this section. Because $y^j\cos^i x\sin x$ is an odd function, we have that

$$\oint_{L_h^+} y^j\cos^i x\sin x\mathrm{d}x=\int_{-\pi}^{\pi} y^j\cos^i x\sin x\mathrm{d}x=0 \tag{4.2}$$

in $(2r,+\infty)$. By using Green's Theorem and the symmetry with respect to the coordinate axes, one has that

$$\begin{aligned}&\oint_{L_h^+} y^j\cos^i x\mathrm{d}y=0,\ \oint_{L_h^0} y^j\cos^i x\sin x\mathrm{d}x=0,\\ &\oint_{L_h^0} y^{2l}\cos^i x\mathrm{d}x=0,\ \oint_{L_h^0} y^j\cos^i x\mathrm{d}y=0.\end{aligned} \tag{4.3}$$

Denote that

$$U_{i,j}^{+}(h)=\oint_{L_h^+} y^j \cos^i x\mathrm{d}x,\ \ h\in(2r,+\infty),$$
$$U_{i,j}^{0}(h)=\oint_{L_h^0} y^j \cos^i x\mathrm{d}x,\ \ h\in(0,2r).$$

Easy computations yield that

$$\begin{aligned}\oint_{L_h^+} y^j\cos^i x\sin x\mathrm{d}y&=\frac{i}{j+1}U_{i-1,j+1}^{+}(h)-\frac{i+1}{j+1}U_{i+1,j+1}^{+}(h),\\ \oint_{L_h^0} y^j\cos^i x\sin x\mathrm{d}y&=\frac{i}{j+1}U_{i-1,j+1}^{0}(h)-\frac{i+1}{j+1}U_{i+1,j+1}^{0}(h).\end{aligned}\tag{4.4}$$

Then, the first order Melnikov functions $M^+(h)$ and $M^0(h)$ of system (4.1) are obtained by an analogous argument using (4.2), (4.3) and (4.4), and are given by

$$\begin{aligned}M^+(h)&=\sum_{i+j=0}^{n}c_{i,j}^{+}U_{i,j}^{+}(h)-\sum_{i+j=0}^{n-1}b_{i,j}^{+}\big[\frac{i}{j+1}U_{i-1,j+1}^{+}(h)-\frac{i+1}{j+1}U_{i+1,j+1}^{+}(h)\big]\\ &=\sum_{i+j=0}^{n+1}\rho_{i,j}^{+}U_{i,j}^{+}(h)\\ &=P^+_{[\frac{n+1}{2}]}(h)+P^+_{[\frac{n}{2}]}(h)U_{0,1}^{+}(h)+P^+_{[\frac{n-1}{2}]}(h)U_{1,1}^{+}(h)+P^+_{[\frac{n-2}{2}]}(h)U_{2,1}^{+}(h),\ \ h\in(2r,+\infty),\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}M^0(h)&=\sum_{i+j=0}^{n}c_{i,j}^{+}U_{i,j}^{0}(h)-\sum_{i+j=0}^{n-1}b_{i,j}^{+}\big[\frac{i}{j+1}U_{i-1,j+1}^{0}(h)-\frac{i+1}{j+1}U_{i+1,j+1}^{0}(h)\big]\\ &=\sum_{i+j=0}^{n+1}\rho_{i,j}^{0}U_{i,j}^{0}(h)\\ &=P^0_{[\frac{n}{2}]}(h)U_{0,1}^{0}(h)+P^0_{[\frac{n-1}{2}]}(h)U_{1,1}^{0}(h)+P^0_{[\frac{n-2}{2}]}(h)U_{2,1}^{0}(h),\ \ h\in(0,2r),\end{aligned}\tag{4.6}$$

where $\rho_{i,j}^{+}$ and $\rho_{i,j}^{0}$ are real constants and $P_k^+(h)$ and $P_k^0(h)$ are polynomials of a degree of at most $k$. Following the same lines as the proof of Theorem 1.1 gives that

$$\begin{aligned}&\#\{h\in(2r,+\infty)|M^+(h)=0\}\le 3n+25[\frac{n+1}{2}]+28,\\ &\#\{h\in(0,2r)|M^0(h)=0\}\le 3n+3[\frac{n+1}{2}]+9.\end{aligned}\tag{4.7}$$

This ends the proof of Theorem 1.2.

## 5 Proof of Theorem 1.3

Let $f_0,f_1,\cdots,f_n$ be analytic functions defined on an open interval $\mathbb{I}$ of $\mathbb{R}$. It is said that $(f_0,f_1,\cdots,f_n)$ is an extended complete Chebyshev system (for shorter, an ECT-system) on $\mathbb{I}$ if, for all $k=0,1,\cdots,n$, any non-trivial linear combination

$$\alpha_0f_0(x)+\alpha_1f_1(x)+\cdots+\alpha_kf_k(x)$$

has at most $k$ isolated zeros on $\mathbb{I}$, counting multiplicity. Here "T" stands for Tchebycheff, which is one of the transcriptions of the Russian name Chebyshev.

