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ON DE FINETTI'S OPTIMAL IMPULSE DIVIDEND CONTROL PROBLEM UNDER CHAPTER 11 BANKRUPTCY*

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Abstract Motivated by recent advances made in the study of dividend control and risk management problems involving the U.S. bankruptcy code, in this paper we follow [44] to revisit the De Finetti dividend control problem under the reorganization process and the regulator's intervention documented in U.S. Chapter 11 bankruptcy. We do this by further accommodating the fixed transaction costs on dividends to imitate the real-world procedure of dividend payments. Incorporating the fixed transaction costs transforms the targeting optimal dividend problem into an impulse control problem rather than a singular control problem, and hence computations and proofs that are distinct from [44] are needed. To account for the financial stress that is due to the more subtle concept of Chapter 11 bankruptcy, the surplus process after dividends is driven by a piece-wise spectrally negative Lévy process with endogenous regime switching. Some explicit expressions of the expected net present values under a double barrier dividend strategy, new to the literature, are established in terms of scale functions. With the help of these expressions, we are able to characterize the optimal strategy among the set of admissible double barrier dividend strategies. When the tail of the Lévy measure is log-convex, this optimal double barrier dividend strategy is then verified as

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the optimal dividend strategy, solving our optimal impulse control problem.

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1 Introduction

As an alternative risk management tool to that of ruin probability, De Finetti's dividend control problem has remained an active research topic in corporate finance and insurance for decades, mainly due to its effectiveness in signaling the health and stability of financial companies. The goal of this type of risk management is to maximize the expected total net present value (NPV) of accumulated dividend payments, which is conventionally referred to as the value of the company. The literature of the past decades has witnessed fruitful research (see [16, 17, 21, 42] for some early works along these lines) on De Finetti's dividend optimization involving stochastic regular, singular, or impulse control problems under a number of risk models, among which the spectrally negative Lévy risk model has been gaining in popularity in insurance applications, due to its capability to model the reserve process of an insurance company that collects premiums continuously and pays claim payments in lump sums. Early works on De Finetti's dividend control problem under the spectrally negative Lévy risk model are [6, 25, 31, 32, 41]; see also the references therein. In particular, the works of [6, 31, 32, 43–45] verified that the optimal dividend strategy yielding the maximum NPV of accumulated dividends is the barrier strategy, here the fluctuation theory of spectrally negative Lévy processes and the standard approach of the Hamilton-Jacobi-Bellman (HJB) equation was adopted. For a more comprehensive review of developments in optimal dividends and the related methodology, we refer to two survey papers, [1] and [3], where thorough and insightful reviews on the classical contributions and recent progress in the dividend control field are provided. In addition, a variety of recent works have taken into account new and different risk factors, control constraints or model generalizations; see, for example, [4, 5, 10, 11, 15, 22, 28, 35, 36, 39, 43, 44, 46], etc..

In recent years, the modelling of the liquidation process (Chapter 7 bankruptcy) and the reorganization process (Chapter 11 bankruptcy) written in the bankruptcy code of the United States has attracted more and more research attention among at the insurance and finance communities; see, [8, 9, 13, 14, 27, 37] as well as the references therein. To get a real-world picture of Chapter 7 and Chapter 11 bankruptcy, [29] used a piece-wise time-homogeneous diffusion process, as well as three constant barrier levels, a, b and c ($a < b < c$), to model the reserve process of an insurance company. The lower barrier a represents the liquidation barrier, i.e., once the reserve process falls below a, the company is liquidated because its assets can no longer cover its debts. The middle barrier b represents the reorganization barrier, i.e., once the reserve process falls below b, the insurer enters a state of insolvency (the businesses of an insolvent insurer are subject to reorganization under the interventions of the regulator), and it may either return to the solvent state (a solvent insurer is free of interventions) if the reserve process recovers to the upper barrier c within the grace period granted by the regulator, or it remains in a state of insolvency and is then liquidated. The upper barrier c represents the solvency barrier, i.e., an insurer who possesses a reserve above this barrier is solvent, since it is able to meet its liabilities, and the solvent insurer will not switch to the insolvent state unless the reserve process falls below b at some future time. In addition, the dynamics of the reserve process, subject to the state of the insurer, switches between two time-homogeneous diffusion processes with different drifts and volatilities. [29] obtained closed-form expressions of the liquidation probability and the Laplace transform of the liquidation time.

Inspired by the above mentioned works on the De Finetti dividend problems and the financial modelling of the liquidation and reorganization process, [44] considered a variant of the De Finetti optimal dividend control problem by incorporating an appealing feature of Chapter 11 bankruptcy to the piece-wise spectrally negative Lévy risk processes embedded with a reorganization barrier b and a solvency barrier c ($b < c$); it turned out that a single barrier dividend strategy is the optimal dividend strategy. In this paper, to better imitate the real-world procedure of dividend payments, we would like to incorporate the real-life factor of fixed transaction costs on dividends into our new targeting variant for the De Finetti dividend optimization problem. In addition, we follow [44] in assuming that the uncontrolled reserve process (i.e., free of dividends) evolves as two spectrally negative Lévy processes switching between each other, where a change in the state of the insurer triggers a switch of the dynamics of the reserve process. To match the real life situation, we also assume that dividends are paid only when the reserve is higher than c. An analytical characterization of the optimal strategy in the set of all double barrier admissible dividend strategies is provided. Then important properties of the optimal double barrier levels are studied. A sufficient condition that the Lévy measure has a log-convex tail is finally found; under that our optimal control problem is solvable, in that the optimal double barrier dividend strategy dominates all admissible impulse dividend strategies. We mention that, since no fixed transaction costs are considered, the dividend optimization problem addressed in [44] is a singular control problem. With the presence of fixed transaction costs, our new control problem becomes an impulse control problem, rather than a singular control problem as in [44]. Therefore, compared with [44], we need distinct computations and arguments to solve our control problem and characterize the optimal impulse dividend strategy; for example, some different deep understandings of the scale functions and delicate computations on generators and slope conditions are needed. Another contribution of the current paper lies in that it helps understanding of how the impulse dividend decision can be affected by reorganization and by regulator's intervention for a concrete example, see Section 4, where detailed discussions are provided.

The rest of the paper is organized as follows: some preliminary results on spectrally negative Lévy processes are presented in Section 2. Section 3 focuses on solving De Finetti's optimal impulse dividend control problem by accommodating fixed transaction costs under Chapter 11 bankruptcy; here the optimal impulse control is shown to fit the double-barrier type dividend strategy by following a "guess-and-verify" procedure. In Section 4, a concrete example is provided and analyzed to illustrate the main results obtained in Section 3.

2 Preliminaries on Spectrally Negative Lévy Processes

We collect in this section some elementary facts on the spectrally negative Lévy processes; interested readers may refer to [23] for more details. A spectrally negative Lévy process is an upward-jump-free stochastic process having stationary and independent increments. Denote by $X = \{X(t); t \geq 0\}$ a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \{\mathcal{F}_t; t \geq 0\}, \mathbb{P})$ satisfying the usual conditions. To avoid trivialities, we assume that X has no monotone paths. Let $\mathbb{P}_x$ be the conditional probability, given that X starts from x, and let $\mathbb{E}_x$ be the corresponding expectation operator. For simplicity, write $\mathbb{P}$ and $\mathbb{E}$ for $\mathbb{P}_0$ and $\mathbb{E}_0$, respectively. The Laplace transform of X is given by

$$\mathbb{E}\big(\mathrm{e}^{\theta X(t)}\big) = \mathrm{e}^{t\psi(\theta)}, \quad \theta \in \mathbb{R}_+,$$

where $\mathbb{R}_+ = [0, \infty)$, and

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (\mathrm{e}^{-\theta z} - 1 + \theta z \mathbf{1}_{(0,1]}(z))\upsilon(\mathrm{d}z), \quad \gamma \in \mathbb{R},\, \sigma \in \mathbb{R}_+,$$

with the σ-finite Lévy measure υ supported on $(0, \infty)$ satisfying that $\int_0^\infty (1 \wedge z^2)\upsilon(\mathrm{d}z) < \infty$. Here, $(\gamma, \sigma, \upsilon)$ is referred to as the Lévy triplet of X. It is known that the Laplace exponent ψ is strictly convex and that $\lim_{\theta\to\infty} \psi(\theta) = \infty$, hence there is a well-defined right inverse function of ψ given as $\Phi_q := \sup\{\theta \in \mathbb{R}_+ : \psi(\theta) = q\}$.

For each $q \in \mathbb{R}_+$, let us follow Chapter 8 of [23] to define the scale function of X, denoted by $W_q : \mathbb{R}_+ \to \mathbb{R}_+$, as the unique continuous and strictly increasing function defined on $\mathbb{R}_+$ such that

$$\int_0^\infty \mathrm{e}^{-\theta x} W_q(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta \in (\Phi_q, \infty).$$

By convention, we may extend the definition of $W_q(x)$ by letting $W_q(x) = 0$ for $x < 0$. Associated to W_q, we have two derivative scale functions, $Z_q(x, \theta)$ and $Z_q(x)$, defined as

$$Z_q(x,\theta) : = \mathrm{e}^{\theta x}\left(1 - (\psi(\theta) - q)\int_0^x \mathrm{e}^{-\theta w} W_q(w)\mathrm{d}w\right), \quad x \in \mathbb{R}_+,\, q \in \mathbb{R}_+,\, \theta \in \mathbb{R}_+, \tag{2.1}$$

and

$$Z_q(x) := Z_q(x, 0) = 1 + q\int_0^x W_q(z)\mathrm{d}z, \quad x \in \mathbb{R}_+,\, q \in \mathbb{R}_+,$$

with $Z_q(x) \equiv 1$ on $(-\infty, 0)$. We write $W := W_0$ and $Z := Z_0$, for short. It is well known that

$$\lim_{x\to\infty} \frac{W_q'(x)}{W_q(x)} = \Phi_q, \quad \lim_{y\to\infty} \frac{W_q(x+y)}{W_q(y)} = \mathrm{e}^{\Phi_q x}. \tag{2.2}$$

We recall that the scale function W_q is left and right differentiable at $x \in (0, \infty)$. Furthermore, W_q is continuously differentiable on $(0, \infty)$ if X has sample paths of unbounded variation or has sample paths of bounded variation and the Lévy measure is atomless; in particular, it is twice continuously differentiable on $(0, \infty)$ if X has a nontrivial Gaussian component. We are refer to [31] for more analytical properties of the scale functions. To make our impulse control problem solvable, we shall assume throughout this paper that the tail of the Lévy measure υ is log-convex, and hence the scale function $W_q(x)$ is continuously differentiable and $W_q'(x)$ is log-convex, implying that $W_q'(x)$ is right and left differentiable over $(0, \infty)$ and is differentiable over $(0, \infty)$ except for countably many points. In the sequel, by $W_q''(x)$, we mean the right derivative of $W_q'(x)$ when $W_q'(x)$ is not differentiable at x.

