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THE INTERIOR TRANSMISSION EIGENVALUE PROBLEM FOR AN ANISOTROPIC MEDIUM BY A PARTIALLY COATED BOUNDARY*

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Abstract We consider the interior transmission eigenvalue problem corresponding to the scattering for an anisotropic medium of the scalar Helmholtz equation in the case where the boundary $\partial\Omega$ is split into two disjoint parts and possesses different transmission conditions. Using the variational method, we obtain the well posedness of the interior transmission problem, which plays an important role in the proof of the discreteness of eigenvalues. Then we achieve the existence of an infinite discrete set of transmission eigenvalues provided that $n \equiv 1$, where a fourth order differential operator is applied. In the case of $n \not\equiv 1$, we show the discreteness of the transmission eigenvalues under restrictive assumptions by the analytic Fredholm theory and the T-coercive method.

Key words interior transmission eigenvalue; anisotropic medium; partially coated boundary; the analytic Fredholm theory; T-coercive method

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1 Introduction

Interior transmission eigenvalue problems have become an important area of research in inverse scattering theory. Now, we investigate the eigenvalue problem for an anisotropic medium by a partially coated boundary. Let Ω be a bounded simply connected open domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$ which is split into two parts Γ_1, Γ_2 and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let A be a 2×2 symmetric matrix-valued function with $L^\infty(\Omega)$ entries such that $\mathrm{Re}(\overline{\xi} \cdot A\xi) \geq \epsilon|\xi|^2$ and $\mathrm{Im}(\overline{\xi} \cdot A\xi) \leq 0$ for all $\xi \in \mathbb{C}^2$, a.e. $x \in \Omega$ and some constant $\epsilon > 0$. Also, $n \in L^\infty(\Omega)$ is a

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complex-valued scalar function such that $\mathrm{Re}(n) > 0$ and $\mathrm{Im}(n) \geq 0$. Then we can formulate the interior transmission problem as follows:

$$\begin{cases} \nabla \cdot A\nabla w + k^2 n w = 0, & \text{in} \quad \Omega, \\ \Delta v + k^2 v = 0, & \text{in} \quad \Omega, \\ \dfrac{\partial w}{\partial \nu_A} = \dfrac{\partial v}{\partial \nu} + \lambda v, & \text{on} \quad \Gamma_1, \\ \dfrac{\partial w}{\partial \nu_A} = \dfrac{\partial v}{\partial \nu}, & \text{on} \quad \Gamma_2, \\ w = v, & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Here $\frac{\partial w}{\partial \nu_A} := \nu \cdot A\nabla w$ denotes the co-normal derivative and ν denotes the unit outward normal. In this work, we will consider the case where $\lambda(x) \in L^\infty(\Gamma_1)$ is a real valued function satisfying that $\lambda \geq c$ (or $\lambda \leq -c$) for some positive constant c.

Definition 1.1 Values of $k \in \mathbb{C}$ for which the homogeneous interior transmission problem (1.1) has a nontrivial solution pair $w \in H^1(\Omega)$ and $v \in H^1(\Omega)$ are called transmission eigenvalues.

The transmission eigenvalue problem is a class of non-selfadjoint eigenvalue problems that first appeared in inverse scattering theory for an inhomogeneous medium, and it is a boundary value problem for a set of equations defined in a bounded domain coinciding with the support of the scattering object. Due to the theoretical importance of transmission eigenvalues in connection with the uniqueness and reconstruction results in inverse scattering theory, the study of transmission eigenvalue problems has recently become an attractive research topic.

The first issue associated with the interior transmission problem is concerned with the case of when the problem is well-posed. There are two main approaches: the boundary integral equation method and the variational method. For the sake of mathematical and computational interest, the boundary integral equation method has been applied to recover the solvability result of isotropic media when the refractive index n is a positive constant different from one (refer to Section 3.1.4 in the book [8], and to the paper [19, 25]). Since there are great limitations in terms of using the boundary integral equation method, the variational method has been widely used for both isotropic and anisotropic obstacles under different assumptions ([5, 6, 8, 11, 14, 26]).

The second issue concerns the discreteness of transmission eigenvalues that we can avoid them in the procedure to reconstruct the boundary from far-field or near-field pattern. Based on the spectral theory of compact operators in Hilbert spaces (see Theorem 6.8 in the book [4]), the discreteness of eigenvalues for an isotropic medium has been shown in [2, 6, 8]. When it comes to the anisotropic medium, we cannot construct a compact operator, so the analytic Fredholm theory (see Section 8.5 in the book [23]) is widely used. And there has been much work based on the analytic Fredholm theory under the case $\lambda = 0$ on Γ_1, which means that the boundary is integral (see [22, 23, 32] for the isotropic inhomogeneous medium, [7] for the anisotropic medium with $n \equiv 1$, [3, 5, 16] for general anisotropic media with $n \not\equiv 1$, [9] for regions with cavities, [10] for absorbing media, [12] for inhomogeneous media containing obstacles, [2, 28, 33] for the conductive boundary, etc).

The third and most difficult issue is whether there exist eigenvalues. The first result about the existence of transmission eigenvalues was obtained by Colton and Monk in 1989 for the case of a spherically stratified medium ([24]). Subsequently, McLaughliny used the knowledge of the transmission eigenvalues to determine a radial scatterer ([29, 30]). Until 2008, Päivärinta and Sylvester ([31]) considered the general isotropic inhomogeneous scatterer without radial symmetry and showed that there exist a finite number of transmission eigenvalues provided that the index of refraction is large enough. Soon after in 2009, Cakoni and Haddar ([15]) extended their ideas to present the existence of transmission eigenvalues for isotropic inhomogeneous media and also for anisotropic media with $n \equiv 1$. And in 2010, they studied a difficult case for regions with cavities ([9]). In the same year, the paper [13] proved the existence of an infinite discrete set of transmission eigenvalues for all of the above cases of the Helmholtz and Maxwell equations, and the paper [12] considered the case for inhomogeneous media containing obstacles. Furthermore, Cakoni and Kirsch ([17]) extended the investigation to the case of anisotropic media with $n \not\equiv 1$, where a fourth order differential operator is no longer applicable. Recently, more and more research has been done for more complicated media ([1, 2, 18, 20, 27, 28]).

In this paper, we consider an anisotropic scatterer whose boundary is split into two parts and possesses different boundary conditions (refer to the problem (1.1) for details). Given this situation, the boundary loses its symmetry and some essential Poincáre inequalities, which will be widely used in the proof, are difficult to obtain. Hence, we assume that $\lambda(x)$ is a real valued function satisfying that $\lambda \geq c$ or $\lambda \leq -c$ for some positive constant c. And we only obtain the existence of infinite discrete eigenvalues under the case $n \equiv 1$ with $a_* > 1$, $\lambda \leq -c$ or $a^* < 1$, $\lambda \geq c$ (Theorem 3.2 and Theorem 3.5). For the case $n \not\equiv 1$, we just show the discreteness result under the assumptions that $a_* > 1$, $n_* > 1$, $\lambda \leq -c$ or $0 < a_* < a^* < 1$, $0 < n_* < n^* < 1$, $\lambda \geq c$ (Theorem 4.2), and the existence is an open problem that will require further investigation.

The rest part of this paper is organized as follows: in Section 2, we will consider the well-posedness of the interior transmission problem by the variational method. Section 3 is devoted to the discreteness and existence of the transmission eigenvalues for $n \equiv 1$ where a fourth order differential operator is applied. In Section 4, we use the analytic Fredholm theory and the T-coercive method to investigate the discreteness of the transmission eigenvalues for $n \not\equiv 1$.

2 The Well-posedness of the Interior Transmission Problem

In this section, we establish the well-posedness of a more general interior transmission problem associated with (1.1); that is, given $\ell_1 \in L^2(\Omega)$, $\ell_2 \in L^2(\Omega)$, $f_1 \in H^{-1/2}(\Gamma_1)$, $f_2 \in H^{-1/2}(\Gamma_2)$ and $f_3 \in H^{1/2}(\partial\Omega)$, find $w \in H^1(\Omega)$ and $v \in H^1(\Omega)$ satisfying that

$$\begin{cases} \nabla \cdot A\nabla w + k^2 n w = \ell_1, & \text{in} \quad \Omega, \\ \Delta v + k^2 v = \ell_2, & \text{in} \quad \Omega, \\ \dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} - \lambda v = f_1, & \text{on} \quad \Gamma_1, \\ \dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} = f_2, & \text{on} \quad \Gamma_2, \\ w - v = f_3, & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Note that $H^{-1/2}(\Gamma_j) := \{ f|_{\Gamma_j} : f \in H^{-1/2}(\partial\Omega) \}$ and the norm is defined by $\|\tilde{f}\|_{H^{-1/2}(\Gamma_j)} := \inf\{ \|f\|_{H^{-1/2}(\partial\Omega)} \ \text{ for } \ f \in H^{-1/2}(\partial\Omega),\ f|_{\Gamma_j} = \tilde{f} \ \}$ $(j = 1, 2)$. Here and below, all notations possess the same meaning as defined in the introduction. We also make the following new notations:

$$a_* := \inf_{x\in\Omega}\inf_{|\xi|=1}\mathrm{Re}(\overline{\xi}\cdot A\xi) > 0, \quad a^* := \sup_{x\in\Omega}\sup_{|\xi|=1}\mathrm{Re}(\overline{\xi}\cdot A\xi) < \infty.$$

