Acta Mathematica Scientia, 2024, **44B**(3): 925–946
https://doi.org/10.1007/s10473-024-0309-6

Mathematica Scientia 数学物理学报
http://actams.apm.ac.cn

THE ASYMPTOTIC BEHAVIOR AND OSCILLATION FOR A CLASS OF THIRD-ORDER NONLINEAR DELAY DYNAMIC EQUATIONS

Xianyong HUANG (黄先勇)
Department of Mathematics, Guangdong University of Education, Guangzhou 510303, China
E-mail: *huangxianyong@gdei.edu.cn*

Xunhuan DENG (邓勋环)*
Department of Mathematics, College of Medical Information Engineering,
Guangdong Pharmaceutical University, Guangzhou 510006, China
E-mail: *deng_xunhuan@163.com*

Qiru WANG (王其如)
School of Mathematics, Sun Yat-sen University, Guangzhou 510275, China
E-mail: *mcswqr@mail.sysu.edu.cn*

Abstract In this paper, we consider a class of third-order nonlinear delay dynamic equations. First, we establish a Kiguradze-type lemma and some useful estimates. Second, we give a sufficient and necessary condition for the existence of eventually positive solutions having upper bounds and tending to zero. Third, we obtain new oscillation criteria by employing the Pötzsche chain rule. Then, using the generalized Riccati transformation technique and averaging method, we establish the Philos-type oscillation criteria. Surprisingly, the integral value of the Philos-type oscillation criteria, which guarantees that all unbounded solutions oscillate, is greater than $\theta_4(t_1, T)$. The results of Theorem 3.5 and Remark 3.6 are novel. Finally, we offer four examples to illustrate our results.

Key words nonlinear delay dynamic equations; nonoscillation; asymptotic behavior; Philos-type oscillation criteria; generalized Riccati transformation

2020 MR Subject Classification 34C10; 34D05; 34K11; 34K42; 34N05; 39A21

1 Introduction

In the paper, we investigate the asymptotic behavior and oscillatory properties of the third-order nonlinear delay dynamic equations of the form

$$\left[b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\Delta}+p(t)|x(\tau(t))|^{\gamma-1}x(\tau(t))=0, \quad t\in\mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale and $\gamma>0$ is a constant. We put forward the following hypotheses:

Received July 28, 2022; revised November 26, 2023. This work was supported by the National Natural Science Foundation of China (12071491, 12001113).

*Corresponding author

(A1) the functions $p \in C_{\rm rd}(\mathbb{T}, \mathbb{R}^+)$, $a \in C_{\rm rd}(\mathbb{T}, \mathbb{R}^+)$ and $b \in C_{\rm rd}(\mathbb{T}, \mathbb{R}^+)$ satisfy that

$$\int_{t_0}^{\infty} \frac{1}{b(s)} \Delta s = \infty, \quad \int_{t_0}^{\infty} \frac{1}{a(u)} \int_{t_0}^{u} \frac{1}{b(s)} \Delta s \Delta u = \infty; \tag{1.2}$$

(A2) the function $\tau \in C_{\rm rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \le t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition 1.1 ([6, p1]) On every time scale we define the forward jump operator by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is a right-scattered point, while if $\rho(t) < t$, we say that t is a left-scattered point. If $\sigma(t) = t$ and $t \ne \sup \mathbb{T}$, we say that t is a right-dense point, while if $\rho(t) = t$ and $t \ne \inf \mathbb{T}$, we say that t is a left-dense point. For more information on this see the monograph [6].

Definition 1.2 ([6, p22]) A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided that it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted as

$$C_{\rm rd} = C_{\rm rd}(\mathbb{T}) = C_{\rm rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{\rm rd} = C^1_{\rm rd}(\mathbb{T}) = C^1_{\rm rd}(\mathbb{T}, \mathbb{R}).$$

Similarly, we can define the set $C^i_{\rm rd}$ $(i = 2, 3, 4, \cdots)$.

Definition 1.3 By a solution x of equation (1.1), we mean a nontrivial real-valued function in $C_{\rm rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ with $ax^{\Delta} \in C^1_{\rm rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $b(ax^{\Delta})^{\Delta} \in C^1_{\rm rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and which satisfies equation (1.1) on interval $[t_0, \infty)_{\mathbb{T}}$.

As usual, a solution of equation (1.1) is called nonoscillatory if it eventually has one sign; otherwise it is said to be oscillatory. Equation (1.1) is oscillatory if all of its solutions are oscillatory.

Oscillations of delay dynamic equations are common in applications, including continuum mechanics, population dynamic behavior, and biology models. For more details, see the monographs [3, 4] and references [14, 34]. For the existence and uniqueness of solutions to delay dynamic equations on time scales, we refer to [7–9, 17, 24, 26, 28, 33].

In recent years, great attention has been paid to the oscillation of third-order dynamic equations. We refer readers to [1, 2, 5, 13, 15, 16, 18–23, 25, 27, 29–31] and the references therein. In 2005, Erbe, *et al* [15] considered the oscillation of a third-order nonlinear dynamic equation

$$\left[c(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\Delta} + p(t)f\left(x(t)\right) = 0, \quad t \in \mathbb{T}, \tag{1.3}$$

where

$$\int_{t_0}^{\infty} \frac{1}{c(t)} \Delta t = \int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty.$$

Using the Riccati transformation techniques, they established some sufficient conditions which ensure that every solution of equation (1.3) is oscillatory or converges to zero.

In 2007, Erbe, *et al* [16] studied the third-order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0,$$

with the assumptions that $x(t) > 0,\ x^{\Delta}(t) > 0,\ x^{\Delta\Delta}(t) > 0,\ x^{\Delta\Delta\Delta}(t) < 0$. They obtained an important estimate, $\liminf_{t\to\infty} \frac{tx(t)}{h_2(t,s)x^{\Delta}(t)} \geq 1$, where the generalized polynomials $\{h_n(t,s)\}_0^{\infty}$ ([6, p38]) on time scales are defined recursively by

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, \qquad s,t \in \mathbb{T}.$$

With the help of this estimate, they obtained Hille and Nehari type oscillation criteria. For when $\mathbb{T} = \mathbb{R}$, $\liminf_{t\to\infty} \frac{tx(t)}{h_2(t,s)x'(t)} \geq 1$ which shows that $\frac{2x(t)}{tx'(t)} \geq 1$ is equivalent to [20, Lemma 3.2 (iv)].

In 2011, Han, *et al* [21] addressed the general form of equation (1.1) as follows:

$$\left[b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\Delta} + p(t)f\left(x(\tau(t))\right) = 0, \quad t \in \mathbb{T}. \tag{1.4}$$

They obtained the oscillation criteria of equation (1.4), which extended and improved the results of [15, 16]. However, [21] only considered the case $f(u)/u > M > 0$ for $u \neq 0$, where M is a constant. [15, 16, 21] did not solve the general case $f(u)/|u|^{\gamma-1}u > M > 0$ for $u \neq 0$. When $f(u) = |u|^{\gamma-1}u$, (1.4) reduces to (1.1).

In 2017, Hassan, *et al* [22] examined the third-order nonlinear functional dynamic equation

$$\left\{r_2(t)\phi_{\alpha_2}(\left[r_1(t)\phi_{\alpha_1}(x^{\Delta}(t))\right]^{\Delta})\right\}^{\Delta} + q(t)\phi_{\alpha}(x(g(t))) = 0, \quad t \in \mathbb{T}, \tag{1.5}$$

where

$$\phi_{\alpha}(u) = |u|^{\alpha-1}u,\ \alpha_1,\ \alpha_2,\ \alpha := \alpha_1\alpha_2 > 0,\ \phi_{\alpha_i}(u) = |u|^{\alpha_i-1}u, \quad i = 1,2.$$

Under the conditions

$$\int_{t_0}^{\infty} r_i^{-\frac{1}{\alpha_i}}(t)\Delta t = \infty, \quad i = 1,2, \tag{1.6}$$

they obtained some new oscillatory criteria of equation (1.5) and got that every solution of equation (1.5) oscillates or converges to zero, which extended and improved the results of [15, 16, 21]. However, $\alpha = \alpha_1\alpha_2$ is a special condition, and the conditions (1.6) are stronger than the conditions (1.2). When $\alpha_1 = \alpha_2 = \alpha = 1$, (1.5) reduces to linear equations.

In 2022, Deng, *et al* [13] considered the third-order nonlinear delay differential equation

$$\left[b(t)\left(a(t)x'(t)\right)'\right]' + p_1(t)\phi_{\gamma}(x(\tau_1(t))) - p_2(t)\phi_{\gamma}(x(\tau_2(t))) + p_3(t)\phi_{\gamma}(x(\tau_3(t))) = 0. \tag{1.7}$$

They derived several sufficient conditions which ensured that every solution of equation (1.7) is either oscillatory or tends to zero as $t \to \infty$, taking on the assumptions of (1.2). However, for when $p_2(t) = p_3(t) = 0$, [13] does not give the oscillation criteria for $0 < \gamma < 1$ with respect to unbounded solutions.

It is well known that the Philos-type oscillation criteria are useful for determining the oscillatory properties of the corresponding equations. Deng, *et al* [10, 11] considered second-order nonlinear dynamic equation

$$[P(t)|z^{\Delta}(t)|^{\gamma-1}z^{\Delta}(t)]^{\Delta} + f(t, x(\tau_1(t))) = 0, \quad t \in \mathbb{T},$$

where $z(t) = x(t) + r(t)x(\tau(t))$. They established the Philos-type oscillation condition

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{H(\sigma(t),\sigma(s))\varphi(s)}{H(\sigma(t),t_1)} - \overline{C}(t,s)\right]\Delta s > \varphi_3(t_1), \tag{1.8}$$

where

$$\overline{C}(t,s) = \frac{\gamma^\gamma([H(\sigma(t),\sigma(s))\varphi_1(s) + H^{\Delta_s}(\sigma(t),s)]_+)^{1+\gamma}}{(1+\gamma)^{1+\gamma}H(\sigma(t),t_1)[H(\sigma(t),\sigma(s))\varphi_2(s)]^\gamma},$$

$$\varphi_3(t_1) = \delta(t_1)\left[\frac{1}{R^\gamma(t_1,T)} + r(t_1)\alpha(t_1)\right].$$

Zhang and Feng [32] considered the equation

$$\left[b(t)\left(\left(a(t)z^\Delta(t)\right)^\Delta\right)^{\gamma_1}\right]^\Delta + f(t,x(\tau(t))) = 0, \tag{1.9}$$

where $z(t) = x(t) + r(t)x^{\gamma_2}(\tau_1(t)), 0 < r(t) \le r < 1$, $f(t,u)/u^{\gamma_3} \ge p(t) > 0$ for $u \ne 0$, and $\gamma_1,\gamma_2,\gamma_3$ are quotients of odd positive integers, $\gamma_3 \ge \gamma_1, 0 < \gamma_2$, $b^\Delta(t) \ge 0$, and

$$\int_{t_0}^\infty \frac{1}{a(s)}\Delta s = \infty, \qquad \int_{t_0}^\infty b^{-\frac{1}{\gamma_2}}(s)\Delta s = \infty. \tag{1.10}$$

When $r(t) = 0, \gamma_1 = 1$, and $\gamma_3 = \gamma$, (1.9) becomes (1.1). Let $D = \{(t,s)|t \ge s \ge t_0\}$ and $D_0 = \{(t,s)|t > s \ge t_0\}$. In order to give the Philos-type oscillation theorem, Zhang and Feng introduced a function $\overline{H}(t,s) \in C(D,\mathbb{R})$ satisfying the following conditions:

(H1) $\overline{H}(t,t) = 0$, $\overline{H}(t,s) > 0$ for $(t,s) \in D_0$;

(H2) $\overline{H}^{\Delta_s}(t,t) = 0$, $\overline{H}^{\Delta_s}(t,s) \le 0$ for $(t,s) \in D_0$.