According to some results in [14, 24, 32], we can see that, for each $k=0,1,\cdots,n$, there exists a real linear combination $f_0,f_1,\cdots,f_n$ with exactly $k$ simple zeros on $\mathbb{I}$ if $(f_0,f_1,\cdots,f_n)$ is an ECT-system on $\mathbb{I}$. A very useful characterization of ECT-system is given in the following lemma (see [14] for instance).

**Lemma 5.1** Let $f_0,f_1,\cdots,f_n$ be analytic functions defined on an open interval $\mathbb{I}$ of $\mathbb{R}$. Then $(f_0,f_1,\cdots,f_n)$ is an ECT-system on $\mathbb{I}$ if and only if, for each $k=0,1,\cdots,n$, and all $x\in\mathbb{I}$, the Wronskian

$$W[f_0,f_1,\cdots,f_k](x)=\begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_k(x)\\ f_0'(x) & f_1'(x) & \cdots & f_k'(x)\\ \cdots & \cdots & \cdots & \cdots\\ f_0^{(k)}(x) & f_1^{(k)}(x) & \cdots & f_k^{(k)}(x)\end{vmatrix}$$

is different from zero.

The following well-known result of linear algebra will else be a key point in our argument.

**Lemma 5.2** Let $v_0,v_1,\cdots,v_n$ be elements of a vectorial space $S$ endowed with an inner product $\langle\cdot,\cdot\rangle$. Then

$$G(v_0,v_1,\cdots,v_n)=\begin{vmatrix} \langle v_0,v_0\rangle & \langle v_0,v_1\rangle & \cdots & \langle v_0,v_n\rangle\\ \langle v_1,v_0\rangle & \langle v_1,v_1\rangle & \cdots & \langle v_1,v_n\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle v_n,v_0\rangle & \langle v_n,v_1\rangle & \cdots & \langle v_n,v_n\rangle\end{vmatrix}\geq 0,$$

and it is zero if and only if the vectors $v_0,v_1,\cdots,v_n$ are linearly dependent.

The determinant $G$ above is usually called the integral Gram determinant; see [5, 23]. We will use this result for $S$ being the space of continuous functions on a closed interval $[a,b]$, and with the inner product $\langle u,v\rangle=\int_a^b u(t)v(t)\mathrm{d}t$.

Now we introduce the family of analytic functions

$$K_{k,\alpha}(y)=\int_a^b \frac{g^k(x)}{\big(1-yg^{2m}(x)\big)^{\alpha}}\mathrm{d}x, \tag{5.1}$$

where $g$ is a continuous function, $k\in\mathbb{N}$ and $\alpha,a,b\in\mathbb{R}$. These functions are defined on the open interval $\mathbb{J}$, where $1-yg^{2m}(x)>0$ for all $x\in[a,b]$. Easy computations yield that

$$K_{k,\alpha}^{(l)}(y)=\prod_{j=0}^{l-1}(\alpha+j)K_{k+2lm,\alpha+l}(y). \tag{5.2}$$

**Lemma 5.3** The following equality holds:

$$K_{k+2lm,\alpha}(y)=y^{-l}\sum_{i=0}^{l}(-1)^iC_l^iK_{k,\alpha-i}(y). \tag{5.3}$$

**Proof** We will prove the quality (5.3) by induction on $l$. Multiplying by $1-yg^{2m}(x)$ the numerator and the denominator of the integrand of $K_{k,\alpha-1}(y)$ gives that

$$K_{k+2m,\alpha}(y)=y^{-1}\big(K_{k,\alpha}(y)-K_{k,\alpha-1}(y)\big), \tag{5.4}$$

which implies that (5.3) is true for $l=1$. Thus, let us assume that the expression (5.3) holds until $l$. Then, one has that

$$\begin{aligned}K_{k,\alpha-l-1}(y) &= \int_a^b \frac{g^k(x)\big(1-yg^{2m}(x)\big)^{l+1}}{\big(1-yg^{2m}(x)\big)^{\alpha}}\mathrm{d}x \\ &= (-1)^{l+1}y^{l+1}K_{k+2(l+1)m,\alpha}(y)+\sum_{i=0}^{l}(-1)^iC_{l+1}^i y^i K_{k+2im,\alpha}(y) \\ &= (-1)^{l+1}y^{l+1}K_{k+2(l+1)m,\alpha}(y)+\sum_{i=0}^{l}(-1)^iC_{l+1}^i \sum_{j=0}^{i}(-1)^jC_i^j K_{k,\alpha-j}(y),\end{aligned}$$

where we have used the induction hypothesis in the last equality of the above expression. Therefore the assertion is proven for $l+1$. This ends the proof. $\square$