For later use, we introduce another spectrally negative Lévy process, $\widetilde{X} = \{\widetilde{X}(t); t \geq 0\}$ with the Lévy triplet $(\widetilde{\gamma}, \widetilde{\sigma}, \widetilde{\upsilon})$, the Laplace exponent $\widetilde{\psi}$, and the inverse Laplace exponent $\widetilde{\Phi}_q$. Denote by $\mathbb{W}_q$ and $\mathbb{Z}_q$ the scale functions of $\widetilde{X}$ defined in a way similar W_q and Z_q, but with

ψ replaced by $\widetilde{\psi}$. Write $\mathbb{W} = \mathbb{W}_0$ and $\mathbb{Z} = \mathbb{Z}_0$. In addition, denote by $\widetilde{\mathbb{P}}_x$ the conditional probability, given that $\widetilde{X}_0$ starts from x, and by $\widetilde{\mathbb{E}}_x$ the corresponding conditional expectation.

3 De Finetti's Optimal Dividend under Chapter 11 Bankruptcy and Fixed Transaction Costs

This section investigates De Finetti's optimal dividend control problem with fixed transaction costs on dividends under the Chapter 11 bankruptcy is written in the U.S. bankruptcy code. To describe the dynamics of switching under the regulator's intervention, we introduce an auxiliary state process $I(t)$, subject to the dividend control, as an indicator of the process of solvency and insolvency states. Supposing that the insurer is solvent at time $t \geq 0$ (i.e., $I(t) = 0$), it remains solvent until the reserve process U falls below the reorganization barrier b, at which time the state of the insurer is switched to that of insolvency. On the other hand, if the insurer is in the insolvent state at time $t \geq 0$ (i.e., $I(t) = 1$), then it remains in the insolvent state until the reserve process climbs up to the safety barrier c, at which period the state of the insurer is switched to that of being solvent. As a result, the dynamics of the reserve process U follows as a spectrally negative Lévy process X deduced with dividends whenever the insurer is solvent, and U is governed by the spectrally negative Lévy process $\widetilde{X}$ whenever the insurer is insolvent. We mention that Chapter 11 bankruptcy takes place if the insurer stays continuously in a state of insolvency for a time interval greater than that of the grace time limit granted by the regulator. It also needs to be mentioned that, due to the additional costs at the time of dividend payment, we shall restrict the admissible controls to impulse dividend strategies, instead of regular or singular dividend strategies.

3.1 Problem Formulation

We assume that transaction costs are paid whenever the cumulative dividend payment process $(D_t)_{t\geq 0}$ increases, i.e., whenever a dividend is distributed, a lump sum of transaction costs with a fixed amount should be paid. To avoid infinitely large transaction costs, the cumulative dividend process $(D_t)_{t\geq 0}$ needs to be a non-decreasing and left-continuous pure-jump process such that $D(t) = \sum\limits_{s<t} \triangle D(t)$ with $\triangle D(t) = D(t+) - D(t)$ for $t \geq 0$ (the so-called impulse control). We conjecture and aim to show that the optimal impulse dividend control under a fixed transaction costs fits this type of double barrier dividend strategy. Let us first construct the piece-wise underlying risk process using an impulse dividend strategy.

Definition 3.1 (The risk process U under an impulse dividend strategy D) Let $D = (D(t))_{t\geq 0}$ be a pure-jump non-decreasing and left-continuous process, i.e., $D(t) = \sum\limits_{s<t} \triangle D(t)$ with $\triangle D(t) = D(t+) - D(t)$ for $t \geq 0$. Recalling that $b < c$, let $U(0) := X(0) \in (b, \infty)$ if $I(0) = 0$, and let $U(0) := \widetilde{X}(0) \in (-\infty, c)$ if $I(0) = 1$. In addition, set that $\mathcal{T}_0 := 0$. The process (U, I) can be constructed recursively as follows:

(a) Suppose that the process (U, I) has been defined on $[0, \mathcal{T}_n]$ for some $n \geq 0$ with $\mathcal{T}_n < \infty$.

- Then, if $I(\mathcal{T}_n) = 0$, we define $U_{n+1} = (U_{n+1}(\mathcal{T}_n + t))_{t\geq 0}$ according to

$$U_{n+1}(\mathcal{T}_n + t) = U(\mathcal{T}_n) + X(\mathcal{T}_n + t) - X(\mathcal{T}_n) - \sum_{s\in[\mathcal{T}_n, \mathcal{T}_n+t)} \mathbf{1}_{\{U_{n+1}(s)-c\geq \triangle D(s)\}} \triangle D(s), t \geq 0,$$

and update $\mathcal{T}_{n+1} := \mathcal{T}_n + \inf\{t \geq 0; U_{n+1}(\mathcal{T}_n + t) < b\}$. We then define the process (U, I) on the time interval $(\mathcal{T}_n, \mathcal{T}_{n+1}]$ by

$$\begin{cases} (U(\mathcal{T}_n + t), I(\mathcal{T}_n + t)) := (U_{n+1}(\mathcal{T}_n + t), 0), & t \in (0, \mathcal{T}_{n+1} - \mathcal{T}_n), \\ (U(\mathcal{T}_{n+1}), I(\mathcal{T}_{n+1})) := (U_{n+1}(\mathcal{T}_{n+1}), 1). \end{cases}$$

- Otherwise, if $I(\mathcal{T}_n) = 1$, we define $\widetilde{U}_{n+1} = (\widetilde{U}_{n+1}(\mathcal{T}_n + t))_{t \geq 0}$ by

$$\widetilde{U}_{n+1}(\mathcal{T}_n + t) := U(\mathcal{T}_n) + \widetilde{X}(\mathcal{T}_n + t) - \widetilde{X}(\mathcal{T}_n), \quad t \geq 0,$$

and update

$$\mathcal{T}_{n+1} := \mathcal{T}_n + \inf\{t \geq 0; \widetilde{U}_{n+1}(\mathcal{T}_n + t) \geq c\}.$$

We then define the process (U, I) on $(\mathcal{T}_n, \mathcal{T}_{n+1}]$ as

$$\begin{cases} (U(\mathcal{T}_n + t), I(\mathcal{T}_n + t)) := \left(\widetilde{U}_{n+1}(\mathcal{T}_n + t), 1\right), & t \in (0, \mathcal{T}_{n+1} - \mathcal{T}_n), \\ (U(\mathcal{T}_{n+1}), I(\mathcal{T}_{n+1})) := \left(\widetilde{U}_{n+1}(\mathcal{T}_{n+1}), 0\right) = (c, 0). \end{cases}$$

(b) Supposing that the process (U, I) has been defined on $[0, \mathcal{T}_n]$ for some $n \geq 0$ with $\mathcal{T}_n = \infty$, we update $\mathcal{T}_{n+1} = \infty$.

Using the path and distributional properties of spectrally negative Lévy processes, we know that it takes the reserve process U a positive amount of time to climb from a level below b up to the level c with a probability of one. Then, one can easily verify that $\lim_{n \to \infty} \mathcal{T}_{2n} = \infty$, and hence the process U is well-defined under impulse controls. In addition, the bi-variate process (U, I) is Markovian. We denote by $\mathbb{P}_{x,0}$ the probability law of (U, I), conditional upon having that $(U(0), I(0)) = (x, 0)$ for $x \in (b, \infty)$, and denote by $\mathbb{P}_{x,1}$ the probability law of (U, I) conditional upon having that $(U(0), I(0)) = (x, 1)$ for $x \in (-\infty, c)$.

Let us consider

$$\kappa := \inf\{k \geq 1 : \mathcal{T}_k + e_\lambda^k < \mathcal{T}_{k+1},\ I(\mathcal{T}_k) = 1\},$$

with the convention that $\inf \emptyset = +\infty$. The Chapter 11 bankruptcy time can be defined by

$$T_D := \begin{cases} \mathcal{T}_\kappa + e_\lambda^\kappa & \text{when} \quad \kappa < \infty, \\ \infty, & \text{when} \quad \kappa = \infty, \end{cases} \tag{3.1}$$

where $\{e_\lambda^k\}_{k \geq 1}$ is a sequence of independent random variables having a common exponential distribution with a mean of $1/\lambda$. In particular, $\{e_\lambda^k\}_{k \geq 1}$ models the sequence of grace times the regulator grants to the managers of financial companies. It is also supposed that X and $\widetilde{X}$ are independent of $\{e_\lambda^k\}_{k \geq 1}$.

Let us denote by $\mathcal{D}$ the set of all admissible impulse dividend controls, which consists of all pure-jump non-decreasing and left-continuous $\mathbf{F}$-adapted processes. For an admissible impulse dividend control $D \in \mathcal{D}$, we consider the next two expected NPVs with the fixed unit transaction cost ϕ such that

$$V_D(x) := \mathbb{E}_{x,0}\left[\sum_{0 \leq s \leq T_D} \mathrm{e}^{-qt}\Big(\triangle D(t) - \phi\Big)\mathbf{1}_{\{U(t) - c \geq \triangle D(t)\}}\right], \quad x \in (b, \infty), \tag{3.2}$$

and

$$\widetilde{V}_D(x) := \mathbb{E}_{x,1}\left[\sum_{0 \leq s \leq T_D} \mathrm{e}^{-qt}\Big(\triangle D(t) - \phi\Big)\mathbf{1}_{\{U(t) - c \geq \triangle D(t)\}}\right], \quad x \in (-\infty, c), \tag{3.3}$$

where T_D is the Chapter 11 bankruptcy time defined by (3.1). Our De Finetti's optimal dividend problem under Chapter 11 bankruptcy and fixed transaction costs is to find the optimal impulse control D^* that attains the maximal value function in the sense that

$$V_{D^*}(x)=\sup_{D\in\mathcal{D}}V_D(x)\text{ for all }x\in(b,\infty)\quad\text{and}\quad \widetilde{V}_{D^*}(x)=\sup_{D\in\mathcal{D}}\widetilde{V}_D(x)\ \text{ for all }\ x\in(-\infty,c). \tag{3.4}$$