To investigate the solvability of problem (2.1), we need to formulate a modified interior transmission problem (2.2), which turns out to be a compact perturbation of our original problem (2.1). Introducing a positive constant γ whose value will change in different cases, given two functions $\ell_1 \in L^2(\Omega)$, $\ell_2 \in L^2(\Omega)$ and boundary data $f_1 \in H^{-1/2}(\Gamma_1)$, $f_2 \in H^{-1/2}(\Gamma_2)$, $f_3 \in H^{1/2}(\partial\Omega)$, find $w \in H^1(\Omega)$ and $v \in H^1(\Omega)$ satisfying that

$$\begin{cases} \nabla\cdot A\nabla w - \gamma w = \ell_1, & \text{in} \quad \Omega, \\ \Delta v - v = \ell_2, & \text{in} \quad \Omega, \\ \dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} - \lambda v = f_1, & \text{on} \quad \Gamma_1, \\ \dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} = f_2, & \text{on} \quad \Gamma_2, \\ w - v = f_3, & \text{on} \quad \partial\Omega. \end{cases} \tag{2.2}$$

First, we consider the case $a_* > 1$ and $\lambda \le -c$ on Γ_1. In order to reformulate (2.2) as an equivalent variational problem, we define the Hilbert space

$$W_1(\Omega) := \Big\{\mathbf{v} \in [L^2(\Omega)]^2 : \nabla\cdot\mathbf{v} \in L^2(\Omega) \text{ and } \nabla\times\mathbf{v} = 0\Big\},$$

equipped with the norm $\|\mathbf{v}\|^2_{W_1(\Omega)} = \|\mathbf{v}\|^2_{[L^2(\Omega)]^2} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)}$.

Next, we multiply the first equation in (2.2) by a test function $\overline{\varphi}$ with $\varphi \in H^1(\Omega)$ and use the transmission boundary conditions:

$$\begin{aligned} \int_\Omega \ell_1\overline{\varphi}\mathrm{d}x &= \int_\Omega(\nabla\cdot A\nabla w - \gamma w)\overline{\varphi}\mathrm{d}x = \int_{\partial\Omega}\frac{\partial w}{\partial\nu_A}\overline{\varphi}\mathrm{d}s - \int_\Omega(A\nabla w\cdot\nabla\overline{\varphi} + \gamma w\overline{\varphi})\mathrm{d}x \\ &= \int_{\Gamma_1}\Big(\frac{\partial v}{\partial\nu} + \lambda v + f_1\Big)\overline{\varphi}\mathrm{d}s + \int_{\Gamma_2}\Big(\frac{\partial v}{\partial\nu} + f_2\Big)\overline{\varphi}\mathrm{d}s - \int_\Omega(A\nabla w\cdot\nabla\overline{\varphi} + \gamma w\overline{\varphi})\mathrm{d}x \\ &= \int_{\Gamma_1} f_1\overline{\varphi}\mathrm{d}s + \int_{\Gamma_2} f_2\overline{\varphi}\mathrm{d}s + \int_{\partial\Omega}(\nu\cdot\mathbf{v})\overline{\varphi}\mathrm{d}s + \int_{\Gamma_1}\lambda w\overline{\varphi}\mathrm{d}s \\ &\quad - \int_{\Gamma_1}\lambda f_3\overline{\varphi}\mathrm{d}s - \int_\Omega(A\nabla w\cdot\nabla\overline{\varphi} + \gamma w\overline{\varphi})\mathrm{d}x. \end{aligned}$$

Here, $\mathbf{v} = \nabla v$, and so $\mathbf{v} \in W_1(\Omega)$. After arranging, we obtain that

$$\begin{aligned} &\int_\Omega(A\nabla w\cdot\nabla\overline{\varphi} + \gamma w\overline{\varphi})\mathrm{d}x - \int_{\partial\Omega}(\nu\cdot\mathbf{v})\overline{\varphi}\mathrm{d}s - \int_{\Gamma_1}\lambda w\overline{\varphi}\mathrm{d}s \\ &= \int_{\Gamma_1} f_1\overline{\varphi}\mathrm{d}s + \int_{\Gamma_2} f_2\overline{\varphi}\mathrm{d}s - \int_{\Gamma_1}\lambda f_3\overline{\varphi}\mathrm{d}s - \int_\Omega \ell_1\overline{\varphi}\mathrm{d}x. \end{aligned}$$

For the second equation in (2.2), we multiply it by a test function $\nabla\cdot\overline{\boldsymbol{\psi}}$ with $\boldsymbol{\psi} \in W_1(\Omega)$. Similarly, integrate in Ω and use the boundary conditions to obtain that

$$\begin{aligned} \int_\Omega(\nabla\cdot\mathbf{v})(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x &= \int_\Omega(\nabla\cdot(\nabla v))(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x = \int_\Omega(v + \ell_2)(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x \\ &= \int_{\partial\Omega} v(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s - \int_\Omega(\nabla v)\cdot\overline{\boldsymbol{\psi}}\mathrm{d}x + \int_\Omega \ell_2(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x \end{aligned}$$

$$= \int_{\partial\Omega} (w - f_3)(\overline{\boldsymbol{\psi}} \cdot \nu) \mathrm{d}s - \int_{\Omega} \mathbf{v} \cdot \overline{\boldsymbol{\psi}} \mathrm{d}x + \int_{\Omega} \ell_2 (\nabla \cdot \overline{\boldsymbol{\psi}}) \mathrm{d}x;$$

that is,

$$\int_{\Omega} \Big[(\nabla \cdot \mathbf{v})(\nabla \cdot \overline{\boldsymbol{\psi}}) + \mathbf{v} \cdot \overline{\boldsymbol{\psi}} \Big] \mathrm{d}x - \int_{\partial\Omega} w(\overline{\boldsymbol{\psi}} \cdot \nu) \mathrm{d}s = \int_{\Omega} \ell_2 (\nabla \cdot \overline{\boldsymbol{\psi}}) \mathrm{d}x - \int_{\partial\Omega} f_3 (\overline{\boldsymbol{\psi}} \cdot \nu) \mathrm{d}s.$$

Now we introduce the sesquilinear form $\mathcal{A}_1(U, V)$ defined on $\{H^1(\Omega) \times W_1(\Omega)\}^2$ by

$$\begin{aligned} \mathcal{A}_1(U,V) = & \int_{\Omega} (A \nabla w \cdot \nabla \overline{\varphi} + \gamma w \overline{\varphi}) \mathrm{d}x + \int_{\Omega} \Big[(\nabla \cdot \mathbf{v})(\nabla \cdot \overline{\boldsymbol{\psi}}) + \mathbf{v} \cdot \overline{\boldsymbol{\psi}} \Big] \mathrm{d}x \\ & - \int_{\partial\Omega} w(\overline{\boldsymbol{\psi}} \cdot \nu) \mathrm{d}s - \int_{\partial\Omega} (\nu \cdot \mathbf{v}) \overline{\varphi} \mathrm{d}s - \int_{\Gamma_1} \lambda w \overline{\varphi} \mathrm{d}s, \end{aligned}$$

where $U := (w, \mathbf{v})$ and $V := (\varphi, \boldsymbol{\psi})$ are in $H^1(\Omega) \times W_1(\Omega)$. Denote by $L_1 : H^1(\Omega) \times W_1(\Omega) \longrightarrow \mathbb{C}$ the bounded antilinear functional given by

$$L_1(V) = \int_{\Gamma_1} (f_1 - \lambda f_3) \overline{\varphi} \mathrm{d}s + \int_{\Gamma_2} f_2 \overline{\varphi} \mathrm{d}s - \int_{\partial\Omega} f_3 (\overline{\boldsymbol{\psi}} \cdot \nu) \mathrm{d}s - \int_{\Omega} \ell_1 \overline{\varphi} \mathrm{d}x + \int_{\Omega} \ell_2 (\nabla \cdot \overline{\boldsymbol{\psi}}) \mathrm{d}x.$$

Therefore, the variational formulation of the problem (2.2) is to find $U = (w, \mathbf{v}) \in H^1(\Omega) \times W_1(\Omega)$ such that

$$\mathcal{A}_1(U, V) = L_1(V), \quad \forall\, V \in H^1(\Omega) \times W_1(\Omega). \tag{2.3}$$

The next theorem states the equivalence between problems (2.2) and (2.3); the detailed proof is the same as Theorem 3.3 in the paper [6] and Theorem 6.5 in the book [5], so we omit the proof for brevity.

Theorem 2.1 The problem (2.2) has a unique solution $(w, v) \in H^1(\Omega) \times H^1(\Omega)$ if and only if the problem (2.3) has a unique solution $U = (w, \mathbf{v}) \in H^1(\Omega) \times W_1(\Omega)$.

