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THREE KINDS OF DENTABILITIES IN BANACH SPACES AND THEIR APPLICATIONS

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Abstract In this paper, we study some dentabilities in Banach spaces which are closely related to the famous Radon-Nikodym property. We introduce the concepts of the weak*-weak denting point and the weak*-weak* denting point of a set. These are the generalizations of the weak* denting point of a set in a dual Banach space. By use of the weak*-weak denting point, we characterize the very smooth space, the point of weak*-weak continuity, and the extreme point of a unit ball in a dual Banach space. Meanwhile, we also characterize an approximatively weak compact Chebyshev set in dual Banach spaces. Moreover, we define the nearly weak dentability in Banach spaces, which is a generalization of near dentability. We prove the necessary and sufficient conditions of the reflexivity by nearly weak dentability. We also obtain that nearly weak dentability is equivalent to both the approximatively weak compactness of Banach spaces and the w-strong proximinality of every closed convex subset of Banach spaces.

Key words weak*-weak denting point; nearly weak dentability; very smooth space; point of weak*-weak continuity; extreme point; approximatively weak compactness; w-strong proximinality; reflexivity

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1 Introduction

In 1966, Rieffel [1] introduced the concept of the dentability of a set in Banach spaces, which obviously has a geometric feature . He proved that every bounded sets of a Banach space are dentable implies the fact that the Banach space has the famous Radon-Nikodym property (RNP for short). The definition of the RNP of Banach spaces can be found in [1]. The concept of dentability builts bridges between the geometric theory and the analysis theory of Banach spaces. Therefore, research on the geometric direction of RNP has drawn the attention of many scholars in functional analysis. A variety of forms of dentabilities have been defined and studied; these include the weak* denting point, the denting point, near dentability and the σ-dentable set, etc.. This paper continues carrying out the research in this direction.

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First, we present some symbols and concepts.

Let X be a Banach space and let X^* be its dual space. $S(X)$ and $B(X)$ are denoted as the unit sphere and the unit ball, respectively. Letting $x \in S(X)$ and $\alpha \in \mathbb{R}$, we denote the weak* half-space $\{x^* \in X^* : x^*(x) \leq \alpha\}$ by $K_{x,\alpha}$, $B[x,\gamma] = \{y \in X : \|y-x\| \leq \gamma\}$, $B(x,\gamma) = \{y \in X : \|y-x\| < \gamma\}$. For $f \in S(X^*)$, we set that $A_f = \{x \in S(X) : f(x) = 1\}$. $\widehat{x}$ denotes the natural embedding of x to X^{**}. w^*, w, and $\|\cdot\|$ stand for the weak*, weak and norm topologies, respectively. We denote the norm-attaining functional in $S(X^*)$ by $NA(X)$. For a subset A^* in X^*(resp. A in X), $\overline{\text{co}}^{w^*}A^*$ (resp. $\overline{\text{co}}A$) denotes the weak* closed convex (resp. closed convex) hull. We denote the weak* interiors of A^* by w^*-int A^*.

For a subset $C \subset X$, the set-valued mapping $P_C : X \to 2^X$ is said to be the metric projection $P_C(x) = \{z \in C : \|x-z\| = d(x,C)\}$, where $d(x,C) = \inf_{y\in C}\|x-y\|$. $C \subset X$ is said to be proximinal if $P_C(x) \neq \emptyset$ for all $x \in X$. C is said to be Chebyshev if $P_C(x)$ contains only one point for all $x \in X$.

A subset C of X is said to be approximatively compact (resp. approximatively weakly compact) if, for every $x \in X$ and for every minimizing sequence $\{x_n\}_{n=1}^\infty$ in C regarding x (i.e. $\|x - x_n\| \to d(x,C)$), $\{x_n\}_{n=1}^\infty$ has a subsequence that converges to an element in C (resp. weakly converges to an element in C). A subset C^* of X^* is said to be approximatively weakly* compact if, for every $x^* \in C^*$ and for every minimizing sequence $\{x_n^*\}_{n=1}^\infty$ regarding x^* (i.e. $\|x^* - x_n^*\| \to d(x^*,C)$), $\{x_n^*\}_{n=1}^\infty$ has a subsequence that weakly* converges to an element in C^*. A Banach space X is said to be approximatively compact (resp. approximatively weakly compact) if every non-empty closed convex subset of X is approximatively compact (resp. approximatively weakly compact); see [2–4].

Next, we recall the weak* denting point of a set and the near dentability of a Banach space and some related properties.

Let C^* be a bounded set in X^*. An element $x^* \in C^*$ is said to be a weak* denting point of C^* if $x^* \notin \overline{\text{co}}^{w^*}(C^* \setminus B(x^*,\varepsilon))$ for any $\varepsilon > 0$; see [5, 6].