**Lemma 5.4** Let $K_{0,\alpha},K_{1,\alpha},\cdots,K_{2n,\alpha}$ be the functions defined in (5.1). Then, for $y\neq 0$,

$$W_1 := y^{-\frac{n(n+1)}{2}}\Lambda_n(\alpha)\begin{vmatrix} K_{0,\alpha} & K_{2,\alpha} & \cdots & K_{2n,\alpha} \\ K_{0,\alpha+1} & K_{2,\alpha+1} & \cdots & K_{2n,\alpha+1} \\ \cdots & \cdots & \cdots & \cdots \\ K_{0,\alpha+n} & K_{2,\alpha+n} & \cdots & K_{2n,\alpha+n}\end{vmatrix}, \tag{5.5}$$

$$W_2 := y^{-\frac{n(n+1)}{2}}\Lambda_n(\alpha)\begin{vmatrix} K_{1,\alpha} & K_{3,\alpha} & \cdots & K_{2n+1,\alpha} \\ K_{1,\alpha+1} & K_{3,\alpha+1} & \cdots & K_{2n+1,\alpha+1} \\ \cdots & \cdots & \cdots & \cdots \\ K_{1,\alpha+n} & K_{3,\alpha+n} & \cdots & K_{2n+1,\alpha+n}\end{vmatrix}, \tag{5.6}$$

where $\Lambda_n(\alpha)=\prod\limits_{i=0}^{n-1}(\alpha+i)^{n-i}$.

**Proof** We only prove (5.5). The proof for (5.6) follows in the same way. Using the expression for the derivatives provided by (5.2), one gets that

$$W_1 = \Lambda_n(\alpha)\begin{vmatrix} K_{0,\alpha} & K_{2,\alpha} & \cdots & K_{2n,\alpha} \\ K'_{0,\alpha} & K'_{2,\alpha} & \cdots & K'_{2n,\alpha} \\ \cdots & \cdots & \cdots & \cdots \\ K^{(n)}_{0,\alpha} & K^{(n)}_{2,\alpha} & \cdots & K^{(n)}_{2n,\alpha}\end{vmatrix} = \Lambda_n(\alpha)\begin{vmatrix} K_{0,\alpha} & K_{2,\alpha} & \cdots & K_{2n,\alpha} \\ K_{2m,\alpha+1} & K_{2m+2,\alpha+1} & \cdots & K_{2m+2n,\alpha+1} \\ \cdots & \cdots & \cdots & \cdots \\ K_{2nm,\alpha+n} & K_{2nm+2,\alpha+n} & \cdots & K_{2nm+2n,\alpha+n}\end{vmatrix}. \tag{5.7}$$

Then, in view of (5.3) and by the elementary properties of the determinants, we obtain (5.5). This completes the proof. $\square$

**Lemma 5.5** For $m=1$ and $y\neq 0$, $W_1$ and $W_2$ can be written as

$$W_1 = y^{-n(n+1)}\Lambda_n(\alpha)\begin{vmatrix} K_{0,\alpha-n} & K_{0,\alpha-n+1} & K_{0,\alpha-n+2} & \cdots & K_{0,\alpha} \\ K_{0,\alpha-n+1} & K_{0,\alpha-n+2} & K_{0,\alpha-n+3} & \cdots & K_{0,\alpha+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{0,\alpha} & K_{0,\alpha+1} & K_{0,\alpha+2} & \cdots & K_{0,\alpha+n}\end{vmatrix}, \tag{5.8}$$

$$W_2 = y^{-n(n+1)}\Lambda_n(\alpha)\begin{vmatrix} K_{1,\alpha-n} & K_{1,\alpha-n+1} & K_{1,\alpha-n+2} & \cdots & K_{1,\alpha} \\ K_{1,\alpha-n+1} & K_{1,\alpha-n+2} & K_{1,\alpha-n+3} & \cdots & K_{1,\alpha+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{1,\alpha} & K_{1,\alpha+1} & K_{1,\alpha+2} & \cdots & K_{1,\alpha+n} \end{vmatrix}. \tag{5.9}$$

**Proof** It follows from (5.3) that

$$K_{2l,\alpha+i} = y^{-l}\Big(\sum_{j=0}^{l-1}(-1)^j C_l^j K_{k,\alpha+i-j} + (-1)^l K_{k,\alpha+i-l}\Big),\ l=1,2,\cdots,n; i=0,1,\cdots,n.$$

Then, using the above equality and the elementary column transformations of determinants, the determinant in (5.5) can be written as

$$\begin{vmatrix} K_{0,\alpha} & -y^{-1}K_{0,\alpha-1} & y^{-2}K_{0,\alpha-2} & \cdots & (-1)^n y^{-n}K_{0,\alpha-n} \\ K_{0,\alpha+1} & -y^{-1}K_{0,\alpha} & y^{-2}K_{0,\alpha-1} & \cdots & (-1)^n y^{-n}K_{0,\alpha-n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{0,\alpha+n} & -y^{-1}K_{0,\alpha+n-1} & y^{-2}K_{0,\alpha+n-2} & \cdots & (-1)^n y^{-n}K_{0,\alpha} \end{vmatrix}, \tag{5.10}$$

which gives the desired result (5.8). The statement (5.9) follows by using the same arguments, which we omit for the sake of brevity. The proof is complete. □