Now we investigate the modified interior transmission problem in the variational formulation (2.3).

Theorem 2.2 Assume that $a_* > 1$, $\gamma \geq a_*$ and $\lambda \leq -c$ on Γ_1. Then the variational problem (2.3) has a unique solution $U = (w, \mathbf{v}) \in H^1(\Omega) \times W_1(\Omega)$ which satisfies that

$$\begin{aligned} \|w\|_{H^1(\Omega)} + \|\mathbf{v}\|_{W_1(\Omega)} \leq 2\,\mathcal{C}\,\frac{a_* + 1}{a_* - 1} \Big(& \|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{-1/2}(\Gamma_1)} \\ & + \|f_2\|_{H^{-1/2}(\Gamma_2)} + \|f_3\|_{H^{1/2}(\partial\Omega)} \Big), \end{aligned}$$

with $\mathcal{C} > 0$ dependent on $\|\lambda\|_{L^\infty(\Gamma_1)}$.

Proof The trace theorems and Schwarz's inequality ensure the continuity of the antilinear functional L_1 on $H^1(\Omega) \times W_1(\Omega)$ and the existence of a constant $\mathcal{C}$ independent of ℓ_1, ℓ_2, f_1, f_2 and f_3 such that

$$\|L_1\| \leq \mathcal{C} \Big(\|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{-1/2}(\Gamma_1)} + \|f_2\|_{H^{-1/2}(\Gamma_2)} + \|f_3\|_{H^{1/2}(\partial\Omega)} \Big).$$

On the other hand, if $U = (w, \mathbf{v}) \in H^1(\Omega) \times W_1(\Omega)$, the assumptions that $a_* > 1$, $\gamma \geq a_*$ and $\lambda \leq -c$ imply

$$\begin{aligned} |\mathcal{A}_1(U, U)| & \geq a_* \|w\|^2_{H^1(\Omega)} + \|\mathbf{v}\|^2_{W_1(\Omega)} - 2\mathrm{Re}\Big(\int_{\partial\Omega} w(\overline{\mathbf{v}} \cdot \nu) \mathrm{d}s \Big) \\ & \geq \frac{a_* - 1}{a_* + 1} \Big(\|w\|^2_{H^1(\Omega)} + \|\mathbf{v}\|^2_{W_1(\Omega)} \Big), \end{aligned}$$

whence $\mathcal{A}_1$ is coercive. The detailed procedure to obtain the last inequality can be found in the paper [6] (Theorem 3.4) or in the book [8] (Lemma 3.30). The continuity of $\mathcal{A}_1$ follows easily from Schwarz's inequality and the classical trace theorems. Theorem 2.2 is now a direct consequence of the Lax-Milgram Lemma applied to (2.3). □

Second, we consider the case $a^* < 1$ and $\lambda \geq c$ on Γ_1. To gain a different equivalent variational form, we define another Hilbert space

$$W_2(\Omega) := \Big\{ \mathbf{w} \in [L^2(\Omega)]^2 : \nabla\cdot\mathbf{w} \in L^2(\Omega) \text{ and } \nabla\times A^{-1}\mathbf{w} = 0 \Big\},$$

equipped with the norm $\|\mathbf{w}\|^2_{W_2(\Omega)} = \|\mathbf{w}\|^2_{[L^2(\Omega)]^2} + \|\nabla\cdot\mathbf{w}\|^2_{L^2(\Omega)}$.

Similar to the first case, we multiply the first equation in (2.2) by a test function $\nabla\cdot\overline{\boldsymbol{\phi}}$ with $\boldsymbol{\phi} \in W_2(\Omega)$, and the second equation by a test function $\overline{\chi}$ with $\chi \in H^1(\Omega)$, then integrate in Ω and use the boundary conditions to obtain that

$$\int_\Omega \Big[(\nabla\cdot\mathbf{w})(\nabla\cdot\overline{\boldsymbol{\phi}}) + \gamma A^{-1}\mathbf{w}\cdot\overline{\boldsymbol{\phi}}\Big]\mathrm{d}x - \int_{\partial\Omega}\gamma v(\overline{\boldsymbol{\phi}}\cdot\nu)\mathrm{d}s = \int_{\partial\Omega}\gamma f_3(\overline{\boldsymbol{\phi}}\cdot\nu)\mathrm{d}s + \int_\Omega \ell_1(\nabla\cdot\overline{\boldsymbol{\phi}})\mathrm{d}x,$$

where $\mathbf{w} = A\nabla w \in W_2(\Omega)$, and

$$\int_\Omega(\nabla v\cdot\nabla\overline{\chi} + v\overline{\chi})\mathrm{d}x + \int_{\Gamma_1}\lambda v\overline{\chi}\mathrm{d}s - \int_{\partial\Omega}(\nu\cdot\mathbf{w})\overline{\chi}\mathrm{d}s = -\int_{\Gamma_1} f_1\overline{\chi}\mathrm{d}s - \int_{\Gamma_2} f_2\overline{\chi}\mathrm{d}s - \int_\Omega \ell_2\overline{\chi}\mathrm{d}x.$$

Introduce the sesquilinear form $\mathcal{A}_2(\mathbb{U},\mathbb{V})$ defined on $\{W_2(\Omega)\times H^1(\Omega)\}^2$ by

$$\begin{aligned}\mathcal{A}_2(\mathbb{U},\mathbb{V}) = &\int_\Omega \Big[(\nabla\cdot\mathbf{w})(\nabla\cdot\overline{\boldsymbol{\phi}}) + \gamma A^{-1}\mathbf{w}\cdot\overline{\boldsymbol{\phi}}\Big]\mathrm{d}x - \int_{\partial\Omega}\gamma v(\overline{\boldsymbol{\phi}}\cdot\nu)\mathrm{d}s\\ &+ \int_\Omega(\nabla v\cdot\nabla\overline{\chi} + v\overline{\chi})\mathrm{d}x - \int_{\partial\Omega}(\nu\cdot\mathbf{w})\overline{\chi}\mathrm{d}s + \int_{\Gamma_1}\lambda v\overline{\chi}\mathrm{d}s,\end{aligned}$$

where $\mathbb{U} := (\mathbf{w}, v)$ and $\mathbb{V} := (\boldsymbol{\phi},\chi)$ are in $W_2(\Omega)\times H^1(\Omega)$. Denote by $L_2 : W_2(\Omega)\times H^1(\Omega) \longrightarrow \mathbb{C}$ the bounded antilinear functional given by

$$L_2(\mathbb{V}) = \int_{\partial\Omega}\gamma f_3(\overline{\boldsymbol{\phi}}\cdot\nu)\mathrm{d}s + \int_\Omega \ell_1(\nabla\cdot\overline{\boldsymbol{\phi}})\mathrm{d}x - \int_{\Gamma_1} f_1\overline{\chi}\mathrm{d}s - \int_{\Gamma_2} f_2\overline{\chi}\mathrm{d}s - \int_\Omega \ell_2\overline{\chi}\mathrm{d}x.$$

Then the variational formulation of the problem (2.2) is to find $\mathbb{U} = (\mathbf{w}, v) \in W_2(\Omega)\times H^1(\Omega)$ such that

$$\mathcal{A}_2(\mathbb{U},\mathbb{V}) = L_2(\mathbb{V}), \quad \forall\ \mathbb{V} \in W_2(\Omega)\times H^1(\Omega). \tag{2.4}$$

Theorem 2.3 The problem (2.2) has a unique solution $(w, v) \in H^1(\Omega)\times H^1(\Omega)$ if and only if the problem (2.4) has a unique solution $\mathbb{U} = (\mathbf{w}, v) \in W_2(\Omega)\times H^1(\Omega)$.

Theorem 2.4 Assume that $a^* < 1$, $a^* \leq \gamma < 1$ and $\lambda \geq c$ on Γ_1. Then the variational problem (2.4) has a unique solution $\mathbb{U} = (\mathbf{w}, v) \in W_2(\Omega)\times H^1(\Omega)$ which satisfies that

$$\begin{aligned}\|\mathbf{w}\|_{W_2(\Omega)} + \|v\|_{H^1(\Omega)} \leq \frac{4\,\mathcal{C}}{1-\gamma}\Big(&\|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{-1/2}(\Gamma_1)}\\ &+ \|f_2\|_{H^{-1/2}(\Gamma_2)} + \|f_3\|_{H^{1/2}(\partial\Omega)}\Big),\end{aligned}$$

with $\mathcal{C} > 0$ dependent on $\|\lambda\|_{L^\infty(\Gamma_1)}$.