Zhang and Feng established the following Philos-type oscillation theorem:

Theorem 1.4 Assume that (1.10) holds. Furthermore, suppose that there exist two positive functions $\eta(t) \in C^1_{\rm rd}(\mathbb{T},\mathbb{R})$ and $h(t,s) \in C_{\rm rd}(\mathbb{D},\mathbb{R})$ satisfying conditions (H1), (H2), and

$$H^{\Delta_s}(\sigma(t),\sigma(s)) + H(\sigma(t),\sigma(s))\frac{(\eta^\Delta(t))_+}{\eta(t)} = -\frac{(h(t,s)}{\eta(t)}H^{\frac{\gamma_1}{\gamma_1+1}}(\sigma(t),\sigma(s)).$$

If, for sufficiently large $t_1 \ge t_0 > 0$,

$$\limsup_{t\to\infty}\frac{1}{\overline{H}(\sigma(t),\sigma(t_1))}\int_{t_1}^t\left[\overline{H}(\sigma(t),\sigma(s))K(s) - \frac{(h_-(t,s)b(\sigma(s)))^{\gamma_1+1}}{(\gamma_1+1)^{\gamma_1+1}(\eta(s)b(s))^{\gamma_1}}\right]\Delta s = \infty \tag{1.11}$$

holds, where $K(t)$ is defined by [32, Theorem 3.1], then every solution x of (1.9) is either oscillatory or $\lim\limits_{t\to\infty} x(t) = 0$.

Almost all the papers mentioned above established Philos-type oscillation criteria which requiring the integral value (the forms are similar to the left hand side of (1.8) and the left hand side of (1.11)) to be ∞.

Based on the above literature review, it is clear that there are several problems to be solved.

(i) Can we obtain a better estimate than $\liminf\limits_{t\to\infty}\frac{tx(t)}{h_2(t,s)x^\Delta(t)} \ge 1$ in [16] and $\frac{2x(t)}{tx'(t)} \ge 1$ in [20, Lemma 3.2 (iv)]? Since the polynomail function $h_2(t,s)$ is dependent on the time scales, it has no uniform expression. Can we obtain a uniform estimate for $x(t)/x^\Delta(t)$?

(ii) Can we derive oscillation criteria for $0 < \gamma < 1$ ([13] did not solve this problem even if $p_2(t) = p_3(t) = 0$ in (1.7)) with respect to unbounded solutions?

(iii) Can we establish a similar Philos-type oscillation criterion that requires the integral value to be greater than some function value $\varphi(t_1)$, as Deng, *et al* established in [10, 11]? Moreover, can we remove the conditions (H2) and $\overline{H}(t,t) = 0$ in (H1)?

On one hand, Deng, *et al* [12] considered the second-order neutral dynamic equation

$$[P(t)z^{\Delta}(t)]^{\Delta}+h(t,x(\sigma(t)),x(\tau_1(t)),x(\tau_2(t)),x(\xi_1(t)),x(\xi_2(t)))=0,\quad t\in\mathbb{T},$$

where $z(t)=x(t)+r(t)x(\tau(t))$, and gave the estimate $z(t)>tz^{\Delta}(t)$. On the other hand, Deng, *et al* [12] also provided a method for establishing the oscillation criteria for $0<\gamma<1$. In view of [13, 15, 16, 21], inspired by [12, Lemma 2.2], in Section 2 we present some useful lemmas, including $x(t)>tx^{\Delta}(t)$, which show problem (i) being solved perfectly. In Section 3, we first give sufficient and necessary conditions for the existence of a solution and we show eventually positive solutions having upper bounds and tending to zero. Then, we investigate the oscillation and asymptotic behavior of equation (1.1) and solve problems (ii) and (iii), inspired by [12, Theorem 3.7] and [10, 11], respectively. Our work generalizes and improves upon the main results of [13, 15, 16, 21, 25, 27, 30, 31] and related results that can be found in the literature. In Section 4, we present four examples to illustrate the validity of our results.

2 Preliminaries

In this section, we will give several useful lemmas. Similarly to [13, Lemmas 2.1–2.3] and [16, Lemmas 2.1–2.3], it is easy to obtain Lemmas 2.1–2.3.

Lemma 2.1 Assume that x is an eventually positive solution of equation (1.1). Then, there exists a sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$ such that, for $t\in[T,\infty)_{\mathbb{T}}$, either

(i) $x\in S_0:=\{x\mid x>0,\ Y(t)=a(t)x^{\Delta}(t)<0,\ b(t)Y^{\Delta}(t)>0,\ [b(t)Y^{\Delta}(t)]^{\Delta}<0\}$,

or

(ii) $x\in S_1:=\{x\mid x>0,\ Y(t)=a(t)x^{\Delta}(t)>0,\ b(t)Y^{\Delta}(t)>0,\ [b(t)Y^{\Delta}(t)]^{\Delta}<0\}$.

Lemma 2.2 Suppose that

$$\int_{t_0}^{\infty}p(t)\Delta t=\infty,\tag{2.1}$$

or

$$\int_{t_0}^{\infty}p(t)\Delta t<\infty,\quad \int_{t_0}^{\infty}\frac{1}{a(v)}\int_{v}^{\infty}\frac{1}{b(s)}\int_{s}^{\infty}p(u)\Delta u\Delta s\Delta v=\infty.\tag{2.2}$$

Then every eventually positive solution $x\in S_0$ of equation (1.1) satisfies that $\lim\limits_{t\to\infty}x(t)=0$.

Lemma 2.3 Assume that $x\in S_1$ is a solution of equation (1.1). Then the estimate

$$a(t)x^{\Delta}(t)\geq B(t,T)b(t)[a(t)x^{\Delta}(t)]^{\Delta}\tag{2.3}$$

holds for large T and $t\geq T$, where

$$B(t,T)=\int_{T}^{t}\frac{1}{b(s)}\Delta s.\tag{2.4}$$

Inspired by the idea [12, Lemma 2.2], we have the following results:

Lemma 2.4 Assume that $x\in S_1$ is an eventually positive solution of equation (1.1). Moreover, suppose that $a^{\Delta}(t)\leq 0$ and

$$\int_{t_0}^{\infty}p(s)\tau^{\gamma}(s)\Delta s=\infty.\tag{2.5}$$

Then there exists a large $T\in[t_0,\infty)_{\mathbb{T}}$ such that the following estimates hold:

(1) $x(t)>tx^{\Delta}(t)$ for $t\in[T,\infty)_{\mathbb{T}}$;

(2) $\frac{x}{t}$ is decreasing, and $\frac{x(\tau(t))}{x(t)}>\frac{\tau(t)}{t}$ for $t\in[T,\infty)_{\mathbb{T}}$.

Proof Let $x \in S_1$ be an eventually positive solution of equation (1.1). Then, for large T and $t \geq T$, by Lemma 2.1 (ii), (A1) and $a^{\Delta}(t) \leq 0$, it is apparent that

$$(a(t)x^{\Delta}(t))^{\Delta} = a^{\Delta}(t)x^{\Delta}(t) + a^{\sigma}(t)x^{\Delta\Delta}(t) > 0$$

implies that $x^{\Delta\Delta}(t) > 0$ for $t \geq T$.

Letting $A(t) := x(t) - tx^{\Delta}(t)$, then $A^{\Delta}(t) = x^{\Delta}(t) - x^{\Delta}(t) - \sigma(t)x^{\Delta\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) < 0$, which shows that $A(t) > 0$ or $A(t) < 0$ for $t \geq T$.

We claim that $A(t) > 0$ on interval $[T, \infty)_{\mathbb{T}}$. If we assume not, then $A(t) < 0$ on $[T, \infty)_{\mathbb{T}}$, and

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{A(t)}{t\sigma(t)} > 0$$

implies that $x(t)/t$ is increasing on $[T, \infty)_{\mathbb{T}}$. Taking $t_1 \in [T, \infty)_{\mathbb{T}}$ such that $\tau(t) \geq T$ for $t \geq t_1$, then we have that

$$\frac{x(\tau(t))}{\tau(t)} \geq \frac{x(\tau(t_1))}{\tau(t_1)} =: d_1 > 0.$$

Note that $b(t)(a(t)x^{\Delta}(t))^{\Delta} > 0$. Integrating equation (1.1) from t_1 to t yields that

$$b(t_1)(a(t_1)x^{\Delta}(t_1))^{\Delta} = b(t)(a(t)x^{\Delta}(t))^{\Delta} + \int_{t_1}^{t} p(s)x^{\gamma}(\tau(s))\Delta s > d_1^{\gamma}\int_{t_1}^{t} p(s)\tau^{\gamma}(s)\Delta s.$$

Letting $t \to \infty$,

$$\frac{1}{d_1^{\gamma}} b(t_1)(a(t_1)x^{\Delta}(t_1))^{\Delta} \geq \int_{t_1}^{\infty} p(s)\tau^{\gamma}(s)\Delta s, \tag{2.6}$$

which contradicts (2.5).

Hence, $A(t) > 0$ and $x(t) > tx^{\Delta}(t)$ for $t \in [T, \infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{A(t)}{t\sigma(t)} < 0, \quad t \geq T.$$

Thus $x(t)/t$ is decreasing on $[T, \infty)_{\mathbb{T}}$, and $\frac{x(\tau(t))}{x(t)} > \frac{\tau(t)}{t}$. □

3 Main Results

In this section, we explore the asymptotic behavior and oscillatory properties of equation (1.1). We are now in a position to derive a sufficient and necessary condition for the existence of eventually positive solutions which have upper bounds and tend to zero.