**Proposition 5.6** Let $W_1$ and $W_2$ be the Wronskian defined in Lemma 5.5. For $n \in \mathbb{N}\backslash\{0\}$ and any $\alpha \in \mathbb{R}\backslash(\mathbb{Z}^- \cup \{0\})$, the ordered sets of functions $(K_{0,\alpha}, K_{2,\alpha}, \cdots, K_{2n,\alpha})$ and $(K_{1,\alpha}, K_{3,\alpha}, \cdots, K_{2n+1,\alpha})$ are ECT-systems on $\Sigma$.

**Proof** Let

$$f_i(x) = \big(1 - yg^2(x)\big)^{\frac{n-\alpha}{2}-i},\ x \in \Sigma,\ i=0,1,\cdots,n.$$

Then, in view of Lemma 5.2, $W_1$ in (5.8) can be written as

$$W_1 = y^{-n(n+1)}\Lambda_n(\alpha)G(f_0, f_1, \cdots, f_n),\ y \neq 0, \tag{5.11}$$

where $G(f_0, f_1, \cdots, f_n)$ is the integral Gram determinant. Notice that $n \in \mathbb{N}\backslash\{0\}$ and $\alpha \in \mathbb{R}\backslash(\mathbb{Z}^- \cup \{0\})$, so one has that $\Lambda_n(\alpha) \neq 0$. Since $g(x)$ is not identically equal to zero, the functions $f_0, f_1, \cdots, f_n$ are linearly independent. Hence, by Lemma 5.2, $W_1$ is different from zero, which, in view of Lemma 5.1, concludes the proof. If $y=0$, then one has that

$$\begin{vmatrix} K_{0,\alpha} & K_{2,\alpha} & \cdots & K_{2n,\alpha} \\ K_{2,\alpha+1} & K_{4,\alpha+1} & \cdots & K_{2+2n,\alpha+1} \\ \cdots & \cdots & \cdots & \cdots \\ K_{2n,\alpha+n} & K_{2n+2,\alpha+n} & \cdots & K_{4n,\alpha+n} \end{vmatrix} = G(1, g, g^2, \cdots, g^n) > 0.$$

It follows from (5.7) that $W_1 \neq 0$ for $y=0$. Thus $W_1$ does not vanish on $\mathbb{J}$. The proof of the case $W_2$ follows by using the same argument. This ends the proof. □

**Proposition 5.7** Let

$$I_{i,l}(h) = \int_0^\pi y^{2l+1}\cos^i x \mathrm{d}x,\ l \in \mathbb{N}.$$

Then, for $n \in \mathbb{N}$ and $p = 2l+1$, the families

$$\big(I_{0,p}(h), I_{2,p}(h), \cdots, I_{2n,p}(h)\big) \text{ and } \big(I_{1,p}(h), I_{3,p}(h), \cdots, I_{2n+1,p}(h)\big)$$

are an ECT-system on $(2r,+\infty)$. Moreover, the same holds for the families

$$\big(I_{0,p}^{(k)}(h), I_{2,p}^{(k)}(h), \cdots, I_{2n,p}^{(k)}(h)\big) \text{ and } \big(I_{1,p}^{(k)}(h), I_{3,p}^{(k)}(h), \cdots, I_{2n+1,p}^{(k)}(h)\big),$$

where $I_{i,p}^{(k)}(h)$ denotes the $k^{th}$-derivative of $I_{i,p}(h)$.

**Proof** Easy computation gives that

$$\begin{aligned} I_{i,p}(h) &= \int_0^\pi \cos^i x(2h+1-\cos^2 x)^{\frac{p}{2}} \\ &= (2h+1)^{\frac{p}{2}} \int_0^\pi \cos^i x\Big(1-\frac{\cos^2 x}{2h+1}\Big)^{\frac{p}{2}} \mathrm{d}x \\ &= (2h+1)^{\frac{p}{2}} K_{i,-\frac{p}{2}}\big(\frac{1}{2h+1}\big), \end{aligned}$$

where $K_{i,-\frac{p}{2}}(y) = \int_0^\pi \cos^i x(1-y\cos^2 x)^{\frac{p}{2}}\mathrm{d}x$. Choosing $g(x)=\cos x$, $m=1$ and $\alpha=-\frac{p}{2}$ in (5.1), it follows from Proposition 5.6 that $\big(K_{0,-\frac{p}{2}}(y), K_{2,-\frac{p}{2}}(y), \cdots, K_{2n,-\frac{p}{2}}(y)\big)$ is an ECT-system on $(0,\frac{1}{4r+1})$. Observe that, for any $k>0$, we have