A Banach space X is said to be nearly dentable if $A_f \neq \emptyset$ and $A_f \cap \overline{\text{co}}(B(X) \setminus U_{A_f}) = \emptyset$ for any $f \in S(X^*)$ and any open set $U_{A_f} \supset A_f$; see [7].

A Banach space X is said to be strongly smooth (resp. very smooth, smooth) if, for any $\{x_n^*\}_{n=1}^\infty \subset S(X^*)$, $x \in S(X)$ with $x_n^*(x) \to 1$, $\{x_n^*\}_{n=1}^\infty$ is convergent (resp. $\{x_n^*\}_{n=1}^\infty$ is weakly convergent, $\{x_n^*\}_{n=1}^\infty$ is weakly* convergent); see [6].

It is well known that a Banach space X is smooth iff, for any $x \in S(X)$, there exists a unique functional $f \in S(X^*)$ such that $f(x) = \|x\|$. An element $x \in C \subset X$ is said to be an extreme point of C if $2x = y + z$ for some $y, z \in C$, so that $y = z$.

An element $x^* \in S(X^*)$ is called a point of weak*-weak continuity ((w^*-w) PC for short) (resp. point of weak*-norm continuity ((w^*-$\|\cdot\|$) PC for short)) of $B(X^*)$ if, for any net $\{x_\alpha^*, \alpha \in D\} \subset B(X^*)$, $x_\alpha^* \xrightarrow{w^*} x^*$ implies that $x_\alpha^* \xrightarrow{w} x^*$ (resp. $x_\alpha^* \to x^*$); see [8].

In 2015, Zhang, Zhou and Liu [9] drew the following conclusion:

Theorem 1.1 Let X be a Banach space. Then the following statements are equivalent:

(1) For any weak* closed convex set A^* in X^* with w^*-int$A^* \neq \emptyset$, A^* is an approximatively compact Chebyshev set.

(2) X is strongly smooth.

(3) If $x^* \in S(X^*)$ attains its norm on $S(X)$, then x^* is a (w^*-$\|\cdot\|$) PC and an extreme point of $B(X^*)$.

(4) If $x^* \in S(X^*)$ attains its norm on $S(X)$, then x^* is a weak* denting point of $B(X^*)$.

In 2011, Shang, Cui and Fu [7] defined the concept of nearly dentable Banach spaces, and proved the following result:

Theorem 1.2 A Banach space X is approximatively compact iff it is nearly dentable and nearly strictly convex.

Furthermore, the authors of [9] showed that the condition regarding the nearly strict convexity of Theorem 1.2 can be removed; they also gave the following main theorem:

Theorem 1.3 A Banach space X is nearly dentable iff it is approximatively compact.

Now, we introduce three kinds of dentabilities in Banach spaces.

Definition 1.4 Let X^* be the dual space of a Banach space X and let C^* be a bounded subset of X^*. An element $x^* \in C^*$ is said to be a weak*-weak denting point (resp. weak*-weak* denting point) of C^* if, for any weak (resp. weak*) neighborhood V^* of the origin in X^*, $x^* \notin \overline{\text{co}}^{w^*}(C^* \setminus (x^* + V^*))$. A bounded subset C^* of X^* is said to be weak*-weak (resp. weak*-weak*) dentable if, for any weak (resp. weak*) neighborhood V^* of the origin in X^*, there exists an element $x^*_{V^*} \in C^*$ such that $x^*_{V^*} \notin \overline{\text{co}}^{w^*}(C^* \setminus (x^*_{V^*} + V^*))$.

Definition 1.5 A Banach space X is said to be nearly weakly dentable if $A_f \neq \emptyset$ and $A_f \bigcap \overline{\text{co}}(B(X) \setminus {}_wU_{A_f}) = \emptyset$ for any $f \in S(X^*)$ and any weakly open set ${}_wU_{A_f} \supset A_f$.

Remark 1.6 (1) Obviously, a weak* denting point implies a weak*-weak denting point, a weak*-weak denting point implies a weak*-weak* denting point, and near dentability implies nearly weak dentability.

(2) By the James theorem, it is trivial to see that nearly weak dentability implies reflexivity.

The structure of the rest of this paper is as follows: in Section 1, we defined three kinds of dentability of Banach spaces. In Section 2, we prove that if $x^* \in S(X)$ attains its norm on $S(X)$, then x^* is a weak*-weak denting point of $B(X^*)$ which is equivalent to the fact that X is a very smooth space, which in turn is equivalent to the fact that x^* is a (w^*-w) PC and an extreme point of $B(X^*)$. We characterize an approximatively weak compact Chebyshev set in dual Banach spaces by a weak*-weak denting point. We obtain analogous conclusions for a weak*-weak* denting point as well. In Section 3, we show that X is nearly weakly dentable if and only if it is reflexive. Furthermore, we obtain that nearly weak dentability is equivalent to both the approximatively weak compactness of X and the w-strong proximinality of every closed convex subset in X.