Theorem 3.1 Suppose that (A1) and (A2) hold. Furthermore, assume that there exists an increasing positive function g with $\lim_{t\to\infty} g(t) = \infty$. Then,

(i) if there exists a sufficiently large T such that, for all $t \geq T$,

$$\int_{t}^{\infty} \frac{1}{a(v)} \int_{v}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} p(u)\frac{1}{g^{\gamma}(\tau(u))}\Delta u\Delta s\Delta v \leq \frac{1}{g(t)}, \tag{3.1}$$

equation (1.1) has an eventually positive solution $x(t) \in \left(0, \frac{1}{g(t)}\right]$ for $t \geq T$ and $\lim_{t\to\infty} x(t) = 0$;

(ii) moreover, if

$$\int_{t_0}^{\infty} \frac{1}{a(v)} \int_{v}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} p(u)\frac{1}{g^{\gamma}(\tau(u))}\Delta u\Delta s\Delta v < \infty \tag{3.2}$$

and

$$\lim_{t\to\infty} \frac{g(t)g^{\sigma}(t)\int_t^{\infty}\frac{1}{b(s)}\int_s^{\infty}p(u)\frac{1}{g^{\gamma}(\tau(u))}\Delta u\Delta s}{a(t)g^{\Delta}(t)} \le 1 \tag{3.3}$$

hold, (3.1) is also a necessary condition for equation (1.1) having an eventually positive solution $x(t) \in \left(0, \frac{1}{g(t)}\right]$.

Proof The assumption (A2) reveals that there exist sufficiently large T and T_1 such that $\tau(t) \ge T$ for $t \ge T_1$.

(i) Introduce the Banach spaces $BC[T,\infty)_{\mathbb{T}}$ and Ω as follows:

$$BC[T,\infty)_{\mathbb{T}} = \left\{x \mid x(t) \in C([T,\infty)_{\mathbb{T}}, \mathbb{R}) \text{ and } ||x|| = \sup_{t\in[T,\infty)_{\mathbb{T}}} |x(t)| < \infty\right\},$$

$$\Omega = \left\{x \middle| x(t) \in BC[T,\infty)_{\mathbb{T}} : 0 < x(t) \le \frac{1}{g(t)}\right\}.$$

Define the mapping I on Ω as

$$(Ix)(t) = \begin{cases} \int_t^{\infty}\frac{1}{a(v)}\int_v^{\infty}\frac{1}{b(s)}\int_s^{\infty}p(u)x^{\gamma}(\tau(u))\Delta u\Delta s\Delta v, & t \in [T_1,\infty), \\ (Ix)(T_1), & t \in [T, T_1]. \end{cases}$$

Similarly to [27, Theorem 3.5], we omit the rest proof.

(ii) We rewrite (3.1) as follows:

$$g(t)\int_t^{\infty}\frac{1}{a(v)}\int_v^{\infty}\frac{1}{b(s)}\int_s^{\infty}p(u)\frac{1}{g^{\gamma}(\tau(u))}\Delta u\Delta s\Delta v \le 1.$$

Therefore, if we show that

$$\lim_{t\to\infty} \frac{\int_t^{\infty}\frac{1}{a(v)}\int_v^{\infty}\frac{1}{b(s)}\int_s^{\infty}p(u)\frac{1}{g^{\gamma}(\tau(u))}\Delta u\Delta s\Delta v}{\frac{1}{g(t)}} \le 1, \tag{3.4}$$

then (3.1) holds for $t \ge T$.

In fact, by (3.2), the left hand side of (3.4) can be written as the indeterminate form $0/0$. By (3.3) and L'Hôpital's Rule, it is easy to check that (3.4) holds. □

Next, we investigate the asymptotic and oscillatory behavior of solutions to equation (1.1). Similarly to [13, Theorem 3.1], we derive the Leighton-Wintner Theorem.

Theorem 3.2 Suppose that (A1) and (A2) hold. If (2.1) holds, then every solution of equation (1.1) is oscillatory or tends to zero.

Theorem 3.3 Suppose that (A1), (A2), (2.2), $a^{\Delta}(t) \le 0$ and (2.5) hold. Then we have the following statements.

(i) Every solution x of equation (1.1) oscillates or converges to zero for $0 < \gamma < 1$ in case that

$$\int_T^{\infty} p(s)[Q(s)]^{\gamma}\Delta s = \infty \tag{3.5}$$

holds for some sufficiently large $T \in \mathbb{T}$, where

$$Q(s) = \frac{B(s,T)\tau(s)}{a(s)},$$

and $B(s,T)$ is defined by (2.4).

(ii) Every solution x of equation (1.1) oscillates or converges to zero for $\gamma > 1$ in case that

$$\int_T^{\infty} \overline{p}(s)[\overline{Q}(s)]^{\gamma}\Delta s = \infty \tag{3.6}$$

holds for some sufficiently large $T \in \mathbb{T}$, where

$$\overline{p}(s) = \int_s^{\infty} p(u)\left(\frac{\tau(u)}{u}\right)^{\gamma}\Delta u, \qquad \overline{Q}(s) = \frac{s}{a(s)b^{\frac{1}{\gamma}}(s)}.$$

(iii) Every solution x of equation (1.1) oscillates or converges to zero for $\gamma = 1$ in case

$$\frac{tB(s,T)}{a(t)}\int_T^{\infty} p(s)\frac{\tau(s)}{s}\Delta s > 1 \tag{3.7}$$

holds for some sufficiently large $T \in \mathbb{T}$, where $B(s,T)$ is defined by (2.4).

Proof Let x be a positive nonoscillatory solution of equation (1.1). Thus, there exists a large T such that

$$x(\tau(t)) > 0,\ x(t) > 0, \quad t \geq T.$$

If $x \in S_0$, by Lemma 2.2, $\lim\limits_{t\to\infty} x(t) = 0$.

If $x \in S_1$, we have that

$$x(t) > 0,\ x^{\Delta}(t) > 0,\ Y^{\Delta}(t) = (a(t)x^{\Delta}(t))^{\Delta} > 0,\ [b(t)Y^{\Delta}(t)]^{\Delta} < 0.$$

From equation (1.1), and noticing that, from Lemma 2.4 (2): $x(\tau(t)) > (\frac{\tau(t)}{t})x(t)$, we get that

$$\left[b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\Delta} + p(t)\left(\frac{\tau(t)}{t}\right)^{\gamma}x^{\gamma}(t) \leq 0. \tag{3.8}$$

Now we prove the conclusions by contradiction.

First of all, we prove the conclusion of the part (i). If $0 < \gamma < 1$, employing the Pötzsche chain rule [12, (3.6)] and the fact that $b(t)(a(t)x^{\Delta}(t))^{\Delta}$ is decreasing on $[T,\infty)_{\mathbb{T}}$, we have, for $t \in [T,\infty)_{\mathbb{T}}$, that

$$\begin{aligned}
&\left([b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta}]^{1-\gamma}\right)^{\Delta} = \left([b(t)Y^{\Delta}(t)]^{1-\gamma}\right)^{\Delta}\\
&= (1-\gamma)\int_0^1 \left(h[b(\sigma(t))Y^{\Delta}(\sigma(t))] + (1-h)[b(t)Y^{\Delta}(t)]\right)^{-\gamma}\mathrm{d}h \times \left[b(t)Y^{\Delta}(t)\right]^{\Delta}\\
&\leq (1-\gamma)\left[b(t)Y^{\Delta}(t)\right]^{-\gamma}\left[b(t)Y^{\Delta}(t)\right]^{\Delta} < 0.
\end{aligned} \tag{3.9}$$

It follows that

$$\frac{\left[b(t)Y^{\Delta}(t)\right]^{\Delta}}{\left[b(t)Y^{\Delta}(t)\right]^{\gamma}} \geq \frac{\left[(b(t)Y^{\Delta}(t))^{1-\gamma}\right]^{\Delta}}{1-\gamma} \tag{3.10}$$

and

$$\frac{\left[b(t)Y^{\Delta}(t)\right]^{\Delta} + p(t)(\frac{\tau(t)}{t})^{\gamma}x^{\gamma}(t)}{\left[b(t)Y^{\Delta}(t)\right]^{\gamma}} \geq \frac{\left[(b(t)Y^{\Delta}(t))^{1-\gamma}\right]^{\Delta}}{1-\gamma} + \frac{p(t)(\frac{\tau(t)}{t})^{\gamma}x^{\gamma}(t)}{\left[b(t)Y^{\Delta}(t)\right]^{\gamma}}. \tag{3.11}$$

In view of Lemma 2.3, we see that

$$b(t)(a(t)x^{\Delta}(t))^{\Delta} = b(t)Y^{\Delta}(t) \leq \frac{a(t)x^{\Delta}(t)}{B(t,T)}.$$

Combining this with (3.11), we have

$$0 \geq \frac{\left([b(t)Y^{\Delta}(t)]^{1-\gamma}\right)^{\Delta}}{1-\gamma} + \frac{p(t)(\frac{\tau(t)}{t}B(t,T))^{\gamma}x^{\gamma}(t)}{a^{\gamma}(t)(x^{\Delta}(t))^{\gamma}}. \tag{3.12}$$

From Lemma 2.4 (1), $x(t) > tx^{\Delta}(t)$, we get that

$$0 \geq \frac{\left([b(t)Y^{\Delta}(t)]^{1-\gamma}\right)^{\Delta}}{1-\gamma} + p(t)\left(\frac{B(t,T)\tau(t)}{a(t)}\right)^{\gamma} =: \frac{\left([b(t)Y^{\Delta}(t)]^{1-\gamma}\right)^{\Delta}}{1-\gamma} + p(t)Q^{\gamma}(t), \tag{3.13}$$

and hence

$$p(t)Q^{\gamma}(t) \leq -\frac{\left([b(t)Y^{\Delta}(t)]^{1-\gamma}\right)^{\Delta}}{1-\gamma}.$$

Integrating this from T to ∞, we get that

$$\int_T^{\infty} p(s)Q^{\gamma}(s)\Delta s \leq -\int_T^{\infty} \frac{\left([b(s)Y^{\Delta}(s)]^{1-\gamma}\right)^{\Delta}}{1-\gamma}\Delta s \leq \frac{[b(T)\left(a(T)x^{\Delta}(T)\right)^{\Delta}]^{1-\gamma}}{1-\gamma}, \tag{3.14}$$

which contradicts (3.5).