$$I_{i,p}^{(k)}(h) = p(p-2)\cdots(p-2k+2)I_{i,p-2k}(h).$$

This ends the proof. □

**Proof of Theorem 1.3** It is well known that the first order Melnikov function of system (1.10) is

$$\begin{aligned} M^+(h) &= \sum_{i=0}^n c_i \int_{L_h^+} y^{2l+1}\cos^{2i} x\mathrm{d}x + \sum_{i=0}^n a_i \int_{L_h^+} y^{2i}\cos^i x\mathrm{d}x + \sum_{i=0}^n b_i \int_{L_h^+} y^{2i}\cos^{i+1} x\mathrm{d}x \\ &= \sum_{i=0}^n 2c_i I_{2i,l}(h) + \sum_{i=0}^n \int_{L_h^+} y^{2i}(a_i\cos^i x + b_i\cos^{i+1} x)\mathrm{d}x, \end{aligned}$$

and direct computations give that

$$\sum_{i=0}^n \int_{L_h^+} y^{2i}(a_i\cos^i x + b_i\cos^{i+1} x)\mathrm{d}x = P_n(h)$$

for a certain polynomial $P_n(h)$ of degree $n$. Therefore, it suffices to show that the family

$$\big(1, h, \cdots, h^n, I_{0,l}(h), I_{2,l}(h), \cdots, I_{2n,l}(h)\big)$$

is an ECT-system. To this end, consider that

$$\phi(h) = \sum_{i=0}^n c_i h^i + \sum_{i=0}^k d_i I_{2i,l}(h),\ k\le n,$$

where $c_i$ and $d_i$ are constants. Then

$$\phi^{(n+1)}(h) = \sum_{i=0}^k d_i^1 I_{2i,l-n-1}(h),$$

and Proposition 5.7 implies that $\phi^{(n+1)}(h)$ has at most $k$ zeros counting multiplicity. By Rolle's Theorem, one obtains that $\phi(h)$ has at most $k+n+1$ zeros counting multiplicity. The proof of Theorem 1.3 is complete. □

## 6 Numerical Simulations

In this section, some numerical simulations show that the perturbed whirling pendulum equation with $f^+(x,y) = f^-(x,y)$ and $g^+(x,y) = g^-(x,y)$ in system (1.4) has limit cycles in the oscillatory and rotary regions in one period annulus for the given values of the coefficients of the perturbation polynomials, as well as $|\varepsilon|$ small when $r = 0$.

When $n = 2$ and $r = 0$, in view of (2.1) and (2.5), one can get the first Melnikov function for $x > 0$ and $h \in (-\frac{1}{2}, 0)$:

$$\begin{aligned} M^*(h) =&(c^+_{0,1} - b^+_{1,0})I^*_{0,1}(h) + (c^+_{1,1} + b^+_{0,0})I^*_{1,1}(h) + 2b^+_{1,0}I^*_{2,1}(h) \\ &+ (d^+_{0,1} - a^+_{1,0})J^*_{0,1}(h) - 2a^+_{2,0}J^*_{1,1}(h). \end{aligned} \tag{6.1}$$

By a straightforward calculation, one has that

$$I^*_{1,1}(h) = J^*_{1,1}(h) = 0, \quad J^*_{0,1}(h) = \pi(2h + 1). \tag{6.2}$$

Letting $u = \cos x$ allows us to rewrite the integrals $I^*_{0,1}(h)$ and $I^*_{2,1}(h)$ as

$$I^*_{0,1}(h) = 4\int_{\arccos\sqrt{2h+1}}^{\frac{\pi}{2}} \sqrt{2h+1-\cos^2 x}\,\mathrm{d}x = 4\sqrt{2h+1}E\Big(\sqrt{2h+1}, \frac{1}{\sqrt{2h+1}}\Big) \tag{6.3}$$

and

$$\begin{aligned} I^*_{2,1}(h) &= 4\int_{\arccos\sqrt{2h+1}}^{\frac{\pi}{2}} \cos^2 x\sqrt{2h+1-\cos^2 x}\,\mathrm{d}x \\ &= \frac{4}{3}\sqrt{2h+1}\Big[2hF\Big(\sqrt{2h+1}, \frac{1}{\sqrt{2h+1}}\Big) - (2h-1)E\Big(\sqrt{2h+1}, \frac{1}{\sqrt{2h+1}}\Big)\Big], \end{aligned} \tag{6.4}$$

where $F(z,k)$ and $E(z,k)$ are the incomplete elliptic integrals of the first and second kind

$$\begin{aligned} F(z,k) &= \int_0^z \frac{1}{\sqrt{1-u^2}\sqrt{1-k^2u^2}}\mathrm{d}u, \\ E(z,k) &= \int_0^z \frac{\sqrt{1-k^2u^2}}{\sqrt{1-u^2}}\mathrm{d}u. \end{aligned}$$