Proof The trace theorems and Schwarz's inequality ensure the continuity of the antilinear functional L_2 on $W_2(\Omega)\times H^1(\Omega)$ and the existence of a constant $\mathcal{C}$ independent of ℓ_1, ℓ_2, f_1, f_2 and f_3 such that

$$\|L_2\| \leq \mathcal{C}\Big(\|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{-1/2}(\Gamma_1)} + \|f_2\|_{H^{-1/2}(\Gamma_2)} + \|f_3\|_{H^{1/2}(\partial\Omega)}\Big).$$

On the other hand, if $\mathbb{U} = (\mathbf{w}, v) \in W_2(\Omega) \times H^1(\Omega)$, the assumptions that $a^* < 1$, $a^* \leq \gamma < 1$ and $\lambda \geq c$ imply that

$$\begin{aligned}|\mathcal{A}_2(\mathbb{U}, \mathbb{U})| &\geq \|\mathbf{w}\|^2_{W_2(\Omega)} + \|v\|^2_{H^1(\Omega)} - (\gamma + 1)\mathrm{Re}\Big(\int_{\partial\Omega} v(\overline{\mathbf{w}} \cdot \nu)\mathrm{d}s\Big) \\ &\geq \|\mathbf{w}\|^2_{W_2(\Omega)} + \|v\|^2_{H^1(\Omega)} - (\gamma + 1)\|\mathbf{w}\|_{W_2(\Omega)}\|v\|_{H^1(\Omega)} \\ &\geq \frac{1-\gamma}{2}\Big(\|\mathbf{w}\|^2_{W_2(\Omega)} + \|v\|^2_{H^1(\Omega)}\Big),\end{aligned}$$

whence $\mathcal{A}_2$ is coercive. The proof is completed. □

Summarizing the above analysis, we can state the following result concerning the solvability of the interior transmission problem (2.1):

Theorem 2.5 Assume that either $a_* > 1$, $\lambda \leq -c$ or $a^* < 1$, $\lambda \geq c$, and that k is not a transmission eigenvalue of the problem (1.1). Then the general interior transmission problem (2.1) has a unique solution $(w, v) \in H^1(\Omega) \times H^1(\Omega)$ which satisfies that

$$\begin{aligned}\|w\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \leq &\mathcal{C}\Big(\|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{-1/2}(\Gamma_1)} \\ &+ \|f_2\|_{H^{-1/2}(\Gamma_2)} + \|f_3\|_{H^{1/2}(\partial\Omega)}\Big),\end{aligned}$$

with $\mathcal{C} > 0$ dependent on $\|\lambda\|_{L^\infty(\Gamma_1)}$.

The proof is completely the same as Theorem 3.6 in the paper [6] and Theorem 3.32 in the book [8], so we omit it here.

Remark 2.6 Note that we obtain the well-posedness of the problem (2.1) under the special condition that the sign of λ has a close relation with the value of A. Consequently, in the following sections, we keep the assumptions of Theorem 2.5 above. And in the future, we want to discard this strict restriction on λ or even consider that $\lambda(x)$ is a complex valued function.

Before we study the transmission eigenvalue problem (1.1), we first establish the uniqueness of a solution to (2.1), i.e., there are no transmission eigenvalues.

Theorem 2.7 Assume that $A \in (L^\infty(\Omega))^{2\times 2}$ and $n \in L^\infty(\Omega)$. If either $\mathrm{Im}(\overline{\xi} \cdot A\xi) < 0$ or $\mathrm{Im}(n) > 0$ almost everywhere in Ω, then the interior transmission problem (2.1) has at most one solution.

Proof Let w and v be a solution pair of the homogeneous interior transmission problem (2.1); that is, $\ell_1 = \ell_2 = f_1 = f_2 = f_3 = 0$. Applying the divergence theorem to $\overline{w}$ and $A\nabla w$, using the boundary condition and applying Green's first identity to $\overline{v}$ and v, we obtain that

$$\begin{aligned}\int_\Omega (\nabla\overline{w} \cdot A\nabla w - k^2 n|w|^2)\mathrm{d}x &= \int_{\partial\Omega} \overline{w}\frac{\partial w}{\partial\nu_A}\mathrm{d}s = \int_{\partial\Omega} \overline{v}\frac{\partial v}{\partial\nu}\mathrm{d}s + \int_{\Gamma_1} \lambda|v|^2\mathrm{d}s \\ &= \int_\Omega (|\nabla v|^2 - k^2|v|^2)\mathrm{d}x + \int_{\Gamma_1} \lambda|v|^2\mathrm{d}s.\end{aligned}$$

Hence,

$$\mathrm{Im}\Big(\int_\Omega \nabla\overline{w} \cdot A\nabla w\mathrm{d}x\Big) = 0, \quad \mathrm{Im}\Big(\int_\Omega n|w|^2\mathrm{d}x\Big) = 0.$$

If $\mathrm{Im}(\overline{\xi} \cdot A\xi) < 0$ almost everywhere in Ω, then $\nabla w = 0$ in Ω and from the equation $w = 0$. From the boundary condition in (2.1) and the integral representation formula, v also vanishes in Ω.

If $\mathrm{Im}(n) > 0$ almost everywhere in Ω, then $w = 0$ in Ω. Similarly, from the boundary condition in (2.1) and the integral representation formula, v also vanishes in Ω. The proof is complete. □

Remark 2.8 If $A \in (C^\infty(\Omega))^{2\times 2}$ and $n \in C(\Omega)$, the result of Theorem 2.7 also holds, but in this case one just needs to assume that either $\mathrm{Im}(\overline{\xi} \cdot A\xi) < 0$ or $\mathrm{Im}(n) > 0$ at a point $x_0 \in \Omega$.

3 The Transmission Eigenvalues for $n \equiv 1$

In this section, we study the discreteness and existence of the transmission eigenvalues under the special case $n \equiv 1$, where we can obtain a fourth-order equation. Here we assume that $\mathrm{Im}(A) = 0$ and either $a_* > 1$, $\lambda \le -c$ or $0 < a^* < 1$, $\lambda \ge c$. Then the transmission eigenvalue problem for $n \equiv 1$ reads as

$$\begin{cases} \nabla \cdot A\nabla w + k^2 w = 0, & \text{in} \quad \Omega, \\ \Delta v + k^2 v = 0, & \text{in} \quad \Omega, \\ \dfrac{\partial w}{\partial \nu_A} = \dfrac{\partial v}{\partial \nu} + \lambda v, & \text{on} \quad \Gamma_1, \\ \dfrac{\partial w}{\partial \nu_A} = \dfrac{\partial v}{\partial \nu}, & \text{on} \quad \Gamma_2, \\ w = v, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

with $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$. We make the following substitutions:

$$\mathbf{w} = A\nabla w \in [L^2(\Omega)]^2 \quad \text{and} \quad \mathbf{v} = \nabla v \in [L^2(\Omega)]^2.$$

Then the transmission problem (3.1) can be written as the equivalent problem

$$\begin{cases} \nabla(\nabla \cdot \mathbf{w}) + k^2 N\mathbf{w} = 0, & \text{in} \quad \Omega, \\ \nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0, & \text{in} \quad \Omega, \\ \nu \cdot \mathbf{w} = \nu \cdot \mathbf{v} - \dfrac{\lambda}{k^2}\nabla \cdot \mathbf{v}, & \text{on} \quad \Gamma_1, \\ \nu \cdot \mathbf{w} = \nu \cdot \mathbf{v}, & \text{on} \quad \Gamma_2, \\ \nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v}, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

where $N := A^{-1}$. Introduce the Sobolev space

$$W = \Big\{ \mathbf{u} \in [L^2(\Omega)]^2 : \nabla \cdot \mathbf{u} \in H_0^1(\Omega), \quad \nu \cdot \mathbf{u} = 0 \text{ on } \Gamma_2 \Big\},$$

equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_W = (\mathbf{u}, \mathbf{v})_{L^2(\Omega)} + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{H^1(\Omega)}.$$

Following the classical procedure, let $\mathbf{u} = \mathbf{w} - \mathbf{v}$. Then we can write (3.2) as an equivalent eigenvalue problem for $\mathbf{u} \in W$ satisfying the fourth order equation

$$\begin{cases} (\nabla\nabla \cdot + k^2 N)(N - I)^{-1}(\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}) = 0, & \text{in} \quad \Omega, \\ \nu \cdot \mathbf{u} = -\dfrac{\lambda}{k^2}\nabla \cdot \mathbf{v}, & \text{on} \quad \Gamma_1. \end{cases} \tag{3.3}$$

Note that $\mathbf{w} = -\frac{1}{k^2}(N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})$ and $\mathbf{v} = -\frac{1}{k^2}(N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2N\mathbf{u})$. Next, we will transform the fourth order equation (3.3) into an equivalent variational formulation. Hence, we multiply the equation in (3.3) by a test function $\overline{\mathbf{u}'}$ with $\mathbf{u}'\in W$ to obtain that

$$\begin{aligned}
0 &= \int_\Omega \left[(\nabla\nabla\cdot + k^2N)(N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})\right]\cdot\overline{\mathbf{u}'}\mathrm{d}x\\
&= \int_{\partial\Omega}\left[\nabla\cdot((N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}))\right](\nu\cdot\overline{\mathbf{u}'})\mathrm{d}s\\
&\quad -\int_{\partial\Omega}\nu\cdot\left[(N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})\right](\nabla\cdot\overline{\mathbf{u}'})\mathrm{d}s\\
&\quad +\int_\Omega (N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})\cdot(\nabla\nabla\cdot\overline{\mathbf{u}'})\mathrm{d}x\\
&\quad +\int_\Omega (N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})\cdot(k^2N\overline{\mathbf{u}'})\mathrm{d}x\\
&= \int_\Omega (N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})\cdot(\nabla\nabla\cdot\overline{\mathbf{u}'}+k^2N\overline{\mathbf{u}'})\mathrm{d}x - k^2\int_{\partial\Omega}(\nabla\cdot\mathbf{w})(\nu\cdot\overline{\mathbf{u}'})\mathrm{d}s.
\end{aligned}$$