3.2 Expected NPVs of Dividends under a Double Barrier Strategy

Let us now consider a double barrier (z_1,z_2) impulse dividend strategy $D_{z_1}^{z_2}$ $(c\le z_1<z_2)$; namely, whenever the surplus level is above the upper level z_2, a lump sum dividend is paid, bringing the surplus value down to the lower level z_1, while no dividend is paid when the surplus level is below z_2. Hence, the risk process U under the (z_1,z_2) impulse dividend strategy can be constructed by Definition 3.1, with D being replaced by the more specific $D_{z_1}^{z_2}$, which can be written as

$$D_{z_1}^{z_2}(\mathcal{T}_n+t)-D_{z_1}^{z_2}(\mathcal{T}_n)=[U(\mathcal{T}_n)\vee z_2-z_1]\mathbf{1}_{\{\sigma_n^1<t,I(\mathcal{T}_n)=0\}}+\sum_{i=2}^{\infty}[z_2-z_1]\mathbf{1}_{\{\sigma_n^i<t,I(\mathcal{T}_n)=0\}},$$
$$t\le\mathcal{T}_{n+1}-\mathcal{T}_n,\ n\ge1,$$

where

$$\sigma_n^i:=\inf\{t\ge0:U(\mathcal{T}_n)+X(\mathcal{T}_n+t)-X(\mathcal{T}_n)>U(\mathcal{T}_n)\vee z_2+(z_2-z_1)(i-1)\},\ i\ge1,\ n\ge1.$$

Let $V_{z_1}^{z_2}(x)$ and $\widetilde{V}_{z_1}^{z_2}(x)$ denote the expected NPVs in (3.2) and (3.3), respectively, when the double barrier (z_1,z_2) impulse dividend strategy is employed. The next result shows that $V_{z_1}^{z_2}(x)$ and $\widetilde{V}_{z_1}^{z_2}(x)$ can be expressed in terms of the scale functions. As in [44], we define the key auxiliary function $\ell_{b,c}^{(q,\lambda)}(x)$ on $\mathbb{R}$ as

$$\begin{aligned}\ell_{b,c}^{(q,\lambda)}(x)=\ &W_q(x-b)(1-\mathrm{e}^{-\widetilde{\Phi}_{q+\lambda}(c-b)}Z_q(c-b,\widetilde{\Phi}_{q+\lambda}))\\&+\mathrm{e}^{-\widetilde{\Phi}_{q+\lambda}(c-b)}W_q(c-b)Z_q(x-b,\widetilde{\Phi}_{q+\lambda}).\end{aligned}\tag{3.5}$$

Actually, $\ell_{b,c}^{(q,\lambda)}$ acts as the scale function of our piece-wise spectrally negative Lévy processes with endogenous regime switching.

Proposition 3.2 We have that

$$V_{z_1}^{z_2}(x)=\begin{cases}\ell_{b,c}^{(q,\lambda)}(x)\dfrac{z_2-z_1-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}, & x\in(b,z_2],\\[2ex] x-z_1-\phi+\dfrac{\ell_{b,c}^{(q,\lambda)}(z_1)\,(z_2-z_1-\phi)}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}, & x\in(z_2,\infty),\end{cases}\tag{3.6}$$

and

$$\widetilde{V}_{z_1}^{z_2}(x)=\frac{\mathrm{e}^{\widetilde{\Phi}_{q+\lambda}(x-c)}\,W_q(c-b)}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}\,(z_2-z_1-\phi)\,,\quad x\in(-\infty,c).\tag{3.7}$$

Proof Set that $T_\lambda:=T_D|_{D\equiv0}$ and $\zeta_z^+:=\inf\{t\ge0;U(t)\ge z\}|_{D\equiv0}$, with T_D and U being defined by (3.1) and Definition 3.1, respectively. By Lemma 3.1 in [44], we have that

$$\mathbb{E}_{x,0}\left[\mathrm{e}^{-q\zeta_z^+}\mathbf{1}_{\{\zeta_z^+<T_\lambda\}}\right]=\frac{\ell_{b,c}^{(q,\lambda)}(x)}{\ell_{b,c}^{(q,\lambda)}(z)},-\infty<b<x<z,\ c\le z<\infty,\tag{3.8}$$

and

$$\mathbb{E}_{x,1}\left[\mathrm{e}^{-q\zeta_z^+}\mathbf{1}_{\{\zeta_z^+<T_\lambda\}}\right]=\mathrm{e}^{\widetilde{\Phi}_{q+\lambda}(x-c)}\frac{\ell_{b,c}^{(q,\lambda)}(c)}{\ell_{b,c}^{(q,\lambda)}(z)}=\frac{\mathrm{e}^{\widetilde{\Phi}_{q+\lambda}(x-c)}W_q(c-b)}{\ell_{b,c}^{(q,\lambda)}(z)},\ -\infty<x<c\leq z<\infty. \tag{3.9}$$

Then, it follows from (3.8) that

$$\begin{aligned}V_{z_1}^{z_2}(x)&=\mathbb{E}_{x,1}\Big(\mathrm{e}^{-q\zeta_{z_2}^+}\mathbf{1}_{\{\zeta_{z_2}^+<T_\lambda\}}\Big)\left(z_2-z_1-\phi+V_{z_1}^{z_2}(z_1)\right)\\&=\frac{\ell_{b,c}^{(q,\lambda)}(x)}{\ell_{b,c}^{(q,\lambda)}(z_2)}\big(z_2-z_1-\phi+V_{z_1}^{z_2}(z_1)\big),\quad x\in(b,z_2],\end{aligned} \tag{3.10}$$

which yields that

$$V_{z_1}^{z_2}(z_1)=\frac{\ell_{b,c}^{(q,\lambda)}(z_1)}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}\left(z_2-z_1-\phi\right). \tag{3.11}$$

In view of (3.11) and (3.10), we readily have (3.6). In addition, (3.7) is a consequence of (3.9) and the fact that

$$\widetilde{V}_{z_1}^{z_2}(x)=\mathbb{E}_{x,1}\left[\mathrm{e}^{-q\zeta_c^+}\mathbf{1}_{\{\zeta_c^+<T_\lambda\}}\right]V_{z_1}^{z_2}(c),\quad x\in(-\infty,c),$$

which completes the proof. □

3.3 Optimal Double Barriers and Verification of the Optimality

For fixed b and c such that $-\infty<b<c<\infty$, let us define the auxiliary function

$$\xi(z_1,z_2):=\frac{z_2-z_1-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)},\quad c\leq z_1\leq z_2-\phi,$$

and consider the set of candidate optimal barriers defined by

$$\mathcal{M}:=\{(z_1^*,z_2^*):c\leq z_1^*\leq z_2^*-\phi\text{ and }\xi(z_1^*,z_2^*)\geq\xi(z_1,z_2)\text{ for all }c\leq z_1\leq z_2-\phi\}, \tag{3.12}$$

which stands for the set of the maximizers of the above bi-variate function ξ. The following lemma investigates some properties of $\mathcal{M}$ that are useful for further computations and analysis; see Proposition 3.2 and the proof of Lemma 3.3, etc.:

Lemma 3.3 The set $\mathcal{M}$ defined in (3.12) is non-empty, and there exists a $z_0\in(c,\infty)$ such that

$$\mathcal{M}\subseteq\{(z_1,z_2):c\leq z_1<z_2-\phi,z_2\leq z_0\}.$$

For $(z_1,z_2)\in\mathcal{M}$, we either have $z_1=c$ and

$$\frac{z_2-z_1-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}=\frac{1}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)}, \tag{3.13}$$

or we have $z_1\in(c,\infty)$ and

$$\frac{z_2-z_1-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}=\frac{1}{\ell_{b,c}^{(q,\lambda)\prime}(z_1)}=\frac{1}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)}. \tag{3.14}$$

Proof From (2.1) and the L'Hôspital's rule, it can be verified that

$$\lim_{x\to\infty}\frac{Z_q(x,\widetilde{\Phi}_{q+\lambda})}{W_q(x)}=\begin{cases}\dfrac{\psi(\widetilde{\Phi}_{q+\lambda})-q}{\widetilde{\Phi}_{q+\lambda}-\Phi_q}, & \widetilde{\Phi}_{q+\lambda}>\Phi_q,\\ \psi'(\Phi_q), & \widetilde{\Phi}_{q+\lambda}=\Phi_q,\\ \dfrac{q-\psi(\widetilde{\Phi}_{q+\lambda})}{\Phi_q-\widetilde{\Phi}_{q+\lambda}}, & \widetilde{\Phi}_{q+\lambda}<\Phi_q.\end{cases}$$

It is then straightforward to check that

$$\begin{aligned}&\lim_{z\uparrow\infty}\frac{\ell_{b,c}^{(q,\lambda)}(z)}{W_q(z-b)}\\ =&\begin{cases}\big[\psi(\widetilde{\Phi}_{q+\lambda})-q\big]\Big[\displaystyle\int_0^{c-b}\mathrm{e}^{-\widetilde{\Phi}_{q+\lambda}w}W_q(w)\mathrm{d}w+\frac{\mathrm{e}^{-\widetilde{\Phi}_{q+\lambda}(c-b)}W_q(c-b)}{\widetilde{\Phi}_{q+\lambda}-\Phi_q}\Big], & \widetilde{\Phi}_{q+\lambda}>\Phi_q,\\ \mathrm{e}^{-\widetilde{\Phi}_{q+\lambda}(c-b)}W_q(c-b)\,\psi'(\Phi_q), & \widetilde{\Phi}_{q+\lambda}=\Phi_q,\\ \big[q-\psi(\widetilde{\Phi}_{q+\lambda})\big]\Big[\dfrac{\mathrm{e}^{-\widetilde{\Phi}_{q+\lambda}(c-b)}W_q(c-b)}{\Phi_q-\widetilde{\Phi}_{q+\lambda}}-\displaystyle\int_0^{c-b}\mathrm{e}^{-\widetilde{\Phi}_{q+\lambda}w}W_q(w)\mathrm{d}w\Big], & \widetilde{\Phi}_{q+\lambda}<\Phi_q,\end{cases}\end{aligned}$$

which, by the arguments used in the proof of Case (ii) of Lemma 3.2 in [44], is positive. It is therefore deduced that

$$\begin{aligned}\lim_{z_2\to\infty}\xi(z_1,z_2)&=\lim_{z_2\to\infty}\frac{1}{\frac{\ell_{b,c}^{(q,\lambda)}(z_2)}{W_q(z_2-b)}-\frac{\ell_{b,c}^{(q,\lambda)}(z_1)}{W_q(z_2-b)}}\frac{z_2-z_1-\phi}{W_q(z_2-b)}\\ &=\frac{1}{\lim\limits_{z_2\to\infty}\frac{\ell_{b,c}^{(q,\lambda)}(z_2)}{W_q(z_2-b)}}\lim_{z_2\to\infty}\frac{1}{W_q'(z_2-b)}=0\end{aligned}\tag{3.15}$$

as $\lim\limits_{x\to\infty}W_q'(x)=\infty$. By the mean value theorem, it holds that

$$\lim_{z_1\to\infty}\xi(z_1,z_2)\le\lim_{z_2\to\infty}\frac{1}{\inf_{z\in[z_1,z_2]}\ell_{b,c}^{(q,\lambda)\prime}(z)}\frac{z_2-z_1-\phi}{z_2-z_1}=0\tag{3.16}$$

as $\lim\limits_{z\to\infty}\ell_{b,c}^{(q,\lambda)\prime}(z)=\infty$. Combining (3.15), (3.16) and the definition of ξ, one can get the desired conclusion: that the non-empty set of maximizers of the function ξ is a subset of $\{(z_1,z_2):c\le z_1<z_2-\phi,z_2\le z_0\}$.