2 Weak*-Weak Dentability

First of all, we present the main theorem of this section.

Theorem 2.1 Let X be a Banach space. Then the following statements are equivalent:

(1) If $x^* \in S(X^*)$ attains its norm on $S(X)$, then x^* is a weak*-weak denting point of $B(X^*)$.

(2) If $x^* \in S(X^*)$ attains its norm on $S(X)$, then x^* is a (w^*-w) PC and an extreme point

of $B(X^*)$.

(3) X is a very smooth space.

Proof (1) $\Rightarrow$ (2). Let $\{x^*_\alpha, \alpha\in D\}\subset B(X^*)$, $x^*_0\in S(X^*)$, $x_0\in S(X)$ and $x^*_0(x_0)=1$, and $x^*_\alpha \xrightarrow{w^*} x^*_0$. If $x^*_\alpha \not\xrightarrow{w} x^*_0$, then there exists a weak neighborhood of the origin V^* such that the following claim holds: for any $\alpha\in D$, there exists $\beta>\alpha$ such that

$$x^*_\beta \notin x^*_0 + V^*.$$

In this way, we obtain a subnet $\{x^*_\beta : \beta\in D'\}$ of $\{x^*_\alpha : \alpha\in D\}$. Clearly, $\{x^*_\beta : \beta\in D'\}\cap(x^*_0+V^*)=\emptyset$, and thus $\{x^*_\beta : \beta\in D'\}\subset B(X^*)\setminus(x^*_0+V^*)$. Since x^*_0 is a weak*-weak denting point, we have that $x^*_0\notin \overline{\mathrm{co}}^{w^*}(B(X^*)\setminus(x^*_0+V^*))$. By the separation theorem of a locally convex space and $(X^*,w^*)^*=X$, there exists an element $y\in X$ such that

$$\widehat{y}(x^*_0) > \sup\{\widehat{y}(x^*) : x^*\in \overline{\mathrm{co}}^{w^*}(B(X^*)\setminus(x^*_0+V^*))\}.$$

Therefore, there exists a constant $\gamma>0$ such that

$$\widehat{y}(x^*_0) - \sup\{\widehat{y}(x^*) : x^*\in \overline{\mathrm{co}}^{w^*}(B(X^*)\setminus(x^*_0+V^*))\} > \gamma.$$

It follows that $x^*_0(y)-x^*_\beta(y)>\gamma$ for any β, which shows that $x^*_\beta \not\xrightarrow{w^*} x^*_0$. This leads to a contradiction with $x^*_\alpha \xrightarrow{w^*} x^*_0$. Therefore, $x^*_\alpha \xrightarrow{w} x^*_0$; that is, x^*_0 is a (w^*-w) PC of $B(X^*)$.

If x^*_0 is not an extreme point of $B(X^*)$, there exist $x^*_1, x^*_2\in B(X^*)$ such that $x^*_0=\frac{x^*_1+x^*_2}{2}$, but $x^*_1\neq x^*_2$. Therefore there exists an element $x\in X$ such that $x^*_1(x)\neq x^*_2(x)$. Since $(\frac{x^*_1+x^*_2}{2})(x_0)=x^*_0(x_0)=1$, $\frac{x^*_1+x^*_2}{2}\in S(X^*)$. On the one hand, by statement (1), we have that $\frac{x^*_1+x^*_2}{2}$ is a weak*-weak denting point of $B(X^*)$. On the other hand, we can let $V^*_1=\{x^*\in X^* : |x^*(x)|<\frac{|x^*_1(x)-x^*_2(x)|}{3}\}$. Note that V^*_1 is a weak* neighborhood of the origin in X^*, which means that $U^*_1=\frac{x^*_1+x^*_2}{2}+V^*_1$ is a weak* neighborhood of $\frac{x^*_1+x^*_2}{2}$. Since

$$|x^*_1(x)-\frac{x^*_1(x)+x^*_2(x)}{2}| = \frac{|x^*_1(x)-x^*_2(x)|}{2} = |x^*_2(x)-\frac{x^*_1(x)+x^*_2(x)}{2}|,$$

we obtain that $x^*_1, x^*_2\notin U^*_1$. Since weak topology is not weaker than weak* topology in X^*, there exists a weak neighborhood of the origin $V^*_0\subset V^*_1$ such that $U^*_0=\frac{x^*_1+x^*_2}{2}+V^*_0$ is a weak neighborhood of $\frac{x^*_1+x^*_2}{2}$. According to $V^*_0\subset V^*_1$, we have that $x^*_1, x^*_2\notin U^*_0$. Due to

$$x^*_1, x^*_2 \in B(X^*)\setminus U^*_0 = B(X^*)\setminus(\frac{x^*_1+x^*_2}{2}+V^*_0),$$

it follows that

$$\frac{x^*_1+x^*_2}{2}\in \overline{\mathrm{co}}(B(X^*)\setminus(\frac{x^*_1+x^*_2}{2}+V^*_0))\subset \overline{\mathrm{co}}^{w^*}(B(X^*)\setminus(\frac{x^*_1+x^*_2}{2}+V^*_0)).$$

This shows that $\frac{x^*_1+x^*_2}{2}$ is not a weak*-weak denting point of $B(X^*)$, which leads to a contradiction.