Thus, by (6.1)–(6.4), one obtains that

$$M^*(h) = \kappa_1\varphi_1(h) + \kappa_2\varphi_2(h) + \kappa_3\varphi_3(h), \tag{6.5}$$

where

$$\begin{aligned} &\kappa_1 = d^+_{0,1} - a^+_{1,0}, \quad \kappa_2 = c^+_{0,1} - b^+_{1,0}, \quad \kappa_3 = b^+_{1,0} \\ &\varphi_1(h) = (2h+1), \quad \varphi_2(h) = 4\sqrt{2h+1}E\Big(\sqrt{2h+1}, \frac{1}{\sqrt{2h+1}}\Big), \\ &\varphi_3(h) = \frac{8}{3}\sqrt{2h+1}\Big[2hF\Big(\sqrt{2h+1}, \frac{1}{\sqrt{2h+1}}\Big) - (2h-1)E\Big(\sqrt{2h+1}, \frac{1}{\sqrt{2h+1}}\Big)\Big]. \end{aligned}$$

Moreover, a simple computation shows that

$$\det\frac{\partial(\kappa_1, \kappa_2, \kappa_3)}{\partial(a^+_{1,0}, c^+_{0,1}, b^+_{1,0})} = -1,$$

which implies that $\kappa_i$, $1 = 1, 2, 3$ can be chosen arbitrarily. We want to determine the existence of the zeros of $M^0(h)$ in (6.5). To this end, we first obtain the asymptotic expansions of functions $\varphi_i(h)$, $i=1,2,3$ from $(-\frac{1}{2},0)$ to $\mathbb{R}$ given in (6.5) in the variable $h$, around $h=-\frac{1}{2}$, as follows:

$$\begin{aligned}
\varphi_1(h) &= 2(h+\frac{1}{2}),\\
\varphi_2(h) &= 2\pi(h+\frac{1}{2})+\frac{\pi}{2}(h+\frac{1}{2})^2+\frac{3\pi}{8}(h+\frac{1}{2})^3+o\Big((h+\frac{1}{2})^4\Big),\\
\varphi_3(h) &= 2\pi(h+\frac{1}{2})^2+\pi(h+\frac{1}{2})^3+o\Big((h+\frac{1}{2})^4\Big).
\end{aligned}$$

Now, applying the above expressions, one can obtain the asymptotic expansion of the following Melnikov function $M^*(h)$ at $h=-\frac{1}{2}$:

$$M^*(h)=\zeta_1(h+\frac{1}{2})+\zeta_2(h+\frac{1}{2})^2+\zeta_3(h+\frac{1}{2})^3+o\Big((h+\frac{1}{2})^4\Big).$$

Here

$$\zeta_1=2\pi(\kappa_1+\kappa_2),\quad \zeta_2=\frac{\pi}{2}\kappa_2+2\pi\kappa_3,\quad \zeta_3=\frac{3\pi}{8}\kappa_1+\pi\kappa_3.$$

Furthermore, one has that

$$\det\frac{\partial(\zeta_1,\zeta_2,\zeta_3)}{\partial(\kappa_1,\kappa_2,\kappa_3)}=\frac{5}{2}\pi^3,$$

which implies that $\zeta_1$, $\zeta_2$ and $\zeta_3$ can be taken as free parameters. Hence, we can choose appropriate $\zeta_1$, $\zeta_2$ and $\zeta_3$ such that $M^*(h)$ can have two simple zeroes near $h=-\frac{1}{2}$.

According to the above discussion, we consider the following perturbed equation:

$$\begin{cases}\dot{x}=y+0.1(\cos x-0.2539385136\cos x\sin x),\\ \dot{y}=\sin x\cos x+0.1\times 0.7460993676y.\end{cases}\tag{6.6}$$

The first Melnikov function $M^*(h)$ of system (6.6) has two zeroes, $-0.4$ and $-0.45$, in $(-\frac{1}{2},0)$ near $-\frac{1}{2}$, which implies that system (6.6) can have two limit cycles for $x>0$; see Figure 5.