To continue, for $(z_1,z_2)\in\mathcal{M}$, we have that

$$0=\frac{\partial}{\partial z_2}\xi(z_1,z_2)=\frac{1}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}-\frac{(z_2-z_1-\phi)\,\ell_{b,c}^{(q,\lambda)\prime}(z_2)}{\Big(\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)\Big)^2},$$

which gives (3.13). Otherwise, if $z_1\in(c,\infty)$, we have that

$$0=\frac{\partial}{\partial z_1}\xi(z_1,z_2)=\frac{-1}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}+\frac{(z_2-z_1-\phi)\,\ell_{b,c}^{(q,\lambda)\prime}(z_1)}{\Big(\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)\Big)^2},$$

which verifies (3.14). □

Putting together Proposition 3.1 and Lemma 3.3, we can immediately get the next Proposition 3.4, which gives more simplified expressions of $V_{z_1}^{z_2}(x)$ and $\widetilde{V}_{z_1}^{z_2}(x)$.

Proposition 3.4 For $(z_1,z_2)\in\mathcal{M}$, we have that

$$V_{z_1}^{z_2}(x)=\begin{cases}\dfrac{\ell_{b,c}^{(q,\lambda)}(x)}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)}, & x\in(b,z_2],\\[2ex] x-z_2+\dfrac{\ell_{b,c}^{(q,\lambda)}(z_2)}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)}, & x\in(z_2,\infty),\end{cases}\tag{3.17}$$

and

$$\widetilde{V}_{z_1}^{z_2}(x)=\frac{\mathrm{e}^{\widetilde{\Phi}_{q+\lambda}(x-c)}\,W_q(c-b)}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)},\quad x\in(-\infty,c).\tag{3.18}$$

By the definition of ξ, Proposition 3.1 and Lemma 3.1, it seems reasonable to take a double barrier $(z_1,z_2)\in\mathcal{M}$ dividend strategy as the candidate optimal strategy among the set of all admissible double barrier dividend strategies. As is stated above, we aim to show that the optimal impulse dividend control under fixed transaction costs fits this type of double barrier dividend strategy. To this end, we guess that the double barrier $(z_1,z_2)\in\mathcal{M}$ dividend strategy is the optimal one among the set of all admissible impulse dividend strategies, and verify in the next lemma that its value function fits the Hamilton-Jacobi-Bellman (HJB) inequality to which the optimal value function should fit.

Lemma 3.5 For $(z_1,z_2)\in\mathcal{M}$, we have that

$$V_{z_1}^{z_2}(y)-V_{z_1}^{z_2}(x)\geq y-x-\phi,\quad c\leq x<y<\infty.\tag{3.19}$$

Proof In view of the non-decreasing property of $V_{z_1}^{z_2}(x)$ in Proposition 3.2 (since $\ell_{b,c}^{(q,\lambda)}(x)$ is increasing), (3.19) holds for (x,y) such that $c\leq x<y<\infty$ and $x>y-\phi$. It is sufficient to show that (3.19) holds true for (x,y) such that $c\leq x<y-\phi<\infty$.

For $(z_1,z_2)\in\mathcal{M}$ and $x<y$, it holds that

$$\frac{1}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)}=\frac{z_2-z_1-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}\geq\frac{y-x-\phi}{\ell_{b,c}^{(q,\lambda)}(y)-\ell_{b,c}^{(q,\lambda)}(x)}.\tag{3.20}$$

From (3.20), it follows that, for $x,y\in[c,z_2]$,

$$\begin{aligned}V_{z_1}^{z_2}(y)-V_{z_1}^{z_2}(x)&=\frac{\ell_{b,c}^{(q,\lambda)}(y)-\ell_{b,c}^{(q,\lambda)}(x)}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)}=\left(\ell_{b,c}^{(q,\lambda)}(y)-\ell_{b,c}^{(q,\lambda)}(x)\right)\frac{z_2-z_1-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}\\&\geq\left(\ell_{b,c}^{(q,\lambda)}(y)-\ell_{b,c}^{(q,\lambda)}(x)\right)\frac{y-x-\phi}{\ell_{b,c}^{(q,\lambda)}(y)-\ell_{b,c}^{(q,\lambda)}(x)}\\&=y-x-\phi,\quad c\leq x<y-\phi<\infty.\end{aligned}$$

Again, by (3.20), we have that, for $x\in[c,z_2]$ and $y\in(z_2,\infty)$,

$$\begin{aligned}V_{z_1}^{z_2}(y)-V_{z_1}^{z_2}(x)&=y-x-\phi-(z_2-x-\phi)+\left(\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(x)\right)\frac{z_2-z_1-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(z_1)}\\&\geq y-x-\phi-(z_2-x-\phi)+\left(\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(x)\right)\frac{z_2-x-\phi}{\ell_{b,c}^{(q,\lambda)}(z_2)-\ell_{b,c}^{(q,\lambda)}(x)}\\&=y-x-\phi,\quad c\leq x\leq z_2<y<\infty,\ x+\phi<y.\end{aligned}$$

In addition, for $x,y\in(z_2,\infty)$, we also know that

$$V_{z_1}^{z_2}(y)-V_{z_1}^{z_2}(x)=y-x\geq y-x-\phi,\quad c\leq z_2<x<y-\phi<\infty.$$

Putting all of the pieces together completes the proof. □

For any $(z_1, z_2) \in \mathcal{M}$, the next lemma characterizes the monotonicity of the function $x \mapsto \ell_{b,c}^{(q,\lambda)\prime}(x)$ over $[z_2, \infty)$, whose proof is partly borrowed from Lemma 3.2 in [44]. We mention that this result plays an critical role in solving our new dividend control problem with fixed transaction costs under Chapter 11 bankruptcy. Before presenting this result, we recall that a function f defined on $(0, \infty)$ is said to be log-convex if the function $\log f(x)$ is convex on $(0, \infty)$, and that the tail of the Lévy measure υ refers to the function $x \mapsto \upsilon(x, \infty)$ for $x \in (0, \infty)$.

Lemma 3.6 For $(z_1, z_2) \in \mathcal{M}$, if the tail of the Lévy measure υ is log-convex, then the function $x \mapsto \ell_{b,c}^{(q,\lambda)\prime}(x)$ for $x \in [c, \infty)$ is increasing on $[z_2, \infty)$.

Proof Thanks to the mean value theorem, there exists a constant $\vartheta \in [z_1, z_2]$ such that

$$\begin{aligned}\frac{1}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)} &= \frac{z_2 - z_1 - \phi}{\ell_{b,c}^{(q,\lambda)}(z_2) - \ell_{b,c}^{(q,\lambda)}(z_1)} = \frac{z_2 - z_1 - \phi}{\ell_{b,c}^{(q,\lambda)\prime}(\vartheta)\,(z_2 - z_1)} \\ &\leq \frac{z_2 - z_1 - \phi}{\min\limits_{x\in[z_1,z_2]} \ell_{b,c}^{(q,\lambda)\prime}(x)\,(z_2 - z_1)} < \frac{1}{\min\limits_{x\in[z_1,z_2]} \ell_{b,c}^{(q,\lambda)\prime}(x)}, \quad (z_1, z_2) \in \mathcal{M},\end{aligned}$$

which implies that

$$\ell_{b,c}^{(q,\lambda)\prime}(z_2) > \min_{x\in[z_1,z_2]} \ell_{b,c}^{(q,\lambda)\prime}(x), \quad (z_1, z_2) \in \mathcal{M}. \tag{3.21}$$

We consider the following cases (i)-(iii) separately:

(i) $\psi(\widetilde{\Phi}_{q+\lambda}) - q > 0$ (or, equivalently, $\widetilde{\Phi}_{q+\lambda} > \Phi_q$).

In this case, by the proof for case (i) of Lemma 3.2 in [44], one knows that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is convex on $[c, \infty)$. Hence, if $z_1 = c$, we have that

- if $\ell_{b,c}^{(q,\lambda)\prime\prime}(c+) < 0$, then letting the $x_0 \in (0, \infty)$ be the unique root of $\ell_{b,c}^{(q,\lambda)\prime\prime}(x) = 0$, convexity of $\ell_{b,c}^{(q,\lambda)\prime}(x)$ implies that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is decreasing on $[c, x_0]$ and is increasing on $[x_0, \infty)$. This, together with (3.21), verifies that $z_2 \in (x_0, \infty)$. Therefore, $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is increasing on $[z_2, \infty)$, as desired;
- if $\ell_{b,c}^{(q,\lambda)\prime\prime}(c+) \geq 0$, then the convexity of $\ell_{b,c}^{(q,\lambda)\prime}(x)$ implies that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is increasing on $[c, \infty)$, and hence, on $[z_2, \infty)$.

Otherwise, if $z_1 \in (c, \infty)$, then $\ell_{b,c}^{(q,\lambda)\prime}(z_2) = \ell_{b,c}^{(q,\lambda)\prime}(z_1)$, and consequently, $z_1 \in (c, x_0)$ and $z_2 \in (x_0, \infty)$. It follows that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is increasing on $[z_2, \infty)$. Actually, we have that $\ell_{b,c}^{(q,\lambda)\prime\prime}(c+) < 0$ in this subcase.