Next, we shall prove the conclusion of part (*ii*). If $\gamma > 1$, integrating (3.8) from t to u, we get

$$b(u)\left(a(u)x^{\Delta}(u)\right)^{\Delta} - b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta} + \int_t^u p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma} x^{\gamma}(s)\Delta s \leq 0;$$

i.e.,

$$b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta} \geq b(u)\left(a(u)x^{\Delta}(u)\right)^{\Delta} + \int_t^u p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma} x^{\gamma}(s)\Delta s.$$

Since x is increasing on $[T,\infty)_{\mathbb{T}}$ and $b(u)\left(a(u)x^{\Delta}(u)\right)^{\Delta} > 0$, letting $u \to \infty$, we have that

$$Y^{\Delta}(t) = \left(a(t)x^{\Delta}(t)\right)^{\Delta} \geq \frac{x^{\gamma}(t)}{b(t)} \int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}\Delta s. \tag{3.15}$$

Noting that $1-\gamma < 0$, using the Pötzsche chain rule and $Y^{\Delta}(t) = (a(t)x^{\Delta}(t))^{\Delta} > 0$ on $[T,\infty)_{\mathbb{T}}$, we have, for $t \in [T,\infty)_{\mathbb{T}}$, that

$$\begin{aligned}\left[(a(t)x^{\Delta}(t))^{1-\gamma}\right]^{\Delta} &= (1-\gamma)\int_0^1 [hY^{\sigma}(t) + (1-h)Y(t)]^{-\gamma}\,\mathrm{d}h \times Y^{\Delta}(t)\\ &\leq (1-\gamma)[Y(\sigma(t))]^{-\gamma}Y^{\Delta}(t) < 0.\end{aligned} \tag{3.16}$$

By (3.15), (3.16), $[a^{\sigma}(t)]^{-\gamma} \geq [a(t)]^{-\gamma}$ and Lemma 2.4 , $\frac{x(\sigma(t))}{x^{\Delta}(\sigma(t))} \geq \sigma(t)$, $\frac{x(t)}{x(\sigma(t))} > \frac{t}{\sigma(t)}$, so we have

$$\begin{aligned}\frac{\left[\left(a(t)x^{\Delta}(t)\right)^{1-\gamma}\right]^{\Delta}}{1-\gamma} &\geq \frac{Y^{\Delta}(t)}{Y^{\gamma}(\sigma(t))} \geq \frac{\int_t^{\infty} p(s)(\frac{\tau(s)}{s})^{\gamma}\Delta s}{a^{\gamma}(\sigma(t))b(t)}\left(\frac{x(\sigma(t))}{x^{\Delta}(\sigma(t))}\right)^{\gamma}\left(\frac{x(t)}{x(\sigma(t))}\right)^{\gamma}\\ &\geq \frac{t^{\gamma}}{a^{\gamma}(t)b(t)}\int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}\Delta s := \overline{p}(t)[\overline{Q}(t)]^{\gamma}.\end{aligned}$$

Integrating this from T to ∞, we obtain

$$\int_T^{\infty} \overline{p}(s)[\overline{Q}(s)]^{\gamma}\Delta s \leq \int_T^{\infty} \frac{\left[(a(s)x^{\Delta}(s))^{1-\gamma}\right]^{\Delta}}{1-\gamma}\Delta s \leq \frac{\left(a(T)x^{\Delta}(T)\right)^{1-\gamma}}{\gamma-1},$$

which contradicts (3.6).

Finally, we prove the conclusion of part (iii). If $\gamma = 1$, integrating the equation (3.8) from t to u, we get

$$b(u)\left(a(u)x^{\Delta}(u)\right)^{\Delta} - b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta} + \int_t^u p(s)\left(\frac{\tau(s)}{s}\right)x(s)\Delta s \leq 0;$$

i.e.,

$$b(t)\left(a(t)x^{\Delta}(t)\right)^{\Delta} \geq b(u)\left(a(u)x^{\Delta}(u)\right)^{\Delta} + \int_t^u p(s)\left(\frac{\tau(s)}{s}\right)x(s)\Delta s.$$

Since x is increasing on $[T,\infty)_{\mathbb{T}}$ and $b(u)\left(a(u)x^{\Delta}(u)\right)^{\Delta} > 0$, letting $u \to \infty$, we have

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} \geq \frac{x(t)}{b(t)}\int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)\Delta s.$$

By Lemma 2.3, $(a(t)x^{\Delta}(t))^{\Delta} \leq \frac{a(t)x^{\Delta}(t)}{B(t,T)b(t)}$, we obtain

$$\frac{a(t)x^{\Delta}(t)}{B(t,T)b(t)} \geq \frac{x(t)}{b(t)}\int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)\Delta s;$$

i.e.,

$$\frac{a(t)}{B(t,T)}\frac{x^{\Delta}(t)}{x(t)} \geq \int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)\Delta s. \tag{3.17}$$

By (3.17) and Lemma 2.4 (1), $\frac{x^{\Delta}(t)}{x(t)} < \frac{1}{t}$, we see that

$$\frac{a(t)}{B(t,T)}\frac{1}{t} > \int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)\Delta s;$$

i.e.,

$$\frac{tB(t,T)}{a(t)}\int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)\Delta s < 1.$$

This contradicts (3.7). The proof is complete. □

We introduce the function family $\overline{\mathfrak{R}}$ in the same way as [11]: $H \in \overline{\mathfrak{R}}$ if $H : D = \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R}$ is continuous, $H(t,s) \geq 0 (\not\equiv 0)$ on D, and for each fixed t, $H^{\Delta_s}(t,s)$ is delta integrable with respect to s. We introduce auxiliary functions

$$\beta(t) = \begin{cases} t/\sigma(t), & 0 < \gamma < 1; \\ (t/\sigma(t))^{\gamma}, & \gamma \geq 1, \end{cases} \tag{3.18}$$

and

$$C_1(t,T) := \frac{\gamma\beta(t)B(t,T)\delta^{\sigma}(t)}{a(t)}, \tag{3.19}$$

where $B(t,T)$ is defined by (2.4).

Theorem 3.4 Suppose that (A1), (A2), (2.2), $a^{\Delta}(t) \leq 0$ and (2.5) hold. Furthermore, assume that there exist a function $\alpha \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ with $(b\alpha)^{\Delta}$ existing, a positive Δ-differentiable function δ, and $H \in \overline{\mathfrak{R}}$ such that, for T large enough,

$$\theta(t) - \frac{(\theta_1(t))^2}{4\theta_2(t)} \geq 0, \quad t \geq T, \tag{3.20}$$

$$-\left(\theta(s) - \frac{(\theta_1(s))^2}{4\theta_2(s)}\right)H(\sigma(t),\sigma(s)) + \theta_3(s,T)[H^{\Delta_s}(\sigma(t),s)]_+ \leq 0, \quad t \geq s \geq T, \tag{3.21}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^t \left[\frac{H(\sigma(t),\sigma(s))\theta(s)}{H(\sigma(t),t_1)} - \frac{[H(\sigma(t),\sigma(s))\theta_1(s) + H^{\Delta_s}(\sigma(t),s)]_+^2}{4H(\sigma(t),t_1)H(\sigma(t),\sigma(s))\theta_2(s)}\right]\Delta s = \infty \tag{3.22}$$

hold for for all constants $k > 0$ and $t_1 \in \mathbb{T},\ t_1 > T$, where

$$C(t,T) := k^{\gamma-1}C_1(t,T),\ [H^{\Delta_s}(\sigma(t),s)]_+ = \max\{0, H^{\Delta_s}(\sigma(t),s)\},$$

$$[H(\sigma(t),\sigma(s))\theta_1(s)+H^{\Delta_s}(\sigma(t),s)]_+=\max\{0,[H(\sigma(t),\sigma(s))\theta_1(s)+H^{\Delta_s}(\sigma(t),s)]\},$$
$$\theta(t):=\delta^\sigma(t)p(t)\Big(\frac{\tau(t)}{\sigma(t)}\Big)^\gamma+C(t,T)(b(t)\alpha(t))^2-\delta^\sigma(t)(b(t)\alpha(t))^\Delta,$$
$$\theta_1(t):=\frac{1}{\delta(t)}\Big[\delta^\Delta(t)+2C(t,T)b(t)\alpha(t)\Big],\ \theta_2(t):=\frac{C(t,T)}{\delta^2(t)},$$
$$\theta_3(t):=\delta(t)\left[\frac{a(t)}{k^{\gamma-1}tB(t,T)}+b(t)\alpha(t)\right]>0,$$

and where $B(s,T)$ is defined by (2.4).

Then,

(i) every solution x of equation (1.1) oscillates or converges to zero for $\gamma\ge 1$;

(ii) any bounded solution x of equation (1.1) oscillates or tends to zero for $0<\gamma<1$.

Proof Proceeding as in the proof of Theorem 3.3, we assume that equation (1.1) has a nonoscillatory solution, say $x(t)>0$, for all $t\ge T$.

If $x\in S_0$, by Lemma 2.2, $\lim\limits_{t\to\infty}x(t)=0$.

Next, we consider $x\in S_1$. Define the function z by the generalized Riccati substitution

$$z(t):=\delta(t)\left[\frac{b(t)(a(t)x^\Delta(t))^\Delta}{x^\gamma(t)}+b(t)\alpha(t)\right],\quad t\ge T. \tag{3.23}$$

From equation (1.1), and noticing that, from Lemma 2.4 (2), $\frac{x(\tau(t))}{x(\sigma(t))}>\frac{\tau(t)}{\sigma(t)}$, we have that

$$\begin{aligned}z^\Delta(t)&=\delta^\Delta(t)\Big[\frac{b(t)(a(t)x^\Delta(t))^\Delta}{x^\gamma(t)}+b(t)\alpha(t)\Big]+\delta^\sigma(t)\Big[\frac{b(t)(a(t)x^\Delta(t))^\Delta}{x^\gamma(t)}+b(t)\alpha(t)\Big]^\Delta\\&=\frac{\delta^\Delta(t)}{\delta(t)}z(t)+\delta^\sigma(t)\Big[\frac{b(t)(a(t)x^\Delta(t))^\Delta}{x^\gamma(t)}\Big]^\Delta+\delta^\sigma(t)\big[b(t)\alpha(t)\big]^\Delta\\&=\frac{\delta^\Delta(t)}{\delta(t)}z(t)+\delta^\sigma(t)\big[b(t)\alpha(t)\big]^\Delta\\&\quad+\delta^\sigma(t)\frac{x^\gamma(t)[b(t)(a(t)x^\Delta(t))^\Delta]^\Delta-(x^\gamma(t))^\Delta\big[b(t)(a(t)x^\Delta(t))^\Delta\big]}{x^\gamma(t)(x^\sigma(t))^\gamma}\\&\le\frac{\delta^\Delta(t)}{\delta(t)}z(t)+\delta^\sigma(t)\left(\big[b(t)\alpha(t)\big]^\Delta-p(t)(\frac{\tau(t)}{\sigma(t)})^\gamma\right)-\delta^\sigma(t)\frac{b(t)(a(t)x^\Delta(t))^\Delta}{x^\gamma(t)}\frac{(x^\gamma(t))^\Delta}{(x^\sigma(t))^\gamma}.\end{aligned} \tag{3.24}$$

Employing the Pötzsche chain rule [12, (3.6)], we have, for $t\in[T,\infty)_{\mathbb{T}}$, that

$$(x^\gamma(t))^\Delta=\gamma\int_0^1[hx^\sigma(t)+(1-h)x(t)]^{\gamma-1}\mathrm{d}h\times x^\Delta(t). \tag{3.25}$$

Case 1 $\gamma\ge 1$. Since $x\in S_1$, then x is increasing for $t\ge T$. By (3.18), (3.25) and the Lemma 2.4 (2), $\frac{x(t)}{x(\sigma(t))}\ge\frac{t}{\sigma(t)}$, we have that

$$\frac{(x^\gamma(t))^\Delta}{(x^\sigma(t))^\gamma}\ge\frac{\gamma x^{\gamma-1}(t)x^\Delta(t)}{(x^\sigma(t))^\gamma}=\frac{\gamma x^\gamma(t)}{(x^\sigma(t))^\gamma}\frac{x^\Delta(t)}{x(t)}\ge\frac{\gamma t^\gamma}{(\sigma(t))^\gamma}\frac{x^\Delta(t)}{x(t)}=\gamma\beta(t)\frac{x^\Delta(t)}{x(t)}. \tag{3.26}$$