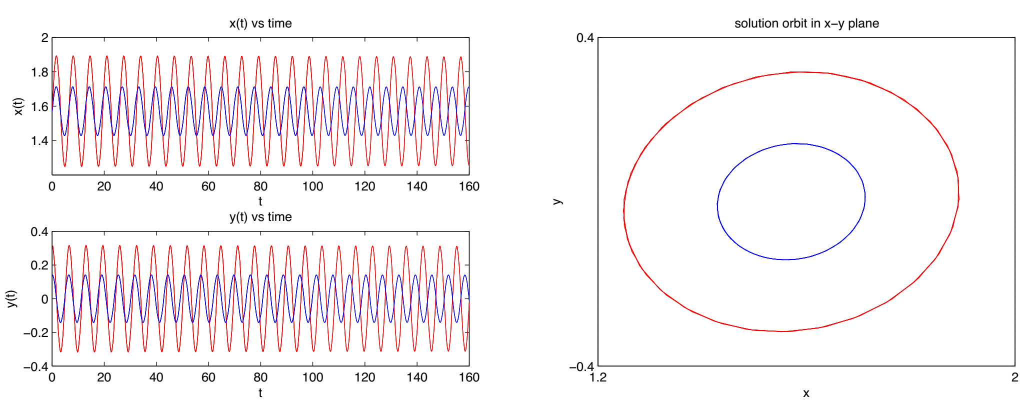


Figure 5 Two limit cycles of system (6.6)

In a similar way, for $x<0$ and $h\in(-\frac{1}{2},0)$, we consider the following perturbed whirling pendulum equation:

$$\begin{cases}\dot{x}=y+0.1(\cos x+0.2539385136\cos x\sin x),\\ \dot{y}=\sin x\cos x-0.1\times 0.7460993676y.\end{cases}\tag{6.7}$$

The first Melnikov function $M^*(h)$ of system (6.7) has two zeroes, $-0.4$ and $-0.45$, in $(-\frac{1}{2}, 0)$ near $-\frac{1}{2}$, which implies that system (6.7) can have two limit cycles for $x < 0$; see Figure 6.

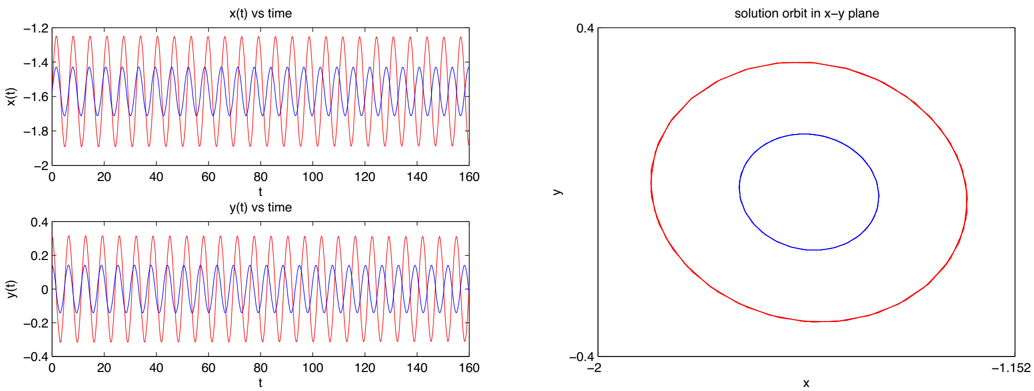


Figure 6 Two limit cycles of system (6.7)

Next we will consider the case $h \in (0, +\infty)$. When $n = 2$ and $y > 0$, by (2.2) and (2.3), one has that

$$M^+(h) = 2\pi c^+_{0,0} + \pi c^+_{0,2} + \pi c^+_{2,0} + 4\pi c^+_{0,2} h + 4(c^+_{0,1} - b^+_{1,0})\sqrt{2h+1}E\Big(\frac{1}{\sqrt{2h+1}}\Big)$$
$$+\frac{8}{3}b^+_{1,0}\sqrt{2h+1}\Big[2hK\Big(\frac{1}{\sqrt{2h+1}}\Big) - (2h-1)E\Big(\frac{1}{\sqrt{2h+1}}\Big)\Big], \tag{6.8}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind

$$K(k) = \int_0^1 \frac{1}{\sqrt{1-u^2}\sqrt{1-k^2u^2}}\mathrm{d}u, \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2u^2}}{\sqrt{1-u^2}}\mathrm{d}u.$$

In a similar way, one can choose the appropriate coefficients of $f^+(x, y)$ and $g^+(x, y)$ such that $M^+(h)$ in (6.8) has two zeroes in $(0, +\infty)$ near 0. Consider the following equation:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \sin x \cos x + 0.001(y - 0.8947516328y^2 - 1.286289275\cos^2 x). \end{cases} \tag{6.9}$$

It is easy to check that the first Melnikov function $M^+(h)$ has two zeros, $h = 0.01$ and $h = 0.05$, which implies that system (6.9) can have two limit cycles in the rotary region; see Figure 7.