(ii) $\psi(\widetilde{\Phi}_{q+\lambda}) - q < 0$, and by following arguments similar to those used in the proof of case (ii) of Lemma 3.2 in [44], we can get that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is strictly increasing on $[c, \infty)$. Consequently, $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is increasing on $[z_2, \infty)$ as $z_2 \geq c$;

(iii) $\psi(\widetilde{\Phi}_{q+\lambda}) - q = 0$, and again, similarly to the proof of case (iii) of Lemma 3.2 in [44], it is easy to see that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ is increasing on $[c, \infty)$, and hence, on $[z_2, \infty)$.

Putting all of the pieces together completes the proof. □

For $f \in C^2(-\infty, \infty)$, let us define the infinitesimal operator $\mathcal{A}f$ by

$$\begin{aligned}\mathcal{A}f(x) := &\ \gamma f'(x) + \frac{1}{2}\sigma^2 f''(x) \\ &+ \int_{(0,\infty)} \left(f(x-y) - f(x) + f'(x)y\mathbf{1}_{(0,1)}(y)\right) \upsilon(\mathrm{d}y),\ x \in (-\infty, \infty).\end{aligned} \tag{3.22}$$

Let $\widetilde{\mathcal{A}}$ be an operator similar to $\mathcal{A}$, where the Lévy triplet (γ,σ,υ) of X is replaced with the Lévy triplet $(\widetilde{\gamma},\widetilde{\sigma},\widetilde{\upsilon})$ of $\widetilde{X}$. Let N $(\widetilde{N})$, $\overline{N}(\overline{\widetilde{N}})$, and B $(\widetilde{B})$ be, respectively, the Poisson random measure, the compensated Poisson random measure, and the Brownian motion in the Lévy-Itô decomposition of X $(\widetilde{X})$. The following theorem proves the verification theorem corresponding to the optimal impulse dividend control problem:

Theorem 3.7 (verification theorem) Suppose that two functions, $V \in C^2[b,\infty)$ and $\widetilde{V} \in C^2(-\infty,c]$, satisfy that $V(x)=\widetilde{V}(x)$ for all $x\in(-\infty,b)\cup\{c\}$, and that

$$\begin{cases} (\mathcal{A}-q)\,V(x)\le 0, & b\le x<\infty,\\ (\widetilde{\mathcal{A}}-q-\lambda)\widetilde{V}(x)\le 0, & -\infty<x\le c,\\ V(x)-V(y)\ge x-y-\phi, & c\le y\le x<\infty. \end{cases} \tag{3.23}$$

When X has paths of unbounded variation, we further suppose that $V(b)=\widetilde{V}(b)$. Then, we have that $V(x)\ge \sup\limits_{D\in\mathcal{D}} V_D(x)$ for $x\in[b,\infty)$, and that $\widetilde{V}(x)\ge \sup\limits_{D\in\mathcal{D}} \widetilde{V}_D(x)$ for $x\in(-\infty,c)$.

Proof For a given strategy $D\in\mathcal{D}$ and the resulting surplus process U, given by Definition 3.1, we denote $(U_c(t))_{t\ge0}$ as the continuous part of $(U(t))_{t\ge0}$. In addition, for a positive integer $N\ge1$, let us define $\eta_{_N}:=\inf\{t\ge0:|U(t)|>N\}$ as the sequence of localizing stopping times. By definition, it holds that, for $t<\eta_{_N}$,

$$-N\le U(t)\le N; \tag{3.24}$$

i.e., both $U(t-)$ and $U(t)$ are restricted to the compact set $[-N,N]$. For simplicity, we write that

$$\mathbb{S}:=\cup_{k\ge0,I(\mathcal{T}_k)=0}\left[\mathcal{T}_k,\mathcal{T}_{k+1}\right),\ \overline{\mathbb{S}}:=\cup_{k\ge0,I(\mathcal{T}_k)=1}\left[\mathcal{T}_k,\mathcal{T}_{k+1}\right),\ J(t):=qt+\lambda\int_0^t \mathbf{1}_{\overline{\mathbb{S}}}(s)\,\mathrm{d}s,\ t\ge0,$$

and

$$\mathcal{G}(U,I)(s):=(\mathcal{A}-q)\,V(U(s))\,\mathbf{1}_{\mathbb{S}}(s)+(\widetilde{\mathcal{A}}-q-\lambda)\widetilde{V}(U(s))\,\mathbf{1}_{\overline{\mathbb{S}}}(s).$$

Note that $V(x)=\widetilde{V}(x)$ for all $x\in(-\infty,b)\cup\{c\}$, and that $V(b)=\widetilde{V}(b)$ when X has paths of unbounded variation. It follows that

- if $I(0)=0$, then

$$\begin{aligned}&\Big[\mathrm{e}^{-J(\eta_{_N})}V\left(U(\eta_{_N})\right)-V(x)\Big]\,\mathbf{1}_{[\mathcal{T}_{2k},\mathcal{T}_{2k+1})}(\eta_{_N})\\ =&\Bigg[\sum_{i=1}^{k}\Big[-\mathrm{e}^{-J(\mathcal{T}_{2i})}V(U(\mathcal{T}_{2i}))+\mathrm{e}^{-J(\mathcal{T}_{2i})}\widetilde{V}(U(\mathcal{T}_{2i}))-\mathrm{e}^{-J(\mathcal{T}_{2i-1})}\widetilde{V}(U(\mathcal{T}_{2i-1}))+\mathrm{e}^{-J(\mathcal{T}_{2i-1})}\\ &\times V(U(\mathcal{T}_{2i-1}))\Big]+\mathrm{e}^{-J(\eta_{_N})}V\left(U(\eta_{_N})\right)-\mathrm{e}^{-J(\mathcal{T}_0)}V(U(\mathcal{T}_0))\Bigg]\,\mathbf{1}_{[\mathcal{T}_{2k},\mathcal{T}_{2k+1})}(\eta_{_N}),\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}&\Big[\mathrm{e}^{-J(\eta_{_N})}\widetilde{V}\left(U(\eta_{_N})\right)-V(x)\Big]\,\mathbf{1}_{[\mathcal{T}_{2k+1},\mathcal{T}_{2k+2})}(\eta_{_N})\\ =&\Bigg[\sum_{i=1}^{k}\Big[-\mathrm{e}^{-J(\mathcal{T}_{2i+1})}\widetilde{V}(U(\mathcal{T}_{2i+1}))+\mathrm{e}^{-J(\mathcal{T}_{2i+1})}V(U(\mathcal{T}_{2i+1}))-\mathrm{e}^{-J(\mathcal{T}_{2i})}V(U(\mathcal{T}_{2i}))\\ &+\mathrm{e}^{-J(\mathcal{T}_{2i})}\widetilde{V}(U(\mathcal{T}_{2i}))\Big]-\mathrm{e}^{-J(\mathcal{T}_1)}\widetilde{V}(U(\mathcal{T}_1))+\mathrm{e}^{-J(\mathcal{T}_1)}V(U(\mathcal{T}_1))\\ &+\mathrm{e}^{-J(\eta_{_N})}\widetilde{V}\left(U(\eta_{_N})\right)-\mathrm{e}^{-J(\mathcal{T}_0)}V(U(\mathcal{T}_0))\Bigg]\,\mathbf{1}_{[\mathcal{T}_{2k+1},\mathcal{T}_{2k+2})}(\eta_{_N});\end{aligned} \tag{3.26}$$

- if $I(0)=1$, then

$$
\begin{aligned}
&\left[\mathrm{e}^{-J(\eta_N)}V\left(U(\eta_N)\right)-\widetilde{V}(x)\right]\mathbf{1}_{[\mathcal{T}_{2k+1},\mathcal{T}_{2k+2})}(\eta_N)\\
=&\left[\sum_{i=1}^{k}\Big[-\mathrm{e}^{-J(\mathcal{T}_{2i+1})}V(U(\mathcal{T}_{2i+1}))+\mathrm{e}^{-J(\mathcal{T}_{2i+1})}\widetilde{V}(U(\mathcal{T}_{2i+1}))-\mathrm{e}^{-J(\mathcal{T}_{2i})}\widetilde{V}(U(\mathcal{T}_{2i}))\right.\\
&+\mathrm{e}^{-J(\mathcal{T}_{2i})}V(U(\mathcal{T}_{2i}))\Big]-\mathrm{e}^{-J(\mathcal{T}_1)}V(U(\mathcal{T}_1))+\mathrm{e}^{-J(\mathcal{T}_1)}\widetilde{V}(U(\mathcal{T}_1))\\
&\left.+\mathrm{e}^{-J(\eta_N)}V\left(U(\eta_N)\right)-\mathrm{e}^{-qJ(\mathcal{T}_0)}\widetilde{V}(U(\mathcal{T}_0))\right]\mathbf{1}_{[\mathcal{T}_{2k+1},\mathcal{T}_{2k+2})}(\eta_N),
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
&\left[\mathrm{e}^{-J(\eta_N)}\widetilde{V}\left(U(\eta_N)\right)-\widetilde{V}(x)\right]\mathbf{1}_{[\mathcal{T}_{2k},\mathcal{T}_{2k+1})}(\eta_N)\\
=&\left[\sum_{i=1}^{k}\Big[-\mathrm{e}^{-J(\mathcal{T}_{2i})}\widetilde{V}(U(\mathcal{T}_{2i}))+\mathrm{e}^{-J(\mathcal{T}_{2i})}V(U(\mathcal{T}_{2i}))-\mathrm{e}^{-J(\mathcal{T}_{2i-1})}V(U(\mathcal{T}_{2i-1}))+\mathrm{e}^{-J(\mathcal{T}_{2i-1})}\right.\\
&\left.\times\widetilde{V}(U(\mathcal{T}_{2i-1}))\Big]+\mathrm{e}^{-J(\eta_N)}\widetilde{V}\left(U(\eta_N)\right)-\mathrm{e}^{-J(\mathcal{T}_0)}\widetilde{V}(U(\mathcal{T}_0))\right]\mathbf{1}_{[\mathcal{T}_{2k},\mathcal{T}_{2k+1})}(\eta_N).
\end{aligned}
\tag{3.28}
$$