In the last equation, we have used the fact that $\nabla\cdot\mathbf{u}'|_{\partial\Omega}=0$, as well as the relation between $\mathbf{w}$ and $\mathbf{u}$. Recalling the boundary condition on $\partial\Omega$ in (3.2), $\nu\cdot\mathbf{u}'|_{\Gamma_2}=0$ and the boundary condition on Γ_1 in (3.3), we have

$$\int_{\partial\Omega}(\nabla\cdot\mathbf{w})(\nu\cdot\overline{\mathbf{u}'})\mathrm{d}s = \int_{\partial\Omega}(\nabla\cdot\mathbf{v})(\nu\cdot\overline{\mathbf{u}'})\mathrm{d}s = \int_{\Gamma_1}(\nabla\cdot\mathbf{v})(\nu\cdot\overline{\mathbf{u}'})\mathrm{d}s = -\int_{\Gamma_1}\frac{k^2}{\lambda}(\nu\cdot\mathbf{u})(\nu\cdot\overline{\mathbf{u}'})\mathrm{d}s.$$

Consequently, the variational form reads as follows: find $\mathbf{u}\in W$ such that

$$\int_\Omega (N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})\cdot(\nabla\nabla\cdot\overline{\mathbf{u}'}+k^2N\overline{\mathbf{u}'})\mathrm{d}x + \int_{\Gamma_1}\frac{k^4}{\lambda}(\nu\cdot\mathbf{u})(\nu\cdot\overline{\mathbf{u}'})\mathrm{d}s = 0. \tag{3.4}$$

In what follows, we introduce the sesquilinear forms $\mathcal{A}_k$, $\widetilde{\mathcal{A}}_k$ and $\mathcal{B}$ given by

$$\begin{aligned}
\mathcal{A}_k(\mathbf{u},\mathbf{u}') &= \Big(N(I-N)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}),(\nabla\nabla\cdot\mathbf{u}'+k^2\mathbf{u}')\Big)_\Omega\\
&\quad + \Big(\nabla\nabla\cdot\mathbf{u},\nabla\nabla\cdot\mathbf{u}'\Big)_\Omega - \Big\langle\frac{k^4}{\lambda}\nu\cdot\mathbf{u},\nu\cdot\mathbf{u}'\Big\rangle_{\Gamma_1},\\
\widetilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}') &= \Big((N-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}),(\nabla\nabla\cdot\mathbf{u}'+k^2\mathbf{u}')\Big)_\Omega\\
&\quad + k^4(\mathbf{u},\mathbf{u}')_\Omega + \Big\langle\frac{k^4}{\lambda}\nu\cdot\mathbf{u},\nu\cdot\mathbf{u}'\Big\rangle_{\Gamma_1},\\
\mathcal{B}(\mathbf{u},\mathbf{u}') &= (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{u}')_\Omega,
\end{aligned}$$

where $(\cdot,\cdot)_\Omega$ denotes the inner product in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{\Gamma_1}$ denotes the dual pairing between $H^{1/2}(\Gamma_1)$ and $\widetilde{H}^{-1/2}(\Gamma_1)$. Then we have

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}')-k^2\mathcal{B}(\mathbf{u},\mathbf{u}')=0,\quad \widetilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}')-k^2\mathcal{B}(\mathbf{u},\mathbf{u}')=0\quad\text{for all }\mathbf{u}'\in W.$$

By means of the Riesz representation theorem, the bounded linear operators $\mathbb{A}_\delta: W\longrightarrow W$, $\widetilde{\mathbb{A}}_\delta: W\longrightarrow W$ and $\mathbb{B}: W\longrightarrow W$ can be defined by

$$(\mathbb{A}_\delta\mathbf{u},\mathbf{u}')=\mathcal{A}_k(\mathbf{u},\mathbf{u}'),\qquad (\widetilde{\mathbb{A}}_\delta\mathbf{u},\mathbf{u}')=\widetilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}'),\qquad (\mathbb{B}\mathbf{u},\mathbf{u}')=\mathcal{B}(\mathbf{u},\mathbf{u}'), \tag{3.5}$$

where $\delta:=k^2$. Then the variational form (3.4) can be rewritten as an operator equation

$$\mathbb{A}_\delta\mathbf{u}-\delta\mathbb{B}\mathbf{u}=0\quad\text{or}\quad\widetilde{\mathbb{A}}_\delta\mathbf{u}-\delta\mathbb{B}\mathbf{u}=0\quad\text{for }\mathbf{u}\in W.$$

Theorem 3.1 Let $\Lambda_1(\Omega)$ be the first Dirichlet eigenvalue of $-\Delta$ in Ω. Then the following hold:

1. for $a_* > 1$ and $\lambda \leq -c$ on Γ_1, real wave numbers $k > 0$ such that $k^2 < \Lambda_1(\Omega)$ are not transmission eigenvalues;

2. for $a^* < 1$ and $\lambda \geq c$ on Γ_1, real wave numbers $k > 0$ such that $k^2 < a_*\Lambda_1(\Omega)$ are not transmission eigenvalues.

Proof First, we recall that for $\nabla\cdot\mathbf{u} \in H_0^1(\Omega)$, using the Poincaré inequality, we have that

$$\|\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)} \leq \frac{1}{\Lambda_1(\Omega)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}, \tag{3.6}$$

where $\Lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$ in Ω.

Now assume that $a_* > 1$ and $\lambda \leq -c$, which implies that $\xi\cdot N(I-N)^{-1}\xi \geq \alpha_1|\xi|^2$ for all $\xi \in \mathbb{R}^2$ and a.e. $x \in \Omega$ with $\alpha_1 = \frac{1}{a^*-1}$. Then we have that

$$\begin{aligned}\mathcal{A}_k(\mathbf{u},\mathbf{u}) &\geq \alpha_1\|\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)} - \int_{\Gamma_1}\frac{k^4}{\lambda}|\nu\cdot\mathbf{u}|^2\mathrm{d}s\\ &\geq \alpha_1\|k^2\mathbf{u}\|^2_{L^2(\Omega)} - 2\alpha_1\|k^2\mathbf{u}\|_{L^2(\Omega)}\|\nabla\nabla\cdot\mathbf{u}\|_{L^2(\Omega)} + (\alpha_1+1)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}\\ &\geq \Big(\alpha_1 - \frac{\alpha_1^2}{\varepsilon}\Big)k^4\|\mathbf{u}\|^2_{L^2(\Omega)} + (1+\alpha_1-\varepsilon)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}\end{aligned}$$

for $\alpha_1 < \varepsilon < \alpha_1 + 1$, and therefore,

$$\begin{aligned}&\mathcal{A}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u})\\ \geq& \Big(\alpha_1 - \frac{\alpha_1^2}{\varepsilon}\Big)k^4\|\mathbf{u}\|^2_{L^2(\Omega)} + (1+\alpha_1-\varepsilon)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)} - \frac{k^2}{\Lambda_1(\Omega)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}\\ \geq& \Big(\alpha_1 - \frac{\alpha_1^2}{\varepsilon}\Big)k^4\|\mathbf{u}\|^2_{L^2(\Omega)} + \Big(1+\alpha_1-\varepsilon-\frac{k^2}{\Lambda_1(\Omega)}\Big)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence, if $k^2 < \Lambda_1(\Omega)(1+\alpha_1-\varepsilon)$ for every $\alpha_1 < \varepsilon < \alpha_1+1$, then $\mathcal{A}_k(\cdot,\cdot) - k^2\mathcal{B}(\cdot,\cdot)$ is coercive. By taking $\varepsilon > 0$ arbitrarily close to α_1, we have that if $k^2 < \Lambda_1(\Omega)$, then k is not a transmission eigenvalue, which proves the first part.