In view of Lemma 2.3, we see that

$$x^\Delta(t)\ge\frac{1}{a(t)}\Big[b(t)\left(a(t)x^\Delta(t)\right)^\Delta\Big]B(t,T).$$

Since $x \in S_1,\ x^{\Delta}(t) > 0$ implies that $x(t) \geq x(T) := k > 0$ for $t \geq T$. By (3.24) and (3.26), we have that

$$z^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} z(t) + \delta^{\sigma}(t)\Big[(b(t)\alpha(t))^{\Delta} - p(t)\Big(\frac{\tau(t)}{\sigma(t)}\Big)^{\gamma}\Big] - C(t,T)\left[\frac{b(t)(a(t)x^{\Delta}(t))^{\Delta}}{x^{\gamma}(t)}\right]^2, \quad (3.27)$$

where $C(t,T) = k^{\gamma-1}C_1(t,T)$. From the definition of $z(t)$, we see that

$$\left[\frac{b(t)(a(t)x^{\Delta}(t))^{\Delta}}{x^{\gamma}(t)}\right]^2 = \left[\frac{z(t)}{\delta(t)} - b(t)\alpha(t)\right]^2 = \left(\frac{z(t)}{\delta(t)}\right)^2 - 2\frac{b(t)\alpha(t)}{\delta(t)} z(t) + (b(t)\alpha(t))^2. \quad (3.28)$$

Substituting (3.28) into (3.27), we obtain that

$$\begin{aligned} z^{\Delta}(t) &\leq -\Big[\delta^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + C(t,T)(b(t)\alpha(t))^2 - \delta^{\sigma}(t)(b(t)\alpha(t))^{\Delta}\Big] \\ &\quad + \left[\frac{\delta^{\Delta}(t)}{\delta(t)} + 2\frac{1}{\delta(t)} b(t)\alpha(t)C(t,T)\right] z(t) - \frac{C(t,T)}{\delta^2(t)} z^2(t) \\ &= -\theta(t) + \theta_1(t)z(t) - \theta_2(t)z^2(t), \end{aligned} \quad (3.29)$$

where

$$\theta(t) = \delta^{\sigma}(t)p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} + C(t,T)(b(t)\alpha(t))^2 - \delta^{\sigma}(t)(b(t)\alpha(t))^{\Delta},$$

$$\theta_1(t) = \frac{1}{\delta(t)}\Big[\delta^{\Delta}(t) + 2C(t,T)b(t)\alpha(t)\Big],\ \theta_2(t) = \frac{C(t,T)}{\delta^2(t)}.$$

By (3.20) and (3.29), we have

$$z^{\Delta}(t) \leq -\theta(t) - \theta_2(t)\left[z(t) - \frac{\theta_1(t)}{2\theta_2(t)}\right]^2 + \frac{\theta_1^2(t)}{4\theta_2(t)} \leq -\theta(t) + \frac{\theta_1^2(t)}{4\theta_2(t)} \leq 0. \quad (3.30)$$

In view of (3.21), (3.30), $x(t) > tx^{\Delta}(t)$ and the definition of $z(t)$,

$$\begin{aligned} &[z(s)H(\sigma(t),s)]^{\Delta_s} \\ &= z^{\Delta}(s)H(\sigma(t),\sigma(s)) + z(s)H^{\Delta_s}(\sigma(t),s) \\ &\leq z^{\Delta}(s)H(\sigma(t),\sigma(s)) + \delta(s)\left[\frac{b(s)(a(s)x^{\Delta}(s))^{\Delta}}{x^{\gamma}(s)} + b(s)\alpha(s)\right][H^{\Delta_s}(\sigma(t),s)]_+ \\ &\leq z^{\Delta}(s)H(\sigma(t),\sigma(s)) + \delta(s)\left[\frac{a(s)x^{\Delta}(s)}{B(s,T)x(s)}\frac{1}{x^{\gamma-1}(s)} + b(s)\alpha(s)\right][H^{\Delta_s}(\sigma(t),s)]_+ \\ &\leq -\left(\theta(s) - \frac{\theta_1^2(s)}{4\theta_2(s)}\right)H(\sigma(t),\sigma(s)) + \theta_3(s,T)[H^{\Delta_s}(\sigma(t),s)]_+ \leq 0. \end{aligned}$$

Thus, $z(s)H(\sigma(t),s)$ is non-increasing with respect to s. Then, we can choose $t \geq t_1 \geq T$ such that $H(\sigma(t),t_1) > 0$ and

$$0 \leq -H(\sigma(t),t)z(t) + H(\sigma(t),t_1)z(t_1) \leq H(\sigma(t),t_1))z(t_1). \quad (3.31)$$

Evaluating both the sides of (3.29) at s, multiplying by $H(\sigma(t),\sigma(s))$ and integrating by parts, we get that

$$\begin{aligned} \int_{t_1}^{t} H(\sigma(t),\sigma(s))\theta(s)\Delta s &\leq -\int_{t_1}^{t} H(\sigma(t),\sigma(s))z^{\Delta}(s)\Delta s + \int_{t_1}^{t} H(\sigma(t),\sigma(s))\theta_1(s)z(s)\Delta s \\ &\quad -\int_{t_1}^{t} H(\sigma(t),\sigma(s))\theta_2(s)z^2(s)\Delta s \end{aligned}$$

$$\le H(\sigma(t),t_1)z(t_1)+\int_{t_1}^{t}[H(\sigma(t),\sigma(s))\theta_1(s)+H^{\Delta_s}(\sigma(t),s)]z(s)\Delta s$$
$$-\int_{t_1}^{t}H(\sigma(t),\sigma(s))\theta_2(s)z^2(s)\Delta s$$
$$\le H(\sigma(t),t_1)z(t_1)+\int_{t_1}^{t}[H(\sigma(t),\sigma(s))\theta_1(s)+H^{\Delta_s}(\sigma(t),s)]_+z(s)\Delta s$$
$$-\int_{t_1}^{t}H(\sigma(t),\sigma(s))\theta_2(s)z^2(s)\Delta s$$
$$\le H(\sigma(t),t_1)z(t_1)+\int_{t_1}^{t}\frac{[H(\sigma(t),\sigma(s))\theta_1(s)+H^{\Delta_s}(\sigma(t),s)]_+^2}{4H(\sigma(t),\sigma(s))\theta_2(s)}\Delta s.$$

This implies that

$$\int_{t_1}^{t}\left[\frac{H(\sigma(t),\sigma(s))\theta(s)}{H(\sigma(t),t_1)}-\frac{[H(\sigma(t),\sigma(s))\theta_1(s)+H^{\Delta_s}(\sigma(t),s)]_+^2}{4H(\sigma(t),t_1)H(\sigma(t),\sigma(s))\theta_2(s)}\right]\Delta s\le z(t_1),$$

which contradicts (3.22).

Case 2 $0<\gamma<1$. Since $x\in S_1$, and $x(t)>0$, letting x eventually be bounded, there exists a constant $\overline{k}>0$ such that $x(t)\le\overline{k}$ for $t\ge T$, and

$$k^{\gamma-1}\ge x^{\gamma-1}(t)\ge\overline{k}^{\gamma-1},\quad t\ge T.$$

Combining (3.25) with (3.18), we also get (3.26). The rest of the proof is similar to the Case 1 $\gamma\ge1$, so we omit it, and the proof is complete. □

Theorem 3.5 Suppose that (A1), (A2), (2.2), $a^{\Delta}(t)\le0$ and (2.5) hold. Furthermore, assume that there exist a function $\alpha\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ with $(b\alpha)^{\Delta}$ existing, a positive Δ-differentiable function δ, and $H\in\overline{\mathfrak{R}}$ such that, for sufficiently large $T\in\mathbb{T}$,

$$\theta^*(t)-\frac{(\theta_1^*(t))^2}{4\theta_2^*(t)}\ge0,\quad t\ge T,\tag{3.32}$$

$$-\left(\theta^*(s)-\frac{(\theta_1^*(s))^2}{4\theta_2^*(s)}\right)H(\sigma(t),\sigma(s))+\theta_4(s,T)[H^{\Delta_s}(\sigma(t),s)]_+\le0,\quad t\ge s\ge T,\tag{3.33}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{H(\sigma(t),\sigma(s))\theta^*(s)}{H(\sigma(t),t_1)}-\frac{[H(\sigma(t),\sigma(s))\theta_1^*(s)+H^{\Delta_s}(\sigma(t),s)]_+^2}{4H(\sigma(t),t_1)H(\sigma(t),\sigma(s))\theta_2^*(s)}\right]\Delta s>\theta_4(t_1,T)\tag{3.34}$$

hold for $t_1\ge T$, where

$$\theta_4(t,T):=\delta(t)\left[\frac{a(t)}{B(t,T)}+b(t)\alpha(t)\right]>0,\ C^*(t,T)=\frac{1}{t}C_1(t,T),$$
$$\theta^*(t):=\delta^{\sigma}(t)p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}+C^*(t,T)(b(t)\alpha(t))^2-\delta^{\sigma}(t)(b(t)\alpha(t))^{\Delta},$$
$$\theta_1^*(t):=\frac{1}{\delta(t)}\left[\delta^{\Delta}(t)+2C^*(t,T)b(t)\alpha(t)\right],\ \theta_2^*(t):=\frac{C^*(t,T)}{\delta^2(t)},$$

and $C_1(t,T)$ is defined by (3.18). Then, every unbounded solution x of equation (1.1) oscillates and all of the bounded solutions tend to zero for $\gamma>0$.

Proof Let $x>0$ be a unbounded nonoscillatory solution of equation (1.1). Then there exists a large T such that

$$x(\tau(t))>0,\ x(t)>0,\quad t\ge T.$$

Since x is a unbounded solution, it is easy to see that $x \in S_1$. Then, by defining $z(t)$ again by (3.23) as in the Theorem 3.4, we have that $z(t) > 0$ and

$$z^{\Delta}(t) \le \frac{\delta^{\Delta}(t)}{\delta(t)} z(t) + \delta^{\sigma}(t)\left[(b(t)\alpha(t))^{\Delta} - p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}\right] - \frac{C_1(t,T)}{x^{1-\gamma}(t)}\left[\frac{b(t)(a(t)x^{\Delta}(t))^{\Delta}}{x^{\gamma}(t)}\right]^2. \tag{3.35}$$

Case 1 $\gamma \ge 1$. Since x is increasing and unbounded, there exists a sufficiently large t_1 such that

$$tx^{\gamma-1}(t) \ge 1, \text{ and } -\frac{1}{x^{1-\gamma}(t)} = -\frac{tx^{\gamma-1}(t)}{t} \le -\frac{1}{t}, \text{ for } t \ge t_1 \ge T.$$

Case 2 $0 < \gamma < 1$. By Lemma 2.3, $x(T) \le x(t) \le \frac{x(T)}{T}t$. Noting that x is increasing and unbounded, for all $t \ge t_1 \ge T$, we have that

$$x^{1-\gamma}(t) \le \left(\frac{x(T)}{T}t\right)^{1-\gamma} \le t, \quad -\frac{1}{x^{1-\gamma}(t)} \le -\frac{1}{t}.$$

Combining this with (3.35), we always obtain that

$$z^{\Delta}(t) \le \frac{\delta^{\Delta}(t)}{\delta(t)} z(t) + \delta^{\sigma}(t)\left[(b(t)\alpha(t))^{\Delta} - p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}\right] - C^*(t,T)\left[\frac{b(t)(a(t)x^{\Delta}(t))^{\Delta}}{x^{\gamma}(t)}\right]^2, \tag{3.36}$$

where $C^*(t,T) = C_1(t,T)/t$.