(2) $\Rightarrow$ (3). Let $\{x^*_n\}_{n=1}^\infty\subset S(X^*)$, $x\in S(X)$ be such that $x^*_n(x)\to 1$. Since $B(X^*)$ is weak* compact, there exists x^* and a subnet $\{x^*_\alpha, \alpha\in D\}$ of $\{x^*_n\}_{n=1}^\infty$ such that $x_\alpha \xrightarrow{w^*} x^*$. Since $x^*_\alpha(x)\to 1$, $x^*(x)=1$. By statement (2), we have that $x^*_\alpha \xrightarrow{w} x^*$. This shows that $\{x^*_n\}_{n=1}^\infty$ is relatively weakly compact. For any $x\in S(X)$, $x^*_1, x^*_2\in S(X^*)$ such that $x^*_1(x)=x^*_2(x)=1$. Setting $x^*=\frac{x^*_1+x^*_2}{2}$, $x^*(x)=1$. By statement (2), x^* is an extreme point of $B(X^*)$, and it follows that $x^*_1=x^*_2$. This implies that X is smooth. Thus, the weak cluster point of $\{x^*_n\}_{n=1}^\infty$ is unique, which means that $x_n \xrightarrow{w} x^*$. This shows that X is very smooth.

(3) $\Rightarrow$ (1). Let $x_0^* \in S(X^*), x_0 \in S(X)$ be such that $x_0^*(x_0) = 1$. We will prove that, for any weak neighborhood V^* of the origin in X^*, there exists a constant $\alpha > 0$ such that

$$x_0^*(x_0) - \alpha > \sup\{y^*(x_0) : y^* \in B(X^*) \setminus (x_0^* + V^*)\}.$$

Otherwise, there would exist $\{y_n^*\}_{n=1}^\infty \subset B(X^*) \setminus (x_0^* + V^*)$ such that $y_n^*(x_0) \to x_0^*(x_0) = 1$. Since X is very smooth, we know that $y_n^* \xrightarrow{w} x_0^*$, which leads to a contradiction with the fact that $y_n^* \in B(X^*) \setminus (x_0^* + V^*)$. According to

$$\begin{aligned} x_0^*(x_0) - \alpha &\geq \sup\{y^*(x_0) : y^* \in B(X^*) \setminus (x_0^* + V^*)\} \\ &= \sup\{y^*(x_0) : y^* \in \mathrm{co}(B(X^*) \setminus (x_0^* + V^*))\} \\ &= \sup\{y^*(x_0) : y^* \in \overline{\mathrm{co}}^{w^*}(B(X^*) \setminus (x_0^* + V^*))\}, \end{aligned}$$

we get that $x_0^* \notin \overline{\mathrm{co}}^{w^*}(B(X^*) \setminus (x_0^* + V^*)))$. Consequently, x_0^* is a weak*-weak denting point of $B(X^*)$. □

As is well-known, the very smoothness of X implies the RNP of X^*. Therefore, we have the following result:

Corollary 2.2 Suppose that every $x^* \in NA(X)$ is weak*-weak denting point. Then X^* has RNP.

Similarly to the proof of Theorem 2.1, we can obtain the following result:

Theorem 2.3 Let X be a Banach space. Then the following statements are equivalent:

(1) If $x^* \in S(X^*)$ attains its norm on $S(X)$, then x^* is a weak*-weak* denting point of $B(X^*)$.

(2) If $x^* \in S(X^*)$ attains its norm on $S(X)$, then x^* is an extreme point of $B(X^*)$.

(3) X is a smooth space.

In [10], Zhang and Liu proved the following formula of distance from a point $x^*(\notin K_{x,\alpha})$ to the weak* half-space $K_{x,\alpha}$:

Lemma 2.4 ([10]) Let X be a Banach space, let $x_0 \in X \setminus \{0\}$ and let $\alpha \in \mathbb{R}$. Then, for any $x^* \in X^* \setminus K_{x_0,\alpha}$,

$$d(x^*, K_{x_0,\alpha}) = \frac{x^*(x_0) - \alpha}{\|x_0\|}.$$

The following theorem establishes the connection between a weak*-weak denting point and an approximatively weakly compact Chebyshev set, which shows that the weak*-weak denting point has important applications for approximation theory:

Theorem 2.5 Let X be a Banach space. The following statements are equivalent:

(1) If $x^* \in S(X^*)$ attains its norm on $S(X)$, then x^* is a weak*-weak denting point of $B(X^*)$.