Next, let $0 < a^* < 1$ and $\lambda \geq c$, which implies that $\xi\cdot(N-I)^{-1}\xi \geq \alpha_2|\xi|^2$ for all $\xi \in \mathbb{R}^2$ and a.e. $x \in \Omega$ with $\alpha_2 = \frac{a_*}{1-a_*}$. Then, in exactly the same way as for the first part, we obtain that

$$\begin{aligned}\widetilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}) &\geq \alpha_2\|\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\|^2_{L^2(\Omega)} + k^4\|\mathbf{u}\|^2_{L^2(\Omega)} + \int_{\Gamma_1}\frac{k^4}{\lambda}|\nu\cdot\mathbf{u}|^2\mathrm{d}s\\ &\geq \alpha_2\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)} - 2\alpha_2\|\nabla\nabla\cdot\mathbf{u}\|_{L^2(\Omega)}\|k^2\mathbf{u}\|_{L^2(\Omega)} + (\alpha_2+1)\|k^2\mathbf{u}\|^2_{L^2(\Omega)}\\ &\geq \Big(\alpha_2 - \frac{\alpha_2^2}{\varepsilon}\Big)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)} + (1+\alpha_2-\varepsilon)k^4\|\mathbf{u}\|^2_{L^2(\Omega)}\end{aligned}$$

for $\alpha_2 < \varepsilon < \alpha_2 + 1$ and

$$\begin{aligned}&\widetilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u})\\ \geq& \Big(\alpha_2 - \frac{\alpha_2^2}{\varepsilon}\Big)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)} + (1+\alpha_2-\varepsilon)k^4\|\mathbf{u}\|^2_{L^2(\Omega)} - \frac{k^2}{\Lambda_1(\Omega)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}\\ \geq& \Big(\alpha_2 - \frac{\alpha_2^2}{\varepsilon} - \frac{k^2}{\Lambda_1(\Omega)}\Big)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)} + (1+\alpha_2-\varepsilon)k^4\|\mathbf{u}\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, if $k^2 < \Lambda_1(\Omega)\Big(\alpha_2 - \frac{\alpha_2^2}{\varepsilon}\Big)$ for every $\alpha_2 < \varepsilon < \alpha_2+1$, then $\widetilde{\mathcal{A}}_k(\cdot,\cdot) - k^2\mathcal{B}(\cdot,\cdot)$ is coercive. By taking $\varepsilon > 0$ arbitrarily close to α_2+1, we have that if $k^2 < \frac{\alpha_2}{\alpha_2+1}\Lambda_1(\Omega) = a_*\Lambda_1(\Omega)$, then k is not a transmission eigenvalue, which proves the second part. □

Theorem 3.2 Assume that $n \equiv 1$, $\mathrm{Im}(A) = 0$ and either $a_* > 1$, $\lambda \le -c$ or $a^* < 1$, $\lambda \ge c$. Then the transmission eigenvalues form a discrete (possibly empty) set with $+\infty$ as the only possible accumulation point.

Proof Let us set

$$\mathcal{H}(\Omega) = \Big\{(w,v) \in H^1(\Omega) \times H^1(\Omega) : \nabla\cdot A\nabla w \in L^2(\Omega) \text{ and } \Delta v \in L^2(\Omega)\Big\},$$

and consider the operator $\mathcal{F}_k$ from $\mathcal{H}(\Omega)$ into $L^2(\Omega)\times L^2(\Omega)\times H^{-1/2}(\Gamma_1)\times H^{-1/2}(\Gamma_2)\times H^{1/2}(\partial\Omega)$ defined by

$$\begin{aligned}\mathcal{F}_k(w,v) = \Big(&\nabla\cdot A\nabla w + k^2 nw, \Delta v + k^2 v, \big(\frac{\partial w}{\partial\nu_A} - \frac{\partial v}{\partial\nu} - \lambda v\big)\Big|_{\Gamma_1},\\ &\big(\frac{\partial w}{\partial\nu_A} - \frac{\partial v}{\partial\nu}\big)\Big|_{\Gamma_2}, (w-v)|_{\partial\Omega}\Big).\end{aligned}$$

Then the family of operators $\mathcal{F}_k$ depends analytically on k. Based on Theorem 3.1, we find that $\mathcal{F}_\kappa$ is injective if $\kappa^2 < \Lambda_1(\Omega)$ when $a_* > 1$, $\lambda \le -c$ or $\kappa^2 < a_*\Lambda_1(\Omega)$ when $a^* < 1$, $\lambda \ge c$. Combining that with Theorem 2.5, we conclude that $\mathcal{F}_\kappa$ is invertible and has a bounded inverse operator $\mathcal{F}_\kappa^{-1}$. So $\mathcal{F}_k = \mathcal{F}_\kappa(I - \mathcal{F}_\kappa^{-1}(\mathcal{F}_\kappa - \mathcal{F}_k))$.

Since $(\mathcal{F}_\kappa - \mathcal{F}_k)(w,v) = ((\kappa^2-k^2)nw, (\kappa^2-k^2)v, 0, 0, 0)$ is compact based on the compact embedding of $H^1(\Omega)$ to $L^2(\Omega)$, we conclude that the transmission eigenvalues are discrete by the analytic Fredholm theory (see Section 8.5 in the book [23]). The proof is complete. □

The following theorem provides the theoretical basis of our analysis regarding the existence of transmission eigenvalues (refer to Theorem 6.15 in the book [5] or Theorem 4.5 in the book [8]):

Theorem 3.3 Let $\delta \longmapsto \mathbb{A}_\delta$ be a continuous mapping from $(0,+\infty)$ to the set of bounded, self-adjoint, and coercive operators on the Hilbert space X and let $\mathbb{B}$ be a self-adjoint and nonnegative compact bounded linear operator on X. We assume that there exist two positive constants $\delta_0 > 0$ and $\delta_1 > 0$ such that the following hold:

1. $\mathbb{A}_{\delta_0} - \delta_0\mathbb{B}$ is positive on X;
2. $\mathbb{A}_{\delta_1} - \delta_1\mathbb{B}$ is nonpositive on an ℓ-dimensional subspace W_j of X.

Then each of the equations $\lambda_j(\delta) = \delta$ for $j = 1,\cdots,\ell$ has at least one solution in $[\delta_0,\delta_1]$ where $\lambda_j(\delta)$ is the j-th eigenvalue (counting multiplicity) of $\mathbb{A}_\delta$ with respect to $\mathbb{B}$, i.e., $\ker(\mathbb{A}_\delta - \lambda_j(\delta)\mathbb{B}) \neq \{0\}$.

Recalling the definitions of $\mathbb{A}_\delta : W \longrightarrow W$, $\widetilde{\mathbb{A}}_\delta : W \longrightarrow W$, $\mathbb{B} : W \longrightarrow W$ in (3.5), and Theorem 3.1, we summarize the properties of these operators.

Lemma 3.4 Assume that $n \equiv 1$, $\mathrm{Im}(A) = 0$ and $a_* > 1$, $\lambda \le -c$ or $a^* < 1$, $\lambda \ge c$. Then

1. $\mathbb{B}$ is a bounded, positive, compact and self-adjoint operator;
2. $\mathbb{A}_\delta$ is a bounded coercive self-adjoint operator provided that $a_* > 1$ and $\lambda \le -c$;
3. $\widetilde{\mathbb{A}}_\delta$ is a bounded coercive self-adjoint operator provided that $a^* < 1$ and $\lambda \ge c$;
4. for $\delta_0 = \kappa^2$, either $\mathbb{A}_{\delta_0} - \delta_0\mathbb{B}$ or $\widetilde{\mathbb{A}}_{\delta_0} - \delta_0\mathbb{B}$ is positive on W (κ is defined in the proof of Theorem 3.2).

To investigate the existence of eigenvalues for the problem (3.1), we need to consider the classical transmission eigenvalue problem for a ball B_ρ of radius ρ centered at the origin with a constant index of refraction $m>0$ and $m\neq 1$, which is formulated as

$$\begin{cases}\Delta w+k^2mw=0, & \text{in}\quad B_\rho,\\ \Delta v+k^2v=0, & \text{in}\quad B_\rho,\\ \dfrac{1}{m}\dfrac{\partial w}{\partial\nu}=\dfrac{\partial v}{\partial\nu}, & \text{on}\quad \partial B_\rho,\\ w=v, & \text{on}\quad \partial B_\rho.\end{cases}\tag{3.7}$$

We denote by $k_{\rho,m}$ the smallest real eigenvalue to (3.7). An eigenfunction corresponding to $k_{\rho,m}$ is $\mathbf{u}^{B_\rho,m}=m^{-1}\nabla w^{B_\rho,m}-\nabla v^{B_\rho,m}$, where $w^{B_\rho,m}$, $v^{B_\rho,m}$ is a nonzero solution pair to (3.7) (refer to subsection 4.3.1 in the book [8]). Furthermore, $\mathbf{u}^{B_\rho,m}$ satisfies that

$$\int_{B_\rho}\frac{1}{m-1}(\nabla\nabla\cdot\mathbf{u}^{B_\rho,m}+k_{\rho,m}^2\mathbf{u}^{B_\rho,m})\cdot(\nabla\nabla\cdot\overline{\mathbf{u}}^{B_\rho,m}+k_{\rho,m}^2m\overline{\mathbf{u}}^{B_\rho,m})\mathrm{d}x=0.\tag{3.8}$$

Theorem 3.5 Assume that $n\equiv 1$, $\mathrm{Im}(A)=0$ and $a_*>1$, $\lambda\leq -c$ or $a^*<1$, $\lambda\geq c$. Then there exists an infinite set of real transmission eigenvalues for the anisotropic medium problem (3.1) with $+\infty$ as the only accumulation point.

Proof Based on the Lemma 3.4, we need only to prove the Assumption 2 in Theorem 3.3 (refer to Theorem 6.20 in [5] or Theorem 4.12 in [8]).