In view of $a(t)x^{\Delta}(t) \ge B(t,T)b(t)[a(t)x^{\Delta}(t)]^{\Delta}$, $x(t) > tx^{\Delta}(t)$ and the definition of $z(t)$, when $\gamma \ge 1$, we obtain that

$$\begin{aligned}
z(t_1) &= \delta(t_1)\Big[\frac{b(t_1)(a(t_1)x^{\Delta}(t_1))^{\Delta}}{x^{\gamma}(t_1)} + b(t_1)\alpha(t_1)\Big] \\
&= \delta(t_1)\Big[\frac{a(t_1)}{B(t_1,T)}\frac{x^{\Delta}(t_1)}{x(t_1)}\frac{1}{x^{\gamma-1}(t_1)} + b(t_1)\alpha(t_1)\Big] \\
&\le \delta(t_1)\Big[\frac{a(t_1)}{B(t_1,T)}\frac{1}{t_1}\frac{1}{x^{\gamma-1}(t_1)} + b(t_1)\alpha(t_1)\Big] \\
&\le \delta(t_1)\Big[\frac{a(t_1)}{B(t_1,T)} + b(t_1)\alpha(t_1)\Big] = \theta_4(t_1,T).
\end{aligned} \tag{3.37}$$

Then, from (3.35), we have that

$$\begin{aligned}
&\frac{1}{H(\sigma(t),t_1)}\int_{t_1}^{t}\left[H(\sigma(t),\sigma(s))\theta(s) - \frac{[H(\sigma(t),\sigma(s))\theta_1(s) + H^{\Delta_s}(\sigma(t),s)]_+^2}{4H(\sigma(t),\sigma(s))\theta_2(s)}\right]\Delta s \\
&\le z(t_1) \le \theta_4(t_1,T),
\end{aligned}$$

which contradicts (3.34).

Similarly, when $0 < \gamma < 1$, we also have that

$$\begin{aligned}
z(t_1) &= \delta(t_1)\Big[\frac{b(t_1)(a(t_1)x^{\Delta}(t_1))^{\Delta}}{x^{\gamma}(t_1)} + b(t_1)\alpha(t_1)\Big] \\
&= \delta(t_1)\Big[\frac{a(t_1)}{B(t_1,T)}\frac{x^{\Delta}(t_1)}{x(t_1)}x^{1-\gamma}(t_1) + b(t_1)\alpha(t_1)\Big] \\
&\le \delta(t_1)\left[\frac{a(t_1)}{B(t_1,T)}\frac{1}{t_1}\left(\frac{x(T)}{T}t_1\right)^{1-\gamma} + b(t_1)\alpha(t_1)\right] \\
&\le \delta(t_1)\left[\frac{a(t_1)}{B(t_1,T)}\frac{1}{t_1}t_1 + b(t_1)\alpha(t_1)\right] \le \delta(t_1)\left[\frac{a(t_1)}{B(t_1,T)} + b(t_1)\alpha(t_1)\right] = \theta_4(t,T).
\end{aligned}$$

The rest proof is similar to the case of $\gamma \geq 1$.

Finally, we show that all bounded solutions of equation (1.1) tend to zero. Now we show that every bounded nonoscillatory solution x of equation (1.1) with $\lim_{t\to\infty} x(t) = l > 0$ does not exist. Let

$$f(t,x) = p(t)|x|^{\gamma-1}x.$$

By [27, Theorem 3.3], (1.1) has a nonoscillatory solution x with $\lim_{t\to\infty} x(t) = l > 0$ if only if there exists a constant K such that

$$\begin{aligned}\int_{t_0}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{1}{a(v)b(s)}f(u,K)\Delta u\Delta s\Delta v &= K^{\gamma}\int_{t_0}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{1}{a(v)b(s)}p(u)\Delta u\Delta s\Delta v\\ &= K^{\gamma}\int_{t_0}^{\infty}\frac{1}{a(v)}\int_{v}^{\infty}\frac{1}{b(s)}\int_{s}^{\infty}p(u)\Delta u\Delta s\Delta v<\infty.\end{aligned}$$

However, (2.2) holds, so we see that $\int_{t_0}^{\infty}\frac{1}{a(v)}\int_{v}^{\infty}\frac{1}{b(s)}\int_{s}^{\infty}p(u)\Delta u\Delta s\Delta v = \infty$ implies that (1.1) does not have any bounded nonoscillatory solution x with $\lim_{t\to\infty} x(t) = l > 0$. □

Remark 3.6 In view of (3.37), if $\gamma \geq 1$, we also have that

$$\begin{aligned}z(t_1) &\leq \delta(t_1)\left[\frac{a(t_1)}{B(t_1,T)}\frac{x^{\Delta}(t_1)}{x(t_1)}\frac{1}{x^{\gamma-1}(t_1)}+b(t_1)\alpha(t_1)\right]\\ &\leq \delta(t_1)\left[\frac{a(t_1)}{t_1B(t_1,T)}+b(t_1)\alpha(t_1)\right] := \theta_4^*(t_1,T).\end{aligned}$$

Thus, if (3.36) is replaced by

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{H(\sigma(t),\sigma(s))\theta^*(s)}{H(\sigma(t),t_1)}-\frac{[H(\sigma(t),\sigma(s))\theta_1^*(s)+H^{\Delta_s}(\sigma(t),s)]_+^2}{4H(\sigma(t),t_1)H(\sigma(t),\sigma(s))\theta_2^*(s)}\right]\Delta s>\theta_4^*(t_1,T),$$

then every unbounded solution of equation (1.1) oscillates for $\gamma \geq 1$. In particular, if we choose that $H(t,s) = \delta(t) = 1$, $\alpha(t) = 0$, then the condition

$$\limsup_{t\to\infty}\int_{t_1}^{t}p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}\Delta s>\frac{a(t_1)}{t_1B(t_1,T)}$$

guarantees that all unbounded solutions of equation (1.1) oscillate for $\gamma \geq 1$.

Moreover, in Theorems 3.4 and 3.5, we remove the conditions (H2) and $\overline{H}(t,t) = 0$ in (H1).

Now we choose another function class $\mathfrak{R}$ as follows: $H_1 \in \mathfrak{R}$ if $H_1 : D = \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R}$ is continuous, $H_1(t,s) \geq 0 (\not\equiv 0)$, $H_1(t,t) = 0$, $H_1^{\Delta_s}(t,s) \leq 0$ on D, and, for each fixed t, $H_1^{\Delta_s}(t,s)$ is delta integrable with respect to s.

Theorem 3.7 Suppose that (A1), (A2), (2.2), $a^{\Delta}(t) \leq 0$ and (2.5) hold. Furthermore, assume that there exist a function $\alpha \in C_{\rm rd}(\mathbb{T},\mathbb{R})$ with $(b\alpha)^{\Delta}$ existing, a positive Δ-differentiable function δ, and $H_1 \in \mathfrak{R}$ such that, for sufficiently large t_1, and all $k > 0$,

$$\limsup_{t\to\infty}\frac{1}{H_1(t,t_1)}\int_{t_1}^{t}H_1(t,s)\left[\theta^{**}(s)-\frac{E_+^2(t,s)}{4\theta_2^{**}(s)}\right]\Delta s=\infty, \tag{3.38}$$

where $\beta(t)$ is defined by (3.18), and

$$C^{**}(t,T) := \gamma k^{\gamma-1}\frac{\beta(t)B(t,T)\delta(t)}{a(t)}, \tag{3.39}$$

$$\theta^{**}(t) := \delta(t)p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}+C^{**}(t,T)(b^{\sigma}(t)\alpha^{\sigma}(t))^2-\delta(t)(b(t)\alpha(t))^{\Delta}, \tag{3.40}$$

$$\theta_1^{**}(t) := \frac{1}{\delta^\sigma(t)}\Big[\delta^\Delta(t) + 2C^{**}(t,T)b^\sigma(t)\alpha^\sigma(t)\Big], \quad \theta_2^{**}(t) := \frac{C^{**}(t,T)}{(\delta^\sigma(t))^2}, \tag{3.41}$$

$$E(t,s) := \frac{H_1^{\Delta_s}(t,s)}{H_1(t,s)} + \theta_1^{**}(s), \quad E_+(t,s) = \max\{0, E(t,s)\}, \tag{3.42}$$

where $B(t,T)$ is defined by (2.4).

Then,

(i) every solution x of equation (1.1) oscillates or converges to zero for $\gamma \geq 1$;

(ii) any bounded solution x of equation (1.1) oscillates or tends to zero for $0 < \gamma < 1$.

Proof Proceeding as in the proof of Theorem 3.4, we assume that equation (1.1) has a nonoscillatory solution, say $x(t) > 0$, for all $t \geq T$.

If $x \in S_0$, by Lemma 2.2, $\lim\limits_{t\to\infty} x(t) = 0$.

Next, we consider $x \in S_1$, and define $z(t)$ by (3.23) as in the Theorem 3.4.

Case 1 $\gamma \geq 1$. We have that $z(t) > 0$ and

$$\begin{aligned} z^\Delta(t) &= \delta(t)\left[\frac{b(t)(a(t)x^\Delta(t))^\Delta}{x^\gamma(t)} + b(t)\alpha(t)\right]^\Delta + \delta^\Delta(t)\left[\frac{b(t)(a(t)x^\Delta(t))^\Delta}{x^\gamma(t)} + b(t)\alpha(t)\right]^\sigma \\ &\leq -\left[\delta(t)p(t)(\frac{\tau(t)}{\sigma(t)})^\gamma + C^{**}(t,T)\left(b^\sigma(t)\alpha^\sigma(t)\right)^2 - \delta(t)(b(t)\alpha(t))^\Delta\right] \\ &\quad + \left[\frac{\delta^\Delta(t)}{\delta^\sigma(t)} + \frac{2}{\delta^\sigma(t)}b^\sigma(t)\alpha^\sigma(t)C^{**}(t,T)\right]z^\sigma(t) - \frac{C^{**}(t,T)}{(\delta^\sigma(t))^2}(z^\sigma(t))^2 \\ &= -\theta^{**}(t) + \theta_1^{**}(t)z^\sigma(t) - \theta_2^{**}(t)(z^\sigma(t))^2, \end{aligned} \tag{3.43}$$

where $C^{**}(t,T)$, $\theta_1^{**}(t)$, $\theta_2^{**}(t)$, $\theta^{**}(t)$ are defined by (3.39), (3.40) and (3.41), respectively.