(2) For any weak* closed convex set A^* in X^* with w^*-int $A^* \neq \emptyset$, A^* is an approximatively weakly compact Chebyshev set.

(3) For any $x \in X$ and $\alpha \in \mathbb{R}$, the weak* half-space $K_{x,\alpha}$ is an approximatively weakly compact Chebyshev set.

Proof (1)$\Rightarrow$(2) can be derived from Theorem 2.1 and [9, Theorem 3.2]. For the sake of completeness, we give a detailed proof as follows: let A^* be a weak* closed convex set with w^*-int$A^* \neq \emptyset$. Without loss of generality, we may assume that $x = 0$. Suppose that $\{y_n^*\}_{n=1}^\infty \subset A^*$

such that $\|0-y_n^*\| \to d(0,A^*)=\gamma$. Taking $y_0^*\in P_{A^*}(0)$, $\|y_0^*\|=\gamma$. Since $B[0,\gamma]\cap w^*\text{-int}A^*=\emptyset$, by the separation theorem of locally convex space and $(X^*,w^*)^*=X$, there exists an element $x\in S(X)$ such that

$$\sup\{x(y^*):y^*\in A^*\}\le \inf\{x(y^*):y^*\in B[0,\gamma]\}=-\|x\|\gamma=-\|x\|\|y_0^*\|.$$

As $y_0^*\in P_{A^*}(0)$, we get that

$$-\|x\|\|y_0^*\|\le x(y_0^*)\le \sup\{x(y^*):y^*\in A^*\}\le \inf\{x(y^*):y^*\in B[0,\gamma]\}=-\|x\|\|y_0^*\|.$$

It follows that $x(y_0^*)\ge x(y_n^*)$. Consequently,

$$\|0-y_0^*\|=x(0-y_0^*)\le x(0-y_n^*)\le \|0-y_n^*\|\to d(0,A^*)=\|y_0^*\|.$$

We derive that $x(\frac{y_n^*}{\|y_0^*\|})\to 1$. Since $\|y_n^*\|\to\|y_0^*\|=d(0,A^*)$, $x(\frac{y_n^*}{\|y_n^*\|})\to 1$. Due to statement (1) and Theorem 2.1, we have that X is very smooth, which implies that $\{y_n^*\}_{n=1}^\infty$ is weakly convergent. This means that A^* is approximatively weakly compact.

If A^* is not a Chebyshev set, then there exist $x^*\in X^*$ and $y_1^*,y_2^*\in A^*$ such that $\|x^*-y_1^*\|=\|x^*-y_2^*\|=d(x^*,A^*)$. Without loss of generality, we assume that $x^*=0$ and $d(0,A^*)=1$. Clearly, $B(X^*)\cap w^*\text{-int}A^*=\emptyset$. Then, according to the separation theorem in locally convex space and, the fact that $(X^*,w^*)^*=X$, there exists an element $x\in S(X)$ such that

$$\sup\{x(x^*):x^*\in B(X^*)\}\le \inf\{x(x^*):x^*\in A^*\}.$$

Then, for any $i=1,2$,

$$x(y_i^*)\le \sup\{x(x^*):x^*\in B(X^*)\}=1\le \inf\{x(x^*):x^*\in A^*\}\le x(y_i^*).$$

Therefore, $x(y_i^*)=1, i=1,2$. This stands in contradiction to the smoothness of X.

(2)⇒(3). For any $x\in X$ and $\alpha\in\mathbb{R}$, we observe that the weak* half-space $K_{x,\alpha}$ is a weakly* closed convex subset with w^*- int $K_{x,\alpha}\ne\emptyset$, which shows that statement (3) is true, by statement (2).

(3)⇒(1). Let $x_0^*\in S(X^*)$, $x_0\in S(X)$ be such that $x_0^*(x_0)=1$. We will prove that, for any weak neighborhood V^* of the origin, there exists an $\alpha>0$ such that

$$x_0^*(x_0)-\alpha>\sup\{y^*(x_0):y^*\in B(X^*)\setminus(x_0^*+V^*)\}.$$

Otherwise, there would exist a sequence $\{y_n^*\}_{n=1}^\infty\subset B(X^*)\setminus(x_0^*+V^*)$ such that $y_n^*(x_0)\to x_0^*(x_0)=1$. Consider the weak* half-space $K_{x_0,-1}=\{x^*\in X^*, x^*(x_0)\le -1\}$. By statement (3), $K_{x_0,-1}$ is approximatively weakly compact, which implies that $K_{x_0,-1}$ is proximinal. We may assume that $y_n^*(x_0)\le 1$. It follows that there exists a sequence $\{z_n^*\}_{n=1}^\infty\subset K_{x_0,-1}$ such that $\|-y_n^*-z_n^*\|=d(-y_n^*,K_{x_0,-1})$. By Lemma 2.4,