For the first case $a_*>1$, $\lambda\leq -c$, we let $k_{1,a_*^{-1}}$ be the first transmission eigenvalue for the ball B_ρ of radius $\rho=1$ and the index of refraction $m=a_*^{-1}$ of the problem (3.7). By a scaling argument, it is obvious that $k_{\varepsilon,a_*^{-1}}=k_{1,a_*^{-1}}/\varepsilon$ is the first transmission eigenvalue of the problem (3.7) corresponding to the ball of radius $\varepsilon>0$ with the index of refraction a_*^{-1}.

Now take $\varepsilon>0$ small enough such that Ω contains $M=M(\varepsilon)\geq 1$ disjoint balls $B_\varepsilon^1, B_\varepsilon^2,\cdots,B_\varepsilon^M$ of radius ε; that is, $\overline{B_\varepsilon^j}\subset\Omega$, $j=1,\cdots,M$, and $\overline{B_\varepsilon^j}\cap\overline{B_\varepsilon^\iota}=\emptyset$ for $j\neq\iota$. Then $k_{\varepsilon,a_*^{-1}}$ is the first transmission eigenvalue for each of these balls with the index of refraction a_*^{-1} and we let $\mathbf{u}^j$, $j=1,\cdots,M$ be the corresponding eigenfunction. The extension by zero $\widetilde{\mathbf{u}}^j$ of $\mathbf{u}^j$ to the whole domain of Ω is obviously in W and $\widetilde{\mathbf{u}}^j|_{\partial\Omega}=\mathbf{0}$. Furthermore, the vectors $\{\widetilde{\mathbf{u}}^1,\widetilde{\mathbf{u}}^2,\cdots,\widetilde{\mathbf{u}}^M\}$ are linearly independent and orthogonal in W since they have disjoint supports. From (3.8), we have that

$$\int_\Omega\frac{1}{a_*^{-1}-1}(\nabla\nabla\cdot\widetilde{\mathbf{u}}^j+k_{\varepsilon,a_*^{-1}}^2\widetilde{\mathbf{u}}^j)\cdot(\nabla\nabla\cdot\overline{\widetilde{\mathbf{u}}}^j+a_*^{-1}k_{\varepsilon,a_*^{-1}}^2\overline{\widetilde{\mathbf{u}}}^j)\mathrm{d}x=0\tag{3.9}$$

for $j=1,\cdots,M$. Let Y denote the M-dimensional subspace of W spanned by $\{\widetilde{\mathbf{u}}^1,\widetilde{\mathbf{u}}^2,\cdots,\widetilde{\mathbf{u}}^M\}$. Since each $\widetilde{\mathbf{u}}^j$, $j=1,\cdots,M$ satisfies (3.9) and they have disjoint supports, then for $\delta_1=k_{\varepsilon,a_*^{-1}}^2$ and for every $\widetilde{\mathbf{u}}\in Y\subset W$, $\widetilde{\mathbf{u}}|_{\partial\Omega}=\mathbf{0}$, we have

$$\begin{aligned}(\mathbb{A}_{\delta_1}\widetilde{\mathbf{u}}-\delta_1\mathbb{B}\widetilde{\mathbf{u}},\widetilde{\mathbf{u}})&=\int_\Omega(I-N)^{-1}(\nabla\nabla\cdot\widetilde{\mathbf{u}}+\delta_1\widetilde{\mathbf{u}})\cdot(\nabla\nabla\cdot\overline{\widetilde{\mathbf{u}}}+\delta_1N\overline{\widetilde{\mathbf{u}}})\mathrm{d}x-\int_{\Gamma_1}\frac{\delta_1^2}{\lambda}(\nu\cdot\widetilde{\mathbf{u}})(\nu\cdot\overline{\widetilde{\mathbf{u}}})\mathrm{d}s\\&\leq\int_\Omega\frac{1}{1-a_*^{-1}}(\nabla\nabla\cdot\widetilde{\mathbf{u}}+\delta_1\widetilde{\mathbf{u}})\cdot(\nabla\nabla\cdot\overline{\widetilde{\mathbf{u}}}+\delta_1a_*^{-1}\overline{\widetilde{\mathbf{u}}})\mathrm{d}x=0.\end{aligned}$$

This means that Assumption 2 of Theorem 3.3 is also satisfied, and therefore we conclude that there are $M(\varepsilon)$ transmission eigenvalues (counting multiplicity) inside $[\delta_0,\delta_1]$. Note that $M(\varepsilon)$ and $k_{\varepsilon,a_*^{-1}}$ both go to $+\infty$ as $\varepsilon\to 0$. Since the multiplicity of each eigenvalue is finite, we have shown, by letting $\varepsilon \to 0$, that there exists an infinite countable set of transmission eigenvalues that accumulate at $+\infty$.

The proof of the second case is the same, so we omit it for brevity. $\square$

4 The Discreteness of Transmission Eigenvalues for $n \not\equiv 1$

In this section, we only consider the discreteness of transmission eigenvalues (1.1) for the general case $n \not\equiv 1$. Based on Theorem 2.7, we assume that $\mathrm{Im}(A) = 0$, $\mathrm{Im}(n) = 0$, and we introduce the notations

$$n_* := \inf_{x\in\Omega} n(x) > 0, \quad n^* := \sup_{x\in\Omega} n(x) < \infty.$$

We multiply the first equation in (1.1) by a test function $\overline{w'}$ with $w' \in H^1(\Omega)$ and the second equation by a test function $\overline{v'}$ with $v' \in H^1(\Omega)$, then integrate in Ω to obtain:

$$0 = \int_\Omega (\nabla\cdot A\nabla w + k^2 nw)\overline{w'}\mathrm{d}x = \int_{\partial\Omega} \frac{\partial w}{\partial \nu_A}\overline{w'}\mathrm{d}s - \int_\Omega (A\nabla w\cdot\nabla\overline{w'} - k^2 nw\overline{w'})\mathrm{d}x,$$
$$0 = \int_\Omega (\Delta v + k^2 v)\overline{v'}\mathrm{d}x = \int_{\partial\Omega} \frac{\partial v}{\partial \nu}\overline{v'}\mathrm{d}s - \int_\Omega (\nabla v\cdot\nabla\overline{v'} - k^2 v\overline{v'})\mathrm{d}x.$$

Recalling the transmission boundary conditions and $w'|_{\partial\Omega} = v'|_{\partial\Omega}$, we have

$$\begin{aligned} 0 &= \int_{\partial\Omega}\Big(\frac{\partial w}{\partial\nu_A} - \frac{\partial v}{\partial\nu}\Big)\overline{v'}\mathrm{d}s - \int_\Omega (A\nabla w\cdot\nabla\overline{w'} - k^2 nw\overline{w'})\mathrm{d}x + \int_\Omega(\nabla v\cdot\nabla\overline{v'} - k^2 v\overline{v'})\mathrm{d}x \\ &= \int_{\Gamma_1}\lambda v\overline{v'}\mathrm{d}s - \int_\Omega (A\nabla w\cdot\nabla\overline{w'} - k^2 nw\overline{w'})\mathrm{d}x + \int_\Omega(\nabla v\cdot\nabla\overline{v'} - k^2 v\overline{v'})\mathrm{d}x. \end{aligned}$$

We observe that k is a transmission eigenvalue to (1.1) if and only if there exists a non-trivial element $(w, v) \in X$ such that

$$a_k((w,v),(w',v')) = 0, \quad \text{for all } (w',v') \in X,$$

where the sesquilinear form is defined by

$$a_k((w,v),(w',v')) = \int_\Omega (A\nabla w\cdot\nabla\overline{w'} - k^2 nw\overline{w'})\mathrm{d}x - \int_\Omega(\nabla v\cdot\nabla\overline{v'} - k^2 v\overline{v'})\mathrm{d}x - \int_{\Gamma_1}\lambda v\overline{v'}\mathrm{d}s,$$

and the Sobolev space X is $X = \big\{(w,v) \in H^1(\Omega)\times H^1(\Omega) : w - v \in H_0^1(\Omega)\big\}$.

By means of the Riesz representation theorem, there exists a bounded linear operator $\mathcal{G}_k : X \longrightarrow X$ defined by

$$(\mathcal{G}_k(w,v),(w',v'))_X = a_k((w,v),(w',v')), \quad \text{for all } (w',v') \in X.$$

Similarly to Section 3, we want to find a $k \in \mathbb{C}$ such that $\mathcal{G}_k$ is invertible. However, differently from the classical form, we find that $a_k(\cdot,\cdot)$ is not coercive for any $k \in \mathbb{C}$. Hence, we will use the T-coercive method ([3, 21]) to show that $a_{i\kappa}(\cdot,\cdot)$ is T-coercive for $\kappa \in \mathbb{R}\backslash\{0\}$.

Theorem 4.1 Assume that either $a_* > 1$, $n_* > 1$, $\lambda \le -c$ or $a^* < 1$, $n^* < 1$, $\lambda \ge c$. Then there exists $k = i\kappa$ with $\kappa \in \mathbb{R}\backslash\{0\}$ such that the operator $\mathcal{G}_k$ is invertible.