By (3.25), (3.26), (3.27), (3.28) and Theorem 4.57 in [20], we deduce that, for $x\in(-\infty,\infty)$,

$$
\begin{aligned}
&\mathrm{e}^{-J(\eta_N)}V\left(U(\eta_N)\right)\mathbf{1}_{\mathbb{S}}(\eta_N)+\mathrm{e}^{-J(\eta_N)}\widetilde{V}\left(U(\eta_N)\right)\mathbf{1}_{\overline{\mathbb{S}}}(\eta_N)-V(x)\mathbf{1}_{\{I(0)=0\}}-\widetilde{V}(x)\mathbf{1}_{\{I(0)=1\}}\\
=&-\int_{0-}^{\eta_N}\left(q+\lambda\mathbf{1}_{\overline{\mathbb{S}}}(s)\right)\mathrm{e}^{-J(s)}\left[V(U(s))\mathbf{1}_{\mathbb{S}}(s)+\widetilde{V}(U(s))\mathbf{1}_{\overline{\mathbb{S}}}(s)\right]\mathrm{d}s\\
&+\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\left[V'(U(s))\mathbf{1}_{\mathbb{S}}(s)+\widetilde{V}'(U(s))\mathbf{1}_{\overline{\mathbb{S}}}(s)\right]\mathrm{d}U(s)\\
&+\frac{1}{2}\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\left[V''(U(s))\mathbf{1}_{\mathbb{S}}(s)+\widetilde{V}''(U(s))\mathbf{1}_{\overline{\mathbb{S}}}(s)\right]\mathrm{d}\langle U_c(\cdot),U_c(\cdot)\rangle_s\\
&+\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}\left(V(U(s-)+\Delta U(s))-V(U(s-))-V'(U(s-))\Delta U(s)\right)\mathbf{1}_{\mathbb{S}}(s)\\
&+\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}\left(\widetilde{V}(U(s-)+\Delta U(s))-\widetilde{V}(U(s-))-\widetilde{V}'(U(s-))\Delta U(s)\right)\mathbf{1}_{\overline{\mathbb{S}}}(s)\\
&+\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}\left(V(U(s+))-V\left(U(s)\right)+V'(U(s))\left(D(s+)-D(s)\right)\right)\mathbf{1}_{\{U(s+)\ge c\}}\\
=&\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\mathcal{G}(U,I)(s)\mathrm{d}s+\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\left(\sigma V'(U(s))\mathbf{1}_{\mathbb{S}}(s)\mathrm{d}B(s)+\widetilde{\sigma}\widetilde{V}'(U(s))\mathbf{1}_{\overline{\mathbb{S}}}(s)\mathrm{d}\widetilde{B}(s)\right)\\
&+\int_{0-}^{\eta_N}\int_0^{\infty}\mathrm{e}^{-J(s)}\left(V(U(s-)-y)-V(U(s-))\right)\mathbf{1}_{\mathbb{S}}(s)\,\overline{N}(\mathrm{d}s,\mathrm{d}y)\\
&+\int_{0-}^{\eta_N}\int_0^{\infty}\mathrm{e}^{-J(s)}\left(V(U(s-)-y)-V(U(s-))\right)\mathbf{1}_{\overline{\mathbb{S}}}(s)\,\overline{\widetilde{N}}(\mathrm{d}s,\mathrm{d}y)\\
&+\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}\big(V(U(s+))-V(U(s+)+D(s+)-D(s))\big)\mathbf{1}_{\{U(s+)\ge c\}}\\
=&\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\mathcal{G}(U,I)(s)\mathrm{d}s+\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\big(\sigma V'(U(s))\mathbf{1}_{\mathbb{S}}(s)\mathrm{d}B(s)+\widetilde{\sigma}\widetilde{V}'(U(s))\mathbf{1}_{\overline{\mathbb{S}}}(s)\mathrm{d}\widetilde{B}(s)\big)\\
&-\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}(D(s+)-D(s))\mathbf{1}_{\{U(s+)\ge c\}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}\Big[V(U(s+))-V(U(s+)+D(s+)-D(s))+(D(s+)-D(s))\Big]\mathbf{1}_{\{U(s+)\ge c\}}\\
&+\int_{0-}^{\eta_N}\int_0^\infty \mathrm{e}^{-J(s)}\left(V(U(s-)-y)-V(U(s-))\right)\mathbf{1}_{\mathbb{S}}(s)\,\overline{N}(\mathrm{d}s,\mathrm{d}y)\\
&+\int_{0-}^{\eta_N}\int_0^\infty \mathrm{e}^{-J(s)}\left(\widetilde{V}(U(s-)-y)-\widetilde{V}(U(s-))\right)\mathbf{1}_{\overline{\mathbb{S}}}(s)\,\overline{\overline{N}}(\mathrm{d}s,\mathrm{d}y),
\end{aligned}\tag{3.29}
$$

where $\Delta U(s)=U(s)-U(s-)$. In light of the third inequality in (3.23), we have for $s\in[0,\eta_N)$, that

$$
\left(V\left(U(s+)\right)-V\left(U(s+)+\left(D(s+)-D(s)\right)\right)+D(s+)-D(s)-\phi\right)\mathbf{1}_{\{U(s+)\ge c\}}(s)\le 0. \tag{3.30}
$$

Hence, by (3.23), (3.29) and (3.30), we have that, for $x\in(-\infty,\infty)$,

$$
\begin{aligned}
&\mathrm{e}^{-J(\eta_N)}V\left(U(\eta_N)\right)-V(x)\\
\le&-\sum_{s\le t\wedge\eta_N}\mathrm{e}^{-J(s)}\left(\Delta D(s)-\phi\right)\mathbf{1}_{\{U(s+)\ge c\}}\\
&+\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\left(\sigma V'(U(s))\mathbf{1}_{\mathbb{S}}(s)\,\mathrm{d}B(s)+\widetilde{\sigma}\widetilde{V}'(U(s))\mathbf{1}_{\overline{\mathbb{S}}}(s)\,\mathrm{d}\widetilde{B}(s)\right)\\
&+\int_{0-}^{\eta_N}\int_0^\infty \mathrm{e}^{-J(s)}\left(V(U(s-)-y)-V(U(s-))\right)\mathbf{1}_{\mathbb{S}}(s)\,\overline{N}(\mathrm{d}s,\mathrm{d}y)\\
&+\int_{0-}^{\eta_N}\int_0^\infty \mathrm{e}^{-J(s)}\left(\widetilde{V}(U(s-)-y)-\widetilde{V}(U(s-))\right)\mathbf{1}_{\overline{\mathbb{S}}}(s)\,\overline{\overline{N}}(\mathrm{d}s,\mathrm{d}y).
\end{aligned}\tag{3.31}
$$

In addition, according to [19, p62], the compensated sums

$$
\int_{0-}^{\eta_N}\int_0^\infty \mathrm{e}^{-J(s)}\left(V(U(s-)-y)-V(U(s-))\right)\mathbf{1}_{\mathbb{S}}(s)\,\overline{N}(\mathrm{d}s,\mathrm{d}y)
$$

and

$$
\int_{0-}^{\eta_N}\int_0^\infty \mathrm{e}^{-J(s)}\left(\widetilde{V}(U(s-)-y)-\widetilde{V}(U(s-))\right)\mathbf{1}_{\overline{\mathbb{S}}}(s)\,\overline{\overline{N}}(\mathrm{d}s,\mathrm{d}y)
$$

have zero means. In fact, the integrands of the above stochastic integrations are bounded from below and above thanks to (3.24) and $\sup\limits_{x\in[-N,N]}|V(x)|<\infty$. Similarly, the integration

$$
\int_{0-}^{\eta_N}\mathrm{e}^{-J(s)}\left(\sigma V'(U(s))\mathbf{1}_{\mathbb{S}}(s)\mathrm{d}B(s)+\widetilde{\sigma}\widetilde{V}'(U(s))\mathbf{1}_{\overline{\mathbb{S}}}(s)\mathrm{d}\widetilde{B}(s)\right)
$$

also has zero mean.

Taking expectations on both sides of (3.31) and then using the monotone convergence theorem as N tends to ∞, we derive that

$$
\begin{aligned}
&V(x)\mathbf{1}_{\{I(0)=0\}}+\widetilde{V}(x)\mathbf{1}_{\{I(0)=1\}}\\
\ge\ &\mathrm{E}_{x,I(0)}\left(\mathrm{e}^{-J(\eta_N)}V\left(U(\eta_N)\right)\mathbf{1}_{\mathbb{S}}(\eta_N)+\mathrm{e}^{-J(\eta_N)}\widetilde{V}\left(U(\eta_N)\right)\mathbf{1}_{\overline{\mathbb{S}}}(\eta_N)\right)\\
&+\mathrm{E}_{x,I(0)}\left(\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}(\Delta D(s)-\phi)\mathbf{1}_{\{U(s+)\ge c\}}\right)\\
\ge\ &\mathrm{E}_{x,I(0)}\left(\sum_{s\le\eta_N}\mathrm{e}^{-J(s)}(\Delta D(s)-\phi)\mathbf{1}_{\{U(s+)\ge c\}}\right)\\
\to\ &\mathrm{E}_{x,I(0)}\left(\sum_{s<\infty}\mathrm{e}^{-J(s)}(\Delta D(s)-\phi)\mathbf{1}_{\{U(s+)\ge c\}}\right)
\end{aligned}
$$

$$= V_D(x)\mathbf{1}_{\{I(0)=0\}} + \widetilde{V}_D(x)\mathbf{1}_{\{I(0)=1\}} \quad \text{for} \quad x \in (-\infty, \infty) \quad \text{as} \quad N \to \infty, \tag{3.32}$$

where the last equality follows by the Poisson method introduced in [30] and the memory-less property of exponential random variables. To wit, we denote by $(T_i)_{i\geq 1}$ the successive arrival times of a Poisson processes with a rate λ that are independent of the four-dimensional process $(X(t), \widetilde{X}(t), U(t), I(t))_{t\geq 0}$, and denote by $\mathcal{F}_{X,\widetilde{X},U,I}$ the smallest sigma field generated by the four-dimensional process $(X(t), \widetilde{X}(t), U(t), I(t))_{t\geq 0}$. It holds that

$$\begin{aligned}
&\mathrm{E}_{x,I(0)}\left(\sum_{s<\infty} \mathrm{e}^{-J(s)}(\Delta D(s)-\phi)\mathbf{1}_{\{U(s+)\geq c\}}\right)\\
&= \mathrm{E}_{x,I(0)}\left(\sum_{t<\infty} \mathrm{e}^{-qt}\,\mathbb{E}\left(\mathbf{1}_{\{((T_i)_{i\geq 1})\bigcap \bar{\mathbb{S}} \bigcap [0,t]=\emptyset\}}\Big|\mathcal{F}_{X,\widetilde{X},U,I}\right)(\Delta D(t)-\phi)\mathbf{1}_{\{U(t+)\geq c\}}\right)\\
&= \mathrm{E}_{x,I(0)}\left(\sum_{t<\infty} \mathrm{e}^{-qt}\,\mathbb{E}\left(\mathbf{1}_{\{T_D>t\}}\Big|\mathcal{F}_{X,\widetilde{X},U,I}\right)(\Delta D(t)-\phi)\mathbf{1}_{\{U(t+)\geq c\}}\right)\\
&= \mathrm{E}_{x,I(0)}\left(\mathbb{E}\left(\sum_{t<\infty} \mathrm{e}^{-qt}\mathbf{1}_{\{T_D>t\}}(\Delta D(t)-\phi)\mathbf{1}_{\{U(t+)\geq c\}}\Big|\mathcal{F}_{X,\widetilde{X},U,I}\right)\right)\\
&= \mathrm{E}_{x,I(0)}\left(\sum_{t<T_D} \mathrm{e}^{-qt}(\Delta D(t)-\phi)\mathbf{1}_{\{U(t+)\geq c\}}\right)\\
&= V_D(x)\mathbf{1}_{\{I(0)=0\}} + \widetilde{V}_D(x)\mathbf{1}_{\{I(0)=1\}}, \quad x \in (-\infty, \infty).
\end{aligned}$$