$$\|-y_n^*-z_n^*\|=d(-y_n^*,K_{x_0,-1})=|-y_n^*(x_0)+1|\to 0.$$

By $\|y_n^*\|\to 1$ and $1\le\|z_n^*\|\le\|y_n^*\|+\|-y_n^*-z_n^*\|\to 1$, we obtain that $\|z_n^*-0\|\to 1=d(0,K_{x_0,-1})$. Due to the approximatively weak compactness of $K_{x_0,-1}$, we know that $\{z_n^*\}_{n=1}^\infty$ is relatively weakly compact. Hence, $\{y_n^*\}_{n=1}^\infty$ is also relatively weakly compact. Without loss of generality, we assume that $y_n^*\xrightarrow{w} y_0^*$. Obviously, $y_0^*(x_0)=1$. We claim that $y_0^*=x_0^*$. Indeed, we have that $-y_0^*,-x_0^*\in K_{x_0,-1}$ and $\|y_0^*\|=\|x_0^*\|=1=d(0,K_{x_0,-1})$, so therefore,

$-y_0^*, -x_0^* \in P_{K_{x_0,-1}}(0)$. Since $K_{x_0,-1}$ is a Chebyshev set, we get that $-y_0^* = -x_0^*$. This shows that $y_n^* \xrightarrow{w} x_0^*$, which leads to a contradiction with $\{y_n^*\}_{n=1}^{\infty} \subset B(X) \setminus (x_0^* + V^*)$. Due to

$$\begin{aligned} x_0^*(x_0) - \alpha &\geq \sup\{y^*(x_0) : y^* \in B(X^*) \setminus (x_0^* + V^*)\} \\ &= \sup\{y^*(x_0) : y^* \in \mathrm{co}(B(X^*) \setminus (x_0^* + V^*))\} \\ &= \sup\{y^*(x_0) : y^* \in \overline{\mathrm{co}}^{w^*}(B(X^*) \setminus (x_0^* + V^*))\}, \end{aligned}$$

we get that $x_0^* \notin \overline{\mathrm{co}^{w^*}}(B(X^*) \setminus (x_0^* + V^*))$. Consequently, x_0^* is a weak*-weak denting point of $B(X^*)$. □

Similarly to the proof of Theorem 2.5, we can deduce the following result:

Theorem 2.6 Let X be a Banach space. Then the following statements are equivalent:

(1) If $x^* \in S(X^*)$ attains its norm on $S(X^*)$, then x^* is a weak*-weak* denting point of $B(X^*)$.

(2) For any weakly* closed convex set A^* of X^* with w^*-int $A^* \neq \emptyset$, A^* is an approximatively weakly* compact Chebyshev set.

(3) For any $x \in X$ and $\alpha \in \mathbb{R}$, the weak* half-space $K_{x,\alpha}$ is an approximatively weakly* compact Chebyshev set.

To conclude Section 2, we will give a necessary and sufficient condition regarding a weak*-weak dentable set in X^*.

Theorem 2.7 Let C^* be a bounded weakly* closed subset in X^*. Then C^* is weak*-weak dentable set if and only if, for any weak neighborhood V^* of the origin in X^*, there exists a weak* slice $S(x, \alpha, C^*)$ of C^* determined by x, α, where $x \subset X, \alpha > 0$ and $x^* \in S(x, \alpha, C^*)$ such that $S(x, \alpha, C^*) \subset x^* + V^*$, where $S(x, \alpha, C^*) = \{y^* \in C^* : y^*(x) > \sup x(C^*) - \alpha\}$.

Proof Sufficiency. Let V^* be a weak neighborhood of the origin in X^*. By assumption, there exists a weak* slice $S(x, \alpha, C^*)$ and $x_0^* \in S(x, \alpha, C^*)$ such that $S(x, \alpha, C^*) \subset x_0^* + V^*$. Therefore, $x_0^*(x) > \sup x(C^*) - \alpha$. Notice that the set $\{y^* \in C^* : y^*(x) \leq \sup x(C^*) - \alpha\}$ is a weakly* closed convex subset in X^*, and that

$$C^* \setminus (x_0^* + V^*) \subset C^* \setminus S(x, \alpha, C^*) = \{y^* \in C^* : y^*(x) \leq \sup x(C^*) - \alpha\},$$

which means that

$$\overline{\mathrm{co}}^{w^*}(C^* \setminus (x_0^* + V^*)) \subset \{y^* \in C^* : y^*(x) \leq \sup x(C^*) - \alpha\}.$$

Therefore, $x_0^* \notin \overline{\mathrm{co}}^{w^*}(C^* \setminus (x_0^* + V^*))$, which implies that C^* is weak*-weak dentable.