Proof For the first case $a_* > 1$, $n_* > 1$, $\lambda \le -c$, we consider the mapping $T : X \longrightarrow X$ defined by $T : (w,v) \longrightarrow (w, -v + 2w)$. Note that $T^2 = I$, and hence T is an isomorphism in X.

Then for all $(w,v)\in X$, we have that

$$
\begin{aligned}
a^T_{i\kappa}((w,v),(w,v)) &= a_{i\kappa}((w,v),T(w,v)) = a_{i\kappa}((w,v),(w,-v+2w))\\
&= \int_\Omega (A\nabla w\cdot\nabla\overline{w}+\kappa^2 nw\overline{w})\mathrm{d}x - \int_{\Gamma_1}\lambda v\overline{(-v+2w)}\mathrm{d}s\\
&\quad - \int_\Omega \Big[\nabla v\cdot\nabla\overline{(-v+2w)}+\kappa^2 v\overline{(-v+2w)}\Big]\mathrm{d}x\\
&= \int_\Omega (A\nabla w\cdot\nabla\overline{w}+|\nabla v|^2-2\nabla v\cdot\nabla\overline{w})\mathrm{d}x - \int_{\Gamma_1}\lambda v\overline{v}\mathrm{d}s\\
&\quad + \kappa^2\int_\Omega (n|w|^2+|v|^2-2v\overline{w})\mathrm{d}x,
\end{aligned}
$$

where $\kappa\in\mathbb{R}\backslash\{0\}$. Using Hölder's inequality and the facts that $\overline{\xi}\cdot A\xi\ge a_*|\xi|^2$ for all $\xi\in\mathbb{C}^2$, $n\ge n_*$ (note that $\mathrm{Im}(A)=0$ and $\mathrm{Im}(n)=0$), we have

$$
\begin{aligned}
2\Big|\int_\Omega \nabla v\cdot\nabla\overline{w}\mathrm{d}x\Big| &\le 2\Big(\int_\Omega|\nabla w|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_\Omega|\nabla v|^2\mathrm{d}x\Big)^{\frac12}\\
&\le \frac{2}{\sqrt{a_*}}\Big(\int_\Omega A\nabla w\cdot\nabla\overline{w}\mathrm{d}x\Big)^{\frac12}\Big(\int_\Omega|\nabla v|^2\mathrm{d}x\Big)^{\frac12}\\
&\le \frac{1}{\sqrt{a_*}}\Big(\int_\Omega A\nabla w\cdot\nabla\overline{w}\mathrm{d}x+\int_\Omega|\nabla v|^2\mathrm{d}x\Big),\\
2\Big|\int_\Omega v\overline{w}\mathrm{d}x\Big| &\le 2\Big(\int_\Omega|w|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_\Omega|v|^2\mathrm{d}x\Big)^{\frac12}\le\frac{2}{\sqrt{n_*}}\Big(\int_\Omega nw\overline{w}\mathrm{d}x\Big)^{\frac12}\Big(\int_\Omega|v|^2\mathrm{d}x\Big)^{\frac12}\\
&\le \frac{1}{\sqrt{n_*}}\Big(\int_\Omega nw\overline{w}\mathrm{d}x+\int_\Omega|v|^2\mathrm{d}x\Big).
\end{aligned}
$$

Based on the above two equalities and the assumptions, we obtain that

$$
\begin{aligned}
\Big|a^T_{i\kappa}((w,v),(w,v))\Big| &= \Big|\int_\Omega (A\nabla w\cdot\nabla\overline{w}+|\nabla v|^2-2\nabla v\cdot\nabla\overline{w})\mathrm{d}x\\
&\quad+\kappa^2\int_\Omega (n|w|^2+|v|^2-2v\overline{w})\mathrm{d}x-\int_{\Gamma_1}\lambda v\overline{v}\mathrm{d}s\Big|\\
&\ge (A\nabla w,\nabla w)+(\nabla v,\nabla v)-2|(\nabla v,\nabla w)|+\kappa^2\big[(nw,w)+(v,v)-2|(v,w)|\big]\\
&\ge \Big(1-\frac{1}{\sqrt{a_*}}\Big)\big[(A\nabla w,\nabla w)+(\nabla v,\nabla v)\big]+\kappa^2\Big(1-\frac{1}{\sqrt{n_*}}\Big)\big[(nw,w)+(v,v)\big].
\end{aligned}
$$

That is,

$$
\Big|a^T_{i\kappa}((w,v),(w,v))\Big|\ge\min\Big\{\Big(1-\frac{1}{\sqrt{a_*}}\Big),\kappa^2\Big(1-\frac{1}{\sqrt{n_*}}\Big)\Big\}\big(\|w\|^2_{H^1(\Omega)}+\|v\|^2_{H^1(\Omega)}\big).
$$

The above estimate proves that $a^T_{i\kappa}$ is coercive over X, which implies that $\mathcal{G}_{i\kappa}$ is invertible.

For the second case $0<a_*<a^*<1$, $0<n_*<n^*<1$ and $\lambda\ge c$, the isomorphism mapping $T:X\longrightarrow X$ is defined by

$$
T:(w,v)\ \longrightarrow\ (w-2v,-v).
$$

For all $(w,v)\in X$, we have

$$
\begin{aligned}
a^T_{i\kappa}((w,v),(w,v)) &= \int_\Omega (A\nabla w\cdot\nabla\overline{w}+|\nabla v|^2-2A\nabla w\cdot\nabla\overline{v})\mathrm{d}x+\int_{\Gamma_1}\lambda v\overline{v}\mathrm{d}s\\
&\quad+\kappa^2\int_\Omega (n|w|^2+|v|^2-2nw\overline{v})\mathrm{d}x.
\end{aligned}
$$

Similarly, we will used the following two inequalities:

$$2\Big|\int_\Omega A\nabla w\cdot\nabla\overline{v}\mathrm{d}x\Big|\le\sqrt{a^*}\Big(\int_\Omega A\nabla w\cdot\nabla\overline{w}\mathrm{d}x+\int_\Omega|\nabla v|^2\mathrm{d}x\Big),$$
$$2\Big|\int_\Omega nw\overline{v}\mathrm{d}x\Big|\le\sqrt{n^*}\Big(\int_\Omega nw\overline{w}\mathrm{d}x+\int_\Omega|v|^2\mathrm{d}x\Big).$$

Then, we obtain that

$$\begin{aligned}\Big|a_{i\kappa}^T((w,v),(w,v))\Big|&\ge(1-\sqrt{a^*})\big[(A\nabla w,\nabla w)+(\nabla v,\nabla v)\big]+\kappa^2(1-\sqrt{n^*})\big[(nw,w)+(v,v)\big]\\&\ge\min\Big\{\big(1-\sqrt{a^*}\big)a_*,\kappa^2\big(1-\sqrt{n^*}\big)n_*\Big\}\big(\|w\|^2_{H^1(\Omega)}+\|v\|^2_{H^1(\Omega)}\big).\end{aligned}$$

The above estimate proves that a_{ik}^T is coercive. This completes the proof. □

Considering $\kappa_0\in\mathbb{R}\backslash\{0\}$, then $\mathcal{G}_{i\kappa_0}$ is invertible. Since $\mathcal{G}_k-\mathcal{G}_{i\kappa_0}$ is a compact operator for all $k\in\mathbb{C}$, using the analytic Fredholm theory (see Section 8.5 in the book [23]), we deduce the following theorem:

Theorem 4.2 Assume that either $a_*>1$, $n_*>1$, $\lambda\le -c$ or $0<a_*<a^*<1$, $0<n_*<n^*<1$, $\lambda\ge c$. Then the transmission eigenvalues of (1.1) form a discrete (possibly empty) set in $\mathbb{C}$ with $+\infty$ as the only possible accumulation point.

Remark 4.3 For the case $n\not\equiv 1$ and $A\not\equiv\mathcal{I}$ ($\mathcal{I}$ is the identity matrix), we only obtain the discreteness under the assumption that $a_*>1$, $n_*>1$, $\lambda\le -c$ or $0<a_*<a^*<1$, $0<n_*<n^*<1$, $\lambda\ge c$. The discreteness for weaker assumptions and the existence of eigenvalues are open problems.

Remark 4.4 In this paper, we consider two cases: $A\not\equiv\mathcal{I}$, $n\equiv 1$ and $A\not\equiv\mathcal{I}$, $n\not\equiv 1$. For the third case, $A\equiv\mathcal{I}$, $n\not\equiv 1$, the discreteness and existence of transmission eigenvalues can be achieved by the same procedure as in the paper [2] where the boundary is integral.

Remark 4.5 If $\Omega\subset\mathbb{R}^3$ simply connected and $\nabla\times\boldsymbol{v}=0$ in Ω, then there exists a potential p such that $\boldsymbol{v}=\nabla p$. Based on this fact, the results in this paper can be extended for the 3-dimensional case.

Conflict of Interest The authors declare no conflict of interest.

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