By (3.32), the arbitrariness of D and the continuity of V and $\widetilde{V}$, we can conclude that $V(x) \geq \sup_{D\in\mathcal{D}} V_D(x)$ for all $x \in (b, \infty)$, and that $\widetilde{V}(x) \geq \sup_{D\in\mathcal{D}} \widetilde{V}_D(x)$ for all $x \in (-\infty, c)$, as desired. □

The next theorem is the main result of this section, and it shows that the double barrier $(z_1, z_2) \in \mathcal{M}$ dividend strategy is the optimal one among all admissible impulse dividend strategies.

Theorem 3.8 Suppose that the tail of the Lévy measure is log-convex. Recall that $\mathcal{M}$ is as defined in (3.12). The double barrier $(z_1, z_2) \in \mathcal{M}$ dividend strategy is the optimal impulse dividend strategy achieving the maximal value function up to the Chapter 11 bankruptcy time.

Proof Let $(z_1, z_2) \in \mathcal{M}$. By Proposition 3.4 and Lemmas 3.4–3.5, one can adapt the techniques of Theorem 4.8 in [43] (or, Lemmas 3.3–3.5 in [44]) to obtain that the candidate optimal value functions $V_{z_1}^{z_2}(x)$ and $\widetilde{V}_{z_1}^{z_2}(x)$ satisfy the HJB inequalities (3.23); i.e., the double barrier $(z_1, z_2) \in \mathcal{M}$ dividend strategy outperforms all other admissible impulse dividend strategies. □

4 An Illustrative Example

Under the mild condition that the Lévy measure of X has a log-convex tail, Theorem 3.8 verifies that the double barrier dividend strategy with barriers $(z_1, z_2) \in \mathcal{M}$ serves as the optimal impulse dividend strategy, and yields the maximum expected discounted total dividends (subtracting with transaction costs). To compute explicitly the two barriers of the optimal impulse dividend strategy, we consider the Cramér-Lundberg process X with exponential jump sizes; namely, a process X defined by a deterministic drift p (the premium income) subtracting a compound Poisson process with jump intensity λ_0 and exponentially distributed jump sizes with a mean of $1/\mu$. It is well known that

$$W_q(x)=p^{-1}(A_+\mathrm{e}^{q_+x}-A_-\mathrm{e}^{q_-x}),$$

where $A_\pm:=\frac{\mu+q_\pm}{q_+-q_-}$ and $q_\pm:=\frac{q+\lambda_0-\mu p\pm\sqrt{(q+\lambda_0-\mu p)^2+4pq\mu}}{2p}$. Together with (3.5), it readily follows that

$$\ell_{b,c}^{(q,\lambda)\prime}(x)=(\psi(\widetilde{\Phi}_{q+\lambda})-q)p^{-1}(B_+\mathrm{e}^{q_+(x-b)}-B_-\mathrm{e}^{q_-(x-b)}),\tag{4.1}$$

where $B_\pm:=\frac{p^{-1}A_\pm q_\pm\left[\mu+\widetilde{\Phi}_{q+\lambda}-(\mu+q_\mp)\mathrm{e}^{(q_\mp-\widetilde{\Phi}_{q+\lambda})(c-b)}\right]}{(q_+-\widetilde{\Phi}_{q+\lambda})(q_--\widetilde{\Phi}_{q+\lambda})}$ with $B_+>0$ and $B_-<0$. By (4.1), one knows that $\ell_{b,c}^{(q,\lambda)\prime\prime\prime}(x)>0$, hence $\ell_{b,c}^{(q,\lambda)\prime\prime}(x)$ is strictly increasing, and $\ell_{b,c}^{(q,\lambda)\prime\prime}(\infty)=\infty$. Denote that $x_0=b+\frac{1}{q_+-q_-}\ln\frac{B_-q_-}{B_+q_+}$, which is the unique zero of $\ell_{b,c}^{(q,\lambda)\prime\prime}(x)$. We first consider the case $\psi(\widetilde{\Phi}_{q+\lambda})>q$ and identify the set $\mathcal{M}$ according to two scenarios, (i) and (ii).

(i) $c\ge x_0$. For this case, $\ell_{b,c}^{(q,\lambda)\prime\prime}(x)>0$ over (c,∞), that is, $\ell_{b,c}^{(q,\lambda)\prime}(x)$ strictly increases over $[c,\infty)$. The function ξ cannot attain a local maximum at an interior point (x,y), since the second equation of (3.14) is violated. Hence, $\mathcal{M}=\{(c,z_2)\}$, with $z_2\in(c+\phi,\infty)$ being the unique solution of (see, (3.13)) $h(y):=\ell_{b,c}^{(q,\lambda)}(y)-\ell_{b,c}^{(q,\lambda)}(c)-(y-c-\phi)\ell_{b,c}^{(q,\lambda)\prime}(y)=0$; i.e.,

$$\frac{(1-(y-c-\phi)q_+)\mathrm{e}^{q_+y}-\mathrm{e}^{q_+c}}{q_+\mathrm{e}^{q_+b}/B_+}-\frac{(1-(y-c-\phi)q_-)\mathrm{e}^{q_-y}-\mathrm{e}^{q_-c}}{q_-\mathrm{e}^{q_-b}/B_-}=0.\tag{4.2}$$

The uniqueness of the solution of the above equation follows due to the fact that $h'(y)=-(y-c-\phi)\times\ell_{b,c}^{(q,\lambda)\prime\prime}(y)<0$ over $(c+\phi,\infty)$, $h(c+\phi)=\ell_{b,c}^{(q,\lambda)}(c+\phi)-\ell_{b,c}^{(q,\lambda)}(c)>0$, and $h(\infty)=-\infty$, because $\ell_{b,c}^{(q,\lambda)\prime}(\infty)=\infty$ and

$$\frac{h(\infty)}{\ell_{b,c}^{(q,\lambda)\prime}(\infty)}:=\lim_{y\to\infty}\frac{h(y)}{\ell_{b,c}^{(q,\lambda)\prime}(y)}\le\lim_{y\to\infty}\left[\frac{\ell_{b,c}^{(q,\lambda)}(y)}{\ell_{b,c}^{(q,\lambda)\prime}(y)}-y+c+\phi\right]=-\infty.$$

(ii) $c<x_0$. For this case, $\ell_{b,c}^{(q,\lambda)\prime\prime}(x)<0$ over $[c,x_0)$ and $\ell_{b,c}^{(q,\lambda)\prime\prime}(x)>0$ over (x_0,∞), that is, $\ell_{b,c}^{(q,\lambda)\prime}(x)$ strictly decreases over $[c,x_0)$ and strictly increases over (x_0,∞). Let x_1 be the unique solution x of $\ell_{b,c}^{(q,\lambda)\prime}(x)=\ell_{b,c}^{(q,\lambda)\prime}(c)$ such that $x>x_0$. Then we have following:

- $c<x_0$ and $\phi\ge x_1-c$. In this case, there cannot be (x,y) satisfying that $y-x-\phi>0$ such that the second equation in (3.14) holds true. Hence, similarly to case (i), $\mathcal{M}$ reduces to a singleton, i.e., $\mathcal{M}=\{(c,z_2)\}$ with $z_2\in(x_0,\infty)$ being the unique solution of (4.2).
- $c<x_0$ and $\phi<x_1-c$. Denote by $y=k(x)$ the implicit function determined by the second equation in (3.14). Then $k'(x)=\frac{\ell_{b,c}^{(q,\lambda)\prime\prime}(x)}{\ell_{b,c}^{(q,\lambda)\prime\prime}(y)}<0$ for all $x\in[c,x_0)$ and $y\in(x_0,\infty)$. Additionally, define that

$$g(x):=\ell_{b,c}^{(q,\lambda)}(k(x))-\ell_{b,c}^{(q,\lambda)}(x)-(k(x)-x-\phi)\,\ell_{b,c}^{(q,\lambda)\prime}(x),\quad x\in[c,x_0],$$
$$G(\varphi):=\ell_{b,c}^{(q,\lambda)}(x_1)-\ell_{b,c}^{(q,\lambda)}(c)-(x_1-c-\varphi)\,\ell_{b,c}^{(q,\lambda)\prime}(x_1),\quad \varphi\in[0,x_1-c].$$

We have that $g'(x)=-(k(x)-x-\phi)\,\ell_{b,c}^{(q,\lambda)\prime\prime}(x)>0$ over $[c,x_0)$, $G'(\varphi)=\ell_{b,c}^{(q,\lambda)\prime}(x_1)>0$, $G(0)<0$, and $G(x_1-c)>0$. Hence, there is a unique $\varphi_0\in(0,x_1-c)$ such that

$$\ell_{b,c}^{(q,\lambda)}(x_1)-\ell_{b,c}^{(q,\lambda)}(c)-(x_1-c-\varphi_0)\,\ell_{b,c}^{(q,\lambda)\prime}(x_1)=0.\tag{4.3}$$

- $c<x_0$ and $\varphi_0\le\phi<x_1-c$. From (4.3) we have that $g(c)>0$, which, together with the fact that g is strictly increasing, yields that $g(x)>0$ for all $x\in[c,x_0]$, which contradicts the first equation in (3.14). Hence, there should not be an interior local maximum point of ξ, which implies that $\mathcal{M}$ reduces to a singleton, i.e., $\mathcal{M} = \{(c, z_2)\}$ with $z_2 \in (x_0, \infty)$ being the unique solution of (4.2).