Necessity. Suppose that C^* is weakly*-weak dentable, and that V^* is a weak neighborhood of the origin in X^*. Then there exists a $x_0^* \in C^*$ such that $x_0^* \notin \overline{\mathrm{co}}^{w^*}(C^* \setminus (x_0^* + V^*))$. According to the seperation theorem of a locally convex space and $(X^*, w^*)^* = X$, there exist $x_0 \in X$ and $r > 0$ such that

$$x_0^*(x_0) - r > \sup \widehat{x_0}(\overline{\mathrm{co}}^{w^*}(C^* \setminus (x_0^* + V^*))). \tag{2.1}$$

Let $\alpha = \sup \widehat{x_0}(C^*) - (x_0^*(x_0) - r)$, $\alpha > 0$. Then

$$x_0(x_0^*) = \sup \widehat{x_0}(C^*) + r - \alpha > \sup \widehat{x_0}(C^*) - \alpha,$$

which implies that $x_0^* \in S(x_0, \alpha, C^*)$, since, for any $y^* \in S(x_0, \alpha, C^*)$, we have that

$$\widehat{x_0}(y^*) > \sup x_0(C^*) - \alpha = x^*(x_0) - r. \tag{2.2}$$

We claim that $y^* \in x_0^* + V^*$. Otherwise,

$$y^* \in C^* \setminus (x_0^* + V^*) \subset \overline{\text{co}}^{w^*}(C^* \setminus (x_0^* + V^*)).$$

Due to (2.1), $x_0^*(x_0) - r > \widehat{x_0}(y^*)$, which stands in contradiction to (2.2). This shows that $S(x_0, \alpha, C^*) \subset x_0^* + V^*$. □

Remark 2.8 When the weak neighborhood V^* of the origin is replaced by a weak* neighborhood of the origin in Theorem 2.7, by using a similar method to that above, we can get an analogous conclusion for a weak*-weak* dentable set.

3 Nearly Weak Dentability

In 2001, Godefroy and Indumathi [11] introduced the concept of the strong proximinality of a subspace in Banach space. In 2008, Bandyopadhyay et al. [12] introduced the strong proximinality and the w-strong proximinality of a set, and generalized the strong proximinality of a subspace to the general subset of a Banach space.

A subset C of X is said to be strongly proximinal (resp. w-strongly proximinal) if C is proximinal, and if, for any $x \in X$ and norm neighborhood V (resp. weak neighborhood V), there exists a constant $\delta > 0$ such that

$$P_C(x, \delta) \subseteq P_C(x) + V,$$

where $P_C(x, \delta) = \{y \in C : \|y - x\| < d(x, C) + \delta\}$.

By [10, Theorem 2.2] and the definitions of strong and w-strong proximinality, the following relation is true:

$$\begin{array}{ccccc} \text{Approximinatively compact} & \rightarrow & \text{Strongly proximinal} & \searrow & \\ \downarrow & & \downarrow & & \text{Proximinal} \\ \text{Approximinatively weakly compact} & \rightarrow & w\text{-strongly proximinal} & \nearrow & \end{array}$$

In [13], Zhang, Liu and Zhou gave four counterexamples to show that the above connection is not converse.

Dutta and Shunmugaraj [14] put forward the following question:

Question 3.1 What are the conditions (necessary or sufficient) that make every closed convex subset of X strongly proximinal?

In order to present their main result, we need a number of concepts; see [14].

Let X be a Banach space. The norm $\|\cdot\|$ is said to be strongly subdifferentiable (in short SSD) at $x \in X$ if

$$\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}$$

is uniform for $h \in S_X$. If the norm $\|\cdot\|$ of X is SSD at all points of S_X, the space X is said to be SSD.

A Banach space X is said to have the (KK) property if $x_n \xrightarrow{w} x$ implies $x_n \to x$ for any sequence $\{x_n\}_{n=1}^{\infty} \subset S(X)$ and $x \in S(X)$.

Dutta and Shunmugaraj then provided their main conclusion, which is as follows:

Theorem 3.2 Let X be a Banach space. Then the following statements are equivalent:

(1) X^* is SSD and A_f is compact for every $f \in S(X)$.

(2) X is reflexive and (KK).

(3) X is approximatively compact.

(4) Every closed convex subset of X is strongly proximinal.

Similarly to Question 3.1, we now come to the following question:

Question 3.3 What are the conditions that make every closed convex subset of X w-strongly proximinal?

Now we establish the main result of this section.

Theorem 3.4 Let X be a Banach space. Then the following statements are equivalent:

(1) X is nearly weakly dentable.

(2) X is reflexive.

(3) X is approximatively weakly compact.

(4) Every closed convex subset of X is w-strongly proximinal.

(5) Every closed convex subset of X is proximinal.

Proof (2)⇔(3)⇔(4)⇔(5) can be deduced from [13, Theorem 1.1].

(1)⇒(2) is evident.