From (3.43), we obtain that

$$\theta^{**}(t) \leq -z^\Delta(t) + \theta_1^{**}(t)z^\sigma(t) - \theta_2^{**}(t)(z^\sigma(t))^2. \tag{3.44}$$

Evaluating both sides of (3.44) at s, and multiplying by $H_1(t,s)$, we get that

$$\begin{aligned} \int_{t_1}^t H_1(t,s)\theta^{**}(s)\Delta s \leq& -\int_{t_1}^t H_1(t,s)z^\Delta(s)\Delta s + \int_{t_1}^t H_1(t,s)\theta_1^{**}(s)z^\sigma(s)\Delta s \\ &- \int_{t_1}^t H_1(t,s)\theta_2^{**}(s)(z^\sigma(s))^2\Delta s. \end{aligned} \tag{3.45}$$

Integrating by parts and using the fact that $H_1(t,t) = 0$, we get that

$$-\int_{t_1}^t H_1(t,s)z^\Delta(s)\Delta s = H_1(t,t_1)z(t_1) + \int_{t_1}^t H_1^{\Delta_s}(t,s)z^\sigma(s)\Delta s.$$

Substituting this into (3.45), we have that

$$\begin{aligned} \int_{t_1}^t H_1(t,s)\theta^{**}(s)\Delta s &\leq H_1(t,t_1)z(t_1) + \int_{t_1}^t [H_1(t,s)\theta_1^{**}(s) + H_1^{\Delta_s}(t,s)]z^\sigma(s)\Delta s \\ &\quad - \int_{t_1}^t H_1(t,s)\theta_2^{**}(s)(z^\sigma(s))^2\Delta s \\ &\leq H_1(t,t_1)z(t_1) + \int_{t_1}^t H_1(t,s)\Big[\theta_1^{**}(s) + \frac{H_1^{\Delta_s}(t,s)}{H_1(t,s)}\Big]z^\sigma(s)\Delta s \\ &\quad - \int_{t_1}^t H_1(t,s)\theta_2^{**}(s)(z^\sigma(s))^2\Delta s \end{aligned}$$

$$\le H_1(t,t_1)z(t_1)+\int_{t_1}^{t}H_1(t,s)E_+(t,s)z^{\sigma}(s)\Delta s$$
$$-\int_{t_1}^{t}H_1(t,s)\theta_2^{**}(s)(z^{\sigma}(s))^2\Delta s,$$

where

$$E(t,s)=\theta_1^{**}(s)+\frac{H_1^{\Delta_s}(t,s)}{H_1(t,s)},\ \ E_+(t,s)=\max\{0,E(t,s)\}.$$

Applying the inequality $a_1y-b_1y^2\le\frac{a_1^2}{4b_1}(b_1>0)$, we obtain that

$$\int_{t_1}^{t}H_1(t,s)\theta^{**}(s)\Delta s\le H_1(t,t_1)z(t_1)+\int_{t_1}^{t}\frac{H_1(t,s)E_+^2(t,s)}{4\theta_2^{**}(s)}\Delta s;$$

i.e.,

$$\frac{1}{H_1(t,t_1)}\int_{t_1}^{t}H_1(t,s)\left[\theta^{**}(s)-\frac{E_+^2(t,s)}{4\theta_2^{**}(s)}\right]\Delta s\le z(t_1), \tag{3.46}$$

which contradicts (3.38).

Case 2 $0<\gamma<1$. Since the proof is similar to the Case 1 $\gamma\ge1$, we omit it. The proof is complete. □

Remark 3.8 Compared with Theorem 3.4, Theorem 3.7 removes the conditions (3.20) and (3.21). However, the function H_1 satisfies conditions $H_1(t,t)=0$ and $H_1^{\Delta_s}(t,s)\le0$ on D, which is stronger than H in theorem 3.4. Moreover, the conclusions of Theorem 3.4 are also true if (3.21) is replaced by $H_1^{\Delta_s}(t,s)\le0$. In particular, Han, *et al* [21, Theorem 2.2] show a special case of Theorem 3.4, and Erbe, *et al* [15, Theorem 2] show a special case of Theorems 3.4 and 3.7.

Remark 3.9 If we choose $H(t,s)=H_1(t,s)=(t-s)^m,\ m\in\mathbb{N}$ in Theorems 3.4, 3.5 and 3.7, then we have that

$$H^{\Delta_s}(t,s)=H_1^{\Delta_s}(t,s)=\big((t-s)^m\big)^{\Delta_s}=\begin{cases}-m(t-s)^{m-1}, & \mu(s)=0,\\ -\dfrac{(t-\sigma(s))^m-(t-s)^m}{\mu(s)}, & \mu(s)>0,\end{cases}$$

and $m\ge1,\ H_1^{\Delta_s}(t,s)<0$ for $t\ge\sigma(s)$. Thus we have the following three statements:

(i) If $H(t,s)=(t-s)^m,\ m\in\mathbb{N}$, and (3.22) is replaced by

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{(\sigma(t)-\sigma(s))^m\theta(s)}{(\sigma(t)-t_1)^m}-\frac{\big[(\sigma(t)-\sigma(s))^m\theta_1(s)+\big((\sigma(t)-s)^m\big)^{\Delta_s}\big]_+^2}{4(\sigma(t)-t_1)^m(\sigma(t)-\sigma(s))^m\theta_2(s)}\right]\Delta s=\infty, \tag{3.47}$$

then the conclusions of Theorem 3.4 are also true.

(ii) If $H(t,s)=(t-s)^m,\ m\in\mathbb{N}$, and (3.34) is replaced by

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{(\sigma(t)-\sigma(s))^m\theta^*(s)}{(\sigma(t)-t_1)^m}-\frac{\big[(\sigma(t)-\sigma(s))^m\theta_1^*(s)+\big((\sigma(t)-s)^m\big)^{\Delta_s}\big]_+^2}{4(\sigma(t)-t_1)^m(\sigma(t)-\sigma(s))^m\theta_2^*(s)}\right]\Delta s>\theta_4(t_1,T), \tag{3.48}$$

then the conclusions of Theorem 3.5 are also true.

(iii) If $H_1(t,s)=(t-s)^m,\ m\in\mathbb{N}$, and (3.38) is replaced by

$$\limsup_{t\to\infty}\frac{1}{(t-t_1)^m}\int_{t_1}^{t}(t-s)^m\left[\theta^{**}(s)-\frac{E_+^2(t,s)}{4\theta_2^{**}(s)}\right]\Delta s=\infty, \tag{3.49}$$

then the conclusions of Theorem 3.7 are also true.

(3.47), (3.48) and (3.49) are called the Kamenev-type oscillation criteria.

4 Examples

In this section, we would like to illustrate the main results obtained in Section 3 with four examples.

Example 4.1 Let $\mathbb{T}=\mathbb{N},\ \gamma=\frac{1}{3}$. Consider the third-order delay difference equation

$$\Delta\left(n^{\frac{1}{2}}\Delta\left(\frac{1}{n}\Delta x_n\right)\right)+p_n|x_{n-2}|^{-\frac{2}{3}}x_{n-2}=0, \tag{4.1}$$

where

$$a_n=\frac{1}{n},\ b_n=n^{\frac{1}{2}},\ \tau_n=n-2,\ \Delta a_n=\frac{-1}{n(n+1)}<0,$$
$$p_n=\frac{\sqrt{n+1}(4n^4+12n^3+10n^2)+\sqrt{n}(4n^4+20n^3+34n^2+24n+8)}{n^2(n+1)^2(n+2)^2}.$$

It is easy to see that (A1), (A2), (2.2) and (2.5) hold. Furthermore, a simple calculation shows that the condition (3.5) becomes

$$\sum_{n=1}^{\infty}p_nQ_n^{\frac{1}{3}}=\sum_{n=1}^{\infty}p_n\left[\frac{\tau_n}{a_n}\sum_{k=1}^{n-1}\frac{1}{b_k}\right]^{\frac{1}{3}}\geq\sum_{n=1}^{\infty}\frac{\sqrt{n}(4n^4+12n^3+10n^2)}{n^2(n+1)^2(n+2)^2}\left[\left(n^2-2n\right)\sum_{k=1}^{n-1}\frac{1}{k^{\frac{1}{2}}}\right]^{\frac{1}{3}}=\infty.$$

Thus, the condition (3.5) is satisfied, and, by Theorem 3.3, every solution of equation (4.1) oscillates or converges to zero. In fact, $x_n=(-1)^n$ is such an oscillatory solution of equation (4.1).

Example 4.2 Let $\mathbb{T}=\mathbb{P}_{a,b}=\bigcup\limits_{k=1}^{\infty}[k(a+b),k(a+b)+a],\ a>0,\ b>0$ and $\gamma\geq1$. Consider the third-order delay dynamic equation

$$\left(t\left(\frac{1}{t}x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+\frac{1}{t^2}|x(\tau(t))|^{\gamma-1}x(\tau(t))=0, \tag{4.2}$$

where

$$a(t)=\frac{1}{t},\ b(t)=t,\ p(t)=\frac{1}{t^2},$$

$$\tau(t)=\begin{cases}0, & t\in\bigcup\limits_{k=1}^{2}\big[k(a+b),k(a+b)+a\big],\\ t-2(a+b), & t\in\bigcup\limits_{k=3}^{\infty}\big[k(a+b),k(a+b)+a\big].\end{cases}$$

It is easy to see that (A1), (A2), (2.2) and (2.5) hold. Set that $\alpha(t)=0,\ \delta(t)=t$ and $H(t,s)=1$ in Theorem 3.4. Then, for large T and $t\geq T$, we have that

$$\delta^{\Delta}(t)=1,\ \theta_1(t)=\frac{1}{t},\ H^{\Delta_s}(t,s)=0,\ [H(t,s)\theta_1(s)+H^{\Delta_s}(t,s)]_+=\frac{1}{s},$$

and

$$\frac{\tau(t)}{\sigma(t)} \geq \frac{1}{2}, \quad \theta(t) = \frac{\sigma(t)}{t^2} \cdot \left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} \geq \frac{1}{2^{\gamma} t}, \quad \theta_3(t,T)[H^{\Delta_s}(t,s)]_+ = 0,$$
$$\frac{t}{\sigma(t)} \geq \frac{1}{2}, \quad \theta_2(t) = \gamma k^{\gamma-1} B(t,T) \left(\frac{t}{\sigma(t)}\right)^{\gamma-1} \geq \gamma \left(\frac{k}{2}\right)^{\gamma-1} B(t,T).$$

It is not difficult to check that (3.18) and (3.19) hold. For $s >> T$, we get that

$$\frac{[\theta_1(s)]^2}{4\theta_2(s)} = \frac{[H(t,s)\theta_1(s) + H^{\Delta_s}(t,s)]_+^2}{4H(t,s)\theta_2(s)} = \frac{1}{s^2} \cdot \frac{1}{4\gamma k^{\gamma-1}(\frac{s}{\sigma(s)})^{\gamma-1} B(s,T)} \leq \frac{2^{\gamma-1}}{4\gamma k^{\gamma-1}} \cdot \frac{1}{s\sigma(s)}.$$

For $s \geq t_1 >> s$, direct calculation shows that the condition (3.20) becomes

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\theta(s) - \frac{[H(t,s)\theta_1(s) + H^{\Delta_s}(t,s)]_+^2}{4H(t,s)\theta_2(s)}\right] \Delta s$$
$$\geq \limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{1}{2^{\gamma} s} - \frac{2^{\gamma-1}}{4\gamma k^{\gamma-1}} \cdot \frac{1}{s\sigma(s)}\right] \Delta s = \infty.$$

Thus, condition (3.20) is satisfied, and hence, by Theorem 3.4, every solution of equation (4.2) oscillates or converges to zero.