- $c < x_0$ and $\phi \in (0, \varphi_0)$. By (4.3), we have that $g(c) < 0$, which, together with $g(x_0) = \phi\,\ell_{b,c}^{(q,\lambda)\prime}(x_0) > 0$ and the strict increasing property of g, yields the existence of a unique solution $z_1' \in (c, x_0)$ of the first equation in (3.14). Hence, $(z_1', z_2' := k(z_1'))$ is the unique solution of (3.14), i.e.

$$\begin{cases} B_+ \mathrm{e}^{-q_+ b}\big(\mathrm{e}^{q_+ x} - \mathrm{e}^{q_+ y}\big) - B_- \mathrm{e}^{-q_- b}\big(\mathrm{e}^{q_- x} - \mathrm{e}^{q_- y}\big) = 0, \\ \dfrac{(1-(y-x-\phi)q_+)\mathrm{e}^{q_+ y} - \mathrm{e}^{q_+ x}}{q_+ \mathrm{e}^{q_+ b}/B_+} - \dfrac{(1-(y-x-\phi)q_-)\mathrm{e}^{q_- y} - \mathrm{e}^{q_- x}}{q_- \mathrm{e}^{q_- b}/B_-} = 0. \end{cases} \tag{4.4}$$

Thus, we have two potential miximizers of ξ, (c, z_2) and (z_1', z_2'). It can be verified that $\xi(c, z_2) \neq \xi(z_1', z_2')$. Otherwise, we have, by (3.13) and (3.14), that

$$\frac{1}{\ell_{b,c}^{(q,\lambda)\prime}(z_2)} = \xi(c, z_2) = \xi(z_1', z_2') = \frac{1}{\ell_{b,c}^{(q,\lambda)\prime}(z_2')}, \quad z_2,\, z_2' \in (x_0, \infty),$$

which, together with the strict increasing property of $\ell_{b,c}^{(q,\lambda)\prime}(x)$ over (x_0, ∞), implies that $z_2 = z_2'$, which, combined with (3.13) and (3.14), further implies that

$$\ell_{b,c}^{(q,\lambda)}(z_1') - \ell_{b,c}^{(q,\lambda)}(c) = (z_1' - c)\,\ell_{b,c}^{(q,\lambda)\prime}(z_1'), \quad z_1' \in (c, x_0),$$

which contradicts the fact that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ strictly decreases over (c, x_0). Therefore, $\mathcal{M} = \{(c, z_2)\}$ when $\xi(c, z_2) > \xi(z_1', z_2')$, and $\mathcal{M} = \{(z_1', z_2')\}$ when $\xi(c, z_2) < \xi(z_1', z_2')$.

Finally, when $\psi(\widetilde{\Phi}_{q+\lambda}) \leq q$, by Lemma 3.6, we know that $\ell_{b,c}^{(q,\lambda)\prime}(x)$ strictly increases over $[c, \infty)$, and hence $\mathcal{M} = \{(c, z_2)\}$ with $z_2 \in (c + \phi, \infty)$ being the unique solution of (4.2).

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References

[1] Albrecher H, Thonhauser S. Optimality results for dividend problems in insurance. Revista de la Real Academia de Ciencias Exactas, Fäsicasy Naturales, Serie A, Matematicas, 2009, **103**(2): 295–320

[2] Artin E. The Gamma Function. New York: Holt, Rinehart and Winston, 1964. English translation of German original, Einführung in die Theorie der Gammafunktion, Teubner, 1931

[3] Avanzi B. Strategies for dividend distribution: A Review. N Am Actuar J, 2009, **13**(2): 217–251

[4] Avanzi B, Lau H, Wong B. Optimal periodic dividend strategies for spectrally positive Lévy risk processes with fixed transaction costs. Scand Actuar J, 2021, **8**: 645–670

[5] Avanzi B, Lau H, Wong B. On the optimality of joint periodic and extraordinary dividend strategies. Eur J Oper Res, 2021, **295**(3): 1189–1210

[6] Avram F, Palmowski Z, Pistorius M. On the optimal dividend problem for a spectrally negative Lévy process. Ann Appl Probab, 2007, **17**(1): 156–180

[7] Avram F, Palmowski Z, Pistorius M. On Gerber-Shiu functions and optimal dividend distribution for a Lévy risk process in the presence of a penalty function. Ann Appl Probab, 2015, **25**(4): 1868–1935

[8] Broadie M, Chernov M, Sundaresan S. Optimal debt and equity values in the presence of chapter 7 and chapter 11. J Finance, 2007, **62**(3): 1341–1377

[9] Broadie M, Kaya O. A binomial lattice method for pricing corporate debt and modeling Chapter 11. J Financ Quant Anal, 2007, **42**(2): 279–312

[10] Cheng X, Jin Z, Yang H. Optimal insurance strategies: a hybrid deep learning markov chain approximation approach. Astin Bull, 2020, **50**(2): 1–29

[11] Cheung E, Wong J. On the dual risk model with Parisian implementation delays in dividend payments. Eur J Oper Res, 2017, **257**(1): 159–173

[12] Constantin N. Convex Functions and Their Applications. A Contemporary Approach. New York: Springer, 2010

[13] Corbae D, D'Erasmo P. Reorganization or liquidation: bankruptcy choice and firm dynamics. Unpublished Working Paper. National Bureau of Economic Research, 2017

[14] Dai M, Jiang L, Lin J. Pricing corporate debt with finite maturity and chapter 11 proceedings. Quant Financ, 2013, **13**(12): 1855–1861

[15] De Angelis T. Optimal dividends with partial information and stopping of a degenerate reflecting diffusion. Financ Stoch, 2020, **24**(1): 71–123

[16] De Finetti B. Su un'impostazion alternativa dell teoria collecttiva del rischio//Trans XVth Internat Congress Actuaries, 1957, **2**: 433–443

[17] Gerber H. Entscheidungskriterien fur den zusammengesetzten Poisson prozess. Mit Verein Schweiz Versicherungsmath, 1969, **69**: 185–227

[18] Højgaard B, Taksar M. Controlling risk exposure and dividends payout schemes: insurance company example. Math Financ, 1999, **9**(2): 153–182

[19] Ikeda N, Watanabe S. Stochastic Differential Equations and Diffusion Processes. New York: North Holland-Kodansha, 1981

[20] Jacod J, Shiryaev A. Limit Theorems for Stochastic Processes. Second ed. Berlin: Springer-Verlag, 2003

[21] Jeanblanc M, Shiryaev A. Optimization of the flow of dividends. Russian Mathematical Surveys, 1995, **50**(2): 257–277

[22] Jin Z, Liao H, Yang Y, Yu X. Optimal dividend strategy for an insurance group with contagious default risk. Scand Actuar J, 2021, **4**: 335–361

[23] Kyprianou A. Introductory Lectures on Fluctuations of Lévy Processes with Applications. Berlin: Springer Science and Business Media, 2014

[24] Kyprianou A, Loeffen R. Refracted Lévy processes. Ann I H Poincare-PR, 2010, **46**(1): 24–44

[25] Kyprianou A, Palmowski Z. Distributional study of de Finetti's dividend problem for a general Lévy insurance risk process. J Appl Probab, 2007, **44**(2): 428–443

[26] Leland H. Corporate debt value, bond covenants, and optimal capital structure. J Finance, 1994, **49**(4): 1213–1252

[27] Li B, Tang Q, Wang L, Zhou X. Liquidation risk in the presence of Chapters 7 and 11 of the US bankruptcy code. J Financ Eng, 2014, **1**(3): 1–19

[28] Li J, Liu G, Zhao J. OPtimal dividend-penalty strategies for insurance risk model with surplus-dependent premium. Acta Math Sci, 2020, **40B**(1): 170–198

[29] Li X, Liu H, Tang Q, Zhu J. Liquidation risk in insurance under contemporary regulatory frameworks. Insur Math Econ, 2020, **93**(1): 36–49

[30] Li Y, Zhou X. On pre-exit joint occupation times for spectrally negative Lévy processes. Stat Probabil Lett, 2014, **94**(1): 48–55

[31] Loeffen R. On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. Ann Appl Probab, 2008, **18**(5), 1669-1680

[32] Loeffen R. An optimal dividends problem with transaction costs for spectrally negative Lévy processes. Insur Math Econ, 2009, **45**(1): 41–48

[33] Loeffen R, Renaud J. De Finetti's optimal dividends problem with an affine penalty function at ruin. Insur Math Econ, 2010, **46**(1): 98–108

[34] Loeffen R, Renaud J, Zhou X. Occupation times of intervals until first passage times for spectrally negative Lévy processes. Stoch Proc Appl, 2014, **124**(3): 1408–1435

[35] Noba K. On the optimality of double barrier strategies for Lévy processes. Stoch Proc Appl, 2021, **131**(1): 73–102

[36] Noba K, Pérez J, Yu X. On the bailout dividend problem for spectrally negative Markov additive models. SIAM J Control Optim, 2020, **58**(2): 1049–1076

[37] Paseka A. Debt valuation with endogenous default and Chapter 11 reorganization. Working paper, University of Arizona, 2003

[38] Pistorius M. On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum. J Theor Probab, 2004, **17**(1): 183–220

[39] Pérez J, Yamazaki K, Yu X. On the bail-out optimal dividend problem. J Optim Theory Appl, 2018, **179**(2): 553–568

[40] Renaud J. On the time spent in the red by a refracted Lévy risk process. J Appl Probab, 2014, **51**(4): 1171–1188

[41] Renaud J, Zhou X. Distribution of the present value of dividend payments in a Lévy risk model. J Appl Probab, 2007, **44**(2): 420–427

[42] Shreve S, Lehoczky J, Gaver D. Optimal consumption for general diffusions with absorbing and reflecting barriers. SIAM J Control Optim, 1984, **22**(1): 55–75

[43] Wang W, Wang Y, Chen P, Wu X. Dividend and capital injection optimization with transaction cost for spectrally negative Lévy risk processes. J Optim Theory Appl, 2022, **194**(3): 924–965

[44] Wang W, Yu X, Zhou X. On optimality of barrier dividend control under endogenous regime switching with application to Chapter 11 bankruptcy. arXiv:2108.01800

[45] Wang W, Zhou X. General drawdown-based de Finetti optimization for spectrally negative Lévy risk processes. J Appl Probab, 2018, **55**(2): 513–542

[46] Zhu J, Siu T, Yang H. Singular dividend optimization for a linear diffusion model with time-inconsistent preferences. Eur J Oper Res, 2020, **285**(1): 66–80