(3)⇒(1). For any $f \in S(X^*)$ and any weakly open set ${}_wU_{A_f} \supset A_f$, we claim that there exists a constant $\alpha > 0$ such that $1 > f(y) + \alpha$ for any $y \in B(X) \setminus {}_wU_{A_f}$, where ${}_wU_{A_f} \supset A_f$. Otherwise, there would exist a sequence $\{z_n\}_{n=1}^{\infty} \subset B(X) \setminus {}_wU_{A_f}$ such that $f(z_n) \to 1$, which would imply that $\|z_n\| \to 1$. By the statement (3), we know that the hyperplane $H = \{x \in X : f(x) = 1\}$ is proximinal. Consequently, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset H$ such that

$$\|z_n - y_n\| = d(z_n, H) = |f(z_n) - 1| \to 0. \tag{3.1}$$

Since $\|z_n\| \to 1$, we have that

$$\|0 - y_n\| = \|y_n\| \to 1 = d(0, H).$$

Since X is approximatively weakly compact, $\{y_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence. Thus, $\{z_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence, by (3.1). Without loss of generality, we assume that $z_n \xrightarrow{w} z_0$. Obviously, $f(z_0) = 1$, i.e., $z_0 \in A_f$. Since ${}_wU_{A_f} \supset A_f$ is weakly open set, we know that $\{z_n\}_{n=1}^{\infty} \subset {}_wU_{A_f}$ for sufficiently large $n \in N$, which leads to a contradiction with the fact that $\{z_n\}_{n=1}^{\infty} \subset B(X) \setminus {}_wU_{A_f}$.

Next, we will show that X is nearly weakly dentable. Since, for any $y \in B(X) \setminus {}_wU_{A_f}$, $1 - \alpha > f(y)$, we deduce that, for any $x \in A_f$,

$$\begin{aligned} f(x) - \alpha &\geq \sup\{f(y) : y \in B(X) \setminus {}_wU_{A_f}\} \\ &= \sup\{f(y) : y \in \mathrm{co}(B(X) \setminus {}_wU_{A_f})\} \\ &= \sup\{f(y) : y \in \overline{\mathrm{co}}(B(X) \setminus {}_wU_{A_f})\}. \end{aligned}$$

It follows that $x \notin \overline{\mathrm{co}}(B(X) \setminus {}_wU_{A_f})$. Hence, $A_f \bigcap \overline{\mathrm{co}}(B(X) \setminus {}_wU_{A_f}) = \varnothing$. This shows that X is nearly weakly dentable. □

As is well known, reflexivity implies that both X and X^* have the RNP. Thus we have the following result:

Corollary 3.5 Suppose that X is nearly weakly dentable. Then both X and X^* have the RNP.

Conflict of Interest The authors declare that no conflict of interest.

References

[1] Rieffel M A. Dentable subsets of Banach spaces, with applications to Radon-Nikodym theorem//Proc Conf Functional Analysis. Washington, DC: Thompson Book Co, 1967: 71–77

[2] Jefimow N W, Stechkin S B. Approximate compactness and Chebyshev sets. Soviet Mathematics, 1961, **2**: 1226–1228

[3] Vlosov L R. Approximate property of sets in normed linear spaces. Russian Math Surveys, 1973, **28**: 1–66

[4] Hudzik H, Wang B X. Approximative compactness in Orlicz space. J Approx Theory, 1998, **95**: 82–89

[5] Diestel J. Geometry of Banach Spaces-Selected Topics. Berlin: Springer-Verlag, 1975

[6] Zhang Z H, Liu C Y, Zhou Y. Geometry of Banach Spaces and Best Approximation. Beijing: Science Press, 2016

[7] Shang S Q, Cui Y A, Fu Y Q. Nearly dentability and approximate compactness and continuity of metric projection in Banach spaces (in Chinese). Sci Sin Math, 2011, **41**(9): 815–825

[8] Bandyopadhyay P, Huang D, Lin B L. Rotund points, nested sequence of ball and smoothness in Banach space. Comment Math Prace Mat, 2004, **44**: 163–186

[9] Zhang Z H, Zhou Y, Liu C Y. Characterization of approximative compactness in Banach spaces (in Chinese). Sci Sin Math, 2015, **45**(12): 1953–1960

[10] Zhang Z H, Liu C Y. The representations and continuity of the metric projections on two class of half-spaces in Banach spaces. Abstr Appl Anal, 2014, **2014**: 908676

[11] Godefroy G, Indumathi V. Strong proximinality and polyhedral spaces. Rev Mat Complut, 2001, **14**(1): 105–125

[12] Bandyopadhyay P, Li Y, Lin B L. Proximinality in Banach spaces. J Math Anal Appl, 2008, **341**(1): 309–317

[13] Zhang Z H, Liu C Y, Zhou Y. Some examples concerning proximinality in Banach spaces. J Approx Theory, 2015, **200**: 136–143

[14] Dutta S, Shunmagaraj P. Strong proximinality of closed convex set. J Approx Theory, 2011, **163**(4): 547–553