Example 4.3 Let $\mathbb{T} = \mathbb{R}$, $0 < \lambda < 1$ and $\gamma \geq 3$. Consider the third-order delay differential equation

$$\left(\frac{1}{t^2} x'(t)\right)'' + \frac{1}{t^4} |x(\lambda t)|^{\gamma-1} x(\lambda t) = 0, \tag{4.3}$$

where

$$a(t) = \frac{1}{t^2}, \ b(t) = 1, \ p(t) = \frac{1}{t^4}, \ \tau(t) = \lambda t, \ a'(t) = -\frac{2}{t^3} < 0.$$

It is easy to see that (A1), (A2), (2.2) and (2.5) hold.

Set that $\alpha(t) = 0$, $\delta(t) = 1$, and $H(t,s) = 1$ in Theorems 3.4 and 3.5. Then we have that

$$\delta'(t) = 0, \ \frac{\partial H(t,s)}{\partial s} = 0, \ \theta_1(t) = \theta_1^*(t) = 0, \ \theta_2^*(t) = \gamma t(t-T),$$
$$\theta_4(t,T) = \frac{a(t)}{B(t,T)} = \frac{1}{t^2(t-T)}, \ \theta(t) = \theta^*(t) = p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} = \frac{\lambda^{\gamma}}{t^4},$$

and

$$\left[H(t,s)\theta_1(s) + \frac{\partial H(t,s)}{\partial s}\right]_+ = \left[H(t,s)\theta_1^*(s) + \frac{\partial H(t,s)}{\partial s}\right]_+ = 0.$$

It is not difficult to check that (3.18), (3.19), (3.30) and (3.31) hold. Furthermore, a simple calculation shows that the conditions (3.20) and (3.32) are the same as the inequality.

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\theta(s) - \frac{\left[H(t,s)\theta_1(s) + \frac{\partial H(t,s)}{\partial s}\right]_+^2}{4H(t,s)\theta_2(s)}\right] ds$$
$$= \limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\theta^*(s) - \frac{\left[H(t,s)\theta_1^*(s) + \frac{\partial H(t,s)}{\partial s}\right]_+^2}{4H(t,s)\theta_2^*(s)}\right] ds$$
$$= \limsup_{t\to\infty} \int_{t_1}^{t} \theta(s) ds = \limsup_{t\to\infty} \int_{t_1}^{t} \theta^*(s) ds = \limsup_{t\to\infty} \int_{t_1}^{t} \frac{\lambda^{\gamma}}{s^4} ds$$
$$= \frac{\lambda^{\gamma}}{3t_1^3} < \theta_4(t_1,T) = \frac{1}{t_1^2(t_1-T)}.$$

Thus, Theorems 3.4 and 3.5 fail if we choose that $\alpha(t)=0,\ \delta(t)=1$, and $H(t,s)=1$. However, in view of Remark 3.6,

$$\limsup_{t\to\infty}\int_{t_1}^{t}p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}\mathrm{d}s=\frac{\lambda^{\gamma}}{3t_1^3}>\theta_4^*(t_1,T)=\frac{a(t_1)}{t_1B(t_1,T)}=\frac{1}{t_1^3(t_1-T)},$$

which implies that every unbounded solution of equation (4.3) oscillates. In [32], when $r(t)=0,\gamma_1=1,\gamma_3=\gamma$, (1.9) becomes (1.1). However, $\overline{H}(t,s)=1$ does not satisfy condition (H1). Our results for Theorem 3.5 and Remark 3.6 do not need the left hand side of (1.11) to be ∞.

Let $f(t,x)=p(t)|x|^{\gamma-1}x$. By [27, Theorem 3.3], equation (4.3) has a nonoscillatory solution x with $\lim\limits_{t\to\infty}x(t)=l$ ($l>0$ is a constant) if and only if there exists a constant K such that

$$\begin{aligned}\int_{t_0}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{1}{a(v)b(s)}f(u,K)\mathrm{d}u\mathrm{d}s\mathrm{d}v&=K^{\gamma}\int_{t_0}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{1}{a(v)b(s)}p(u)\mathrm{d}u\mathrm{d}s\mathrm{d}v\\&=K^{\gamma}\int_{t_0}^{\infty}\frac{1}{a(v)}\int_{v}^{\infty}\frac{1}{b(s)}\int_{s}^{\infty}p(u)\mathrm{d}u\mathrm{d}s\mathrm{d}v<\infty.\end{aligned}$$

However, $\int_{t_0}^{\infty}\frac{1}{a(v)}\int_{v}^{\infty}\frac{1}{b(s)}\int_{s}^{\infty}p(u)\mathrm{d}u\mathrm{d}s\mathrm{d}v=\infty$ implies that equation (4.3) has no nonoscillatory solution x with $\lim\limits_{t\to\infty}x(t)=l>0$, by Theorem 3.5.

Now, we pick up on the fact that $g(t)=\mathrm{e}^t$ in Theorem 3.1, so

$$\begin{aligned}\int_{t}^{\infty}\frac{1}{a(v)}\int_{v}^{\infty}\frac{1}{b(s)}\int_{s}^{\infty}p(u)\frac{1}{\mathrm{e}^{\gamma u}}\mathrm{d}u\mathrm{d}s\mathrm{d}v&<\int_{t}^{\infty}\frac{1}{a(v)}\int_{v}^{\infty}\frac{1}{b(s)}\int_{s}^{\infty}\frac{1}{\mathrm{e}^{\gamma u}}\mathrm{d}u\mathrm{d}s\mathrm{d}v\\&=\frac{1}{\gamma^2}\int_{t}^{\infty}\frac{1}{a(v)}\frac{1}{\mathrm{e}^{\gamma t}}\mathrm{d}u\mathrm{d}s\mathrm{d}v\\&=\left(\frac{1}{\gamma}t^2+\frac{2}{\gamma}t+\frac{1}{\gamma}\right)\mathrm{e}^{-\gamma t}<\mathrm{e}^{-t},\end{aligned}$$

which implies that (3.1) and (3.2) hold. Moreover, if $\gamma\lambda\geq 1$, it is easy to see that

$$\lim_{t\to\infty}\frac{g^2(t)\int_t^{\infty}\frac{1}{b(s)}\int_s^{\infty}p(u)\frac{1}{\mathrm{e}^{\gamma\lambda u}}\mathrm{d}u\mathrm{d}s}{a(t)g'(t)}=0<1.$$

Hence, for $\gamma\lambda\geq 1$, we have given a necessary and sufficient condition which guarantees that equation (4.3) has an eventually positive solution $x(t)\in\left(0,\mathrm{e}^{-t}\right]$.

In summary, the equation (4.3) oscillates or has at least a nonoscillatory solution $x(t)\in\left(0,\mathrm{e}^{-t}\right]$.

Example 4.4 Let $\mathbb{T}=q^{\mathbb{N}},\ q>1,\ \gamma>3>\gamma_1+1>2$. Consider the third-order delay dynamic equation

$$x^{\Delta\Delta\Delta}(t)+\frac{1}{t^{\gamma_1}}|x(t)|^{\gamma-1}x(t)=0,\tag{4.4}$$

where

$$a(t)=1,\ b(t)=1,\ p(t)=\frac{1}{t^{\gamma_1}},\ \tau(t)=t,\ a^{\Delta}(t)=0.$$

It is easy to see that (A1), (A2), (2.2) and (2.5) hold.

Set that $\alpha(t)=0,\ \delta(t)=t$, and define that H: $H_1(t,t)=0,\ t\geq t_0,\ H_1(t,s)=1,\ t>s\geq t_0$. Then we have that

$$\begin{aligned}&\delta^{\Delta}(t)=1,\ C^{**}(t,T)=\gamma k^{\gamma-1}q^{-\gamma}t(t-T),\ \theta^{**}(t)=\frac{1}{q^{\gamma}t^{\gamma_1-1}},\\&\theta_1^{**}(t)=\frac{1}{qt},\ \theta_2^{**}(t)=\frac{C^{**}(t,T)}{q^2t^2},\end{aligned}$$

and

$$H_1^{\Delta_s}(t,s)=\begin{cases}\dfrac{-q}{(q-1)t}<0, & s=\rho(t)<t;\\ 0, & t_1\le s<\rho(t)<t,\end{cases}$$

$$E(t,s)=\begin{cases}-\dfrac{1}{(q-1)t}<0, & s=\rho(t)<t;\\ \theta_1^{**}(s), & t_1\le s<\rho(t)<t.\end{cases}$$

A simple calculation shows that the condition (3.36) becomes

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H_1(t,t_1)}\int_{t_1}^{t}H_1(t,s)\left[\theta^{**}(s)-\frac{E_+^2(t,s)}{4\theta_2^{**}(s)}\right]\Delta s\\ =&\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{1}{q^{\gamma}s^{\gamma_1-1}}-\frac{E_+^2(t,s)q^2s^2}{4C^{**}(s,T)}\right]\Delta s\\ =&\limsup_{t\to\infty}\left(\int_{t_1}^{\rho(t)}\left[\frac{1}{q^{\gamma}s^{\gamma_1-1}}-\frac{1}{4C^{**}(s,T)}\right]\Delta s+\int_{\rho(t)}^{t}\frac{1}{q^{\gamma}s^{\gamma_1-1}}\Delta s\right)\\ =&\limsup_{t\to\infty}\left(\int_{t_1}^{\rho(t)}\left[\frac{1}{q^{\gamma}s^{\gamma_1-1}}-\frac{1}{4C^{**}(s,T)}\right]\Delta s+\frac{q-1}{q^{\gamma+\gamma_1-2}}t^{2-\gamma_1}\right)=\infty.\end{aligned}$$

Thus, condition (3.38) is satisfied, so every solution of equation (4.4) oscillates or converges to zero, by Theorem 3.7.

Conflict of Interest The authors declare no conflict of interest.

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