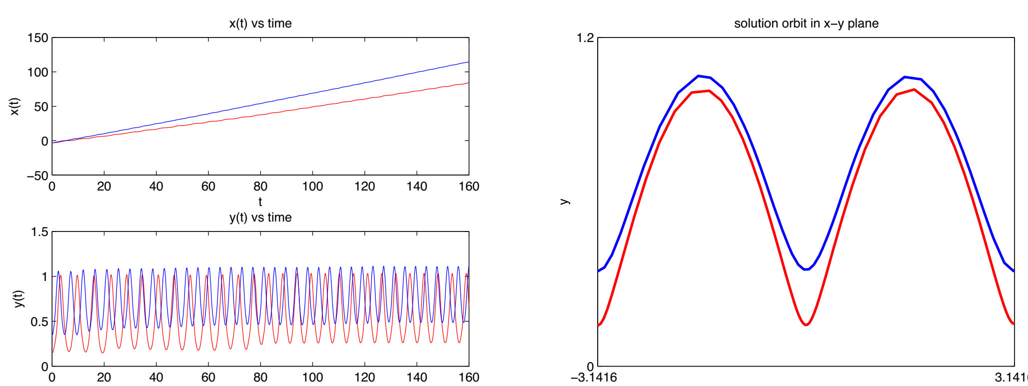


Figure 7 Two limit cycles of system (6.9)

In order to see this result more intuitively, we can roll up the phase plane between the two straight lines of $x = \pi$ and $x = -\pi$, and glue the two straight lines of $x = \pm\pi$ together, so as to form a cylindrical surface, which is called a phase cylinder. All the motions of the pendulum can be represented by the phase trajectories on the phase cylinder. The phase trajectories corresponding to the rotary region are the trajectories around the phase cylinder, as shown in Figure 8.

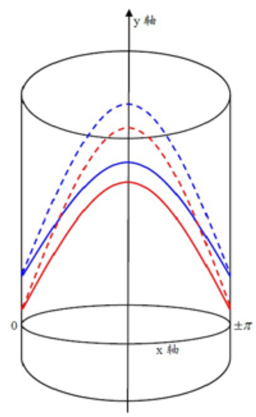


Figure 8 Two limit cycles in the phase cylinder of system (6.9)

Similarly, for $y < 0$ and $h \in (0, +\infty)$, we consider the following perturbed whirling pendulum equation:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \sin x \cos x + 0.001(y + 0.8947516328y^2 + 1.286289275 \cos^2 x). \end{cases} \tag{6.10}$$

The first Melnikov function $M^+(h)$ of system (6.10) has two zeroes, 0.01 and 0.05 in $h \in (0, +\infty)$, near 0, which implies that system (6.10) can have two limit cycles for $y < 0$; see Figure 9 and Figure 10.

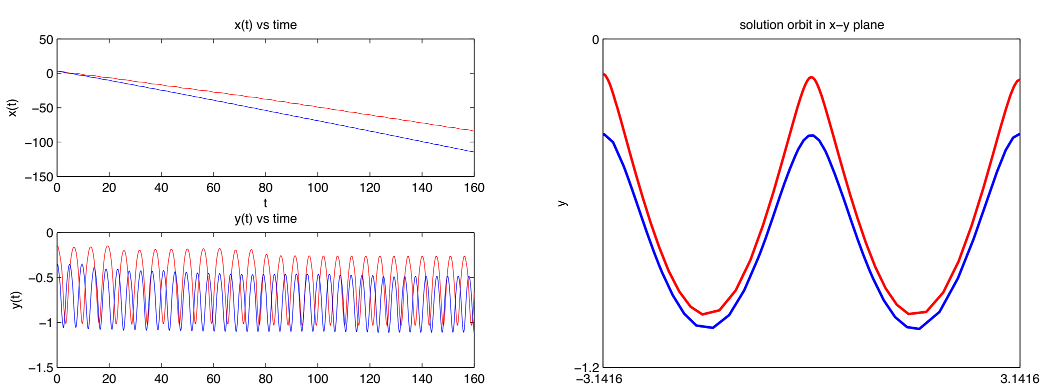


Figure 9 Two limit cycles of system (6.10)

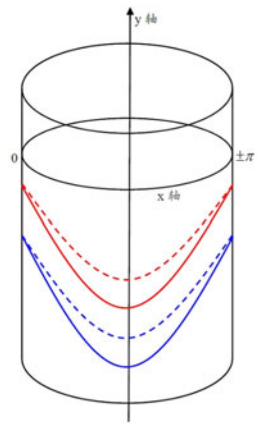


Figure 10 Two limit cycles in the phase cylinder of system (6.10)

## 7 Conclusion and Discussion

In this paper, we have given the upper bounds for the number of limit cycles bifurcating from the oscillatory and rotary regions of the whirling pendulum equation under piecewise polynomial perturbations. For some particular smooth perturbations in the rotary region, we obtained the optimal bounds. The main tools used are the Picard-Fuchs equations, the Chebyshev criterion, the Gram determinant and combination techniques. However, there still exist some open problems on the optimal bounds for the general case, since the Melnikov functions contain the complete and incomplete elliptic integrals and anti-trigonometric functions. It is difficult to solve these problems by applying recent by discovered methods and techniques, even if $n$ is small. We need improve the algebraic criterion and develop more efficient approaches.

**Conflict of Interest** The author declares no conflict of interest.

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