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GLOBAL SOLUTIONS TO 1D COMPRESSIBLE NAVIER-STOKES/ALLEN-CAHN SYSTEM WITH DENSITY-DEPENDENT VISCOSITY AND FREE-BOUNDARY*

Shijin DING (丁时进) *Yinghua LI* (李颖花)† *Yu WANG* (王喻)
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
E-mail: *dingsj@scnu.edu.cn; yinghua@scnu.edu.cn; yuwang@m.scnu.edu.cn*

Abstract This paper is concerned with the Navier-Stokes/Allen-Cahn system, which is used to model the dynamics of immiscible two-phase flows. We consider a 1D free boundary problem and assume that the viscosity coefficient depends on the density in the form of $\eta(\rho) = \rho^{\alpha}$. The existence of unique global H^{2m}-solutions ($m \in \mathbb{N}$) to the free boundary problem is proven for when $0 < \alpha < \frac{1}{4}$. Furthermore, we obtain the global C^{∞}-solutions if the initial data is smooth.

Key words Navier-Stokes/Allen-Cahn system; density-dependent viscosity; free boundary; global solutions

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1 Introduction

In [1], Blesgen proposed a diffuse interface model for macroscopically immiscible two-phase flows. Compared with the classical sharp interface models, this can naturally describe topological transitions on the interface, such as droplet formation, coalescence or the break-up of droplets. A phase field variable ϕ was introduced to describe the interaction between two fluids. This model couples together Navier-Stokes equations and Allen-Cahn equations, and has been widely accepted and successfully used in numerical simulations [15, 28].

It is well known that the Navier-Stokes equations can be derived from the Boltzmann equation through a Chapman-Enskog expansion to the second order [13], where the viscosity coefficient depends on temperature. For the isentropic case, this dependence is transferred into the dependence on the density by laws of Boyle and Gay-Lussac for an ideal gas [22]. From the deduction of the compressible Navier-Stokes/Allen-Cahn equations [1, 10, 16], we can see that the viscosity also depends on the density. In this paper, we consider the 1D compressible

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†Corresponding author

Navier-Stokes/Allen-Cahn equations with density-dependent viscosity and free boundaries

$$\begin{cases}\rho_\tau + (\rho u)_y = 0,\\ \rho u_\tau + \rho u u_y + P(\rho)_y = (\eta(\rho) u_y)_y - \dfrac{\delta}{2}\left(\phi_y^2\right)_y,\\ \rho\phi_\tau + \rho u \phi_y = -\mu,\\ \rho\mu = \dfrac{\rho}{\delta}(\phi^3-\phi) - \delta\phi_{yy}\end{cases} \tag{1.1}$$

for $(y,\tau)\in(a(\tau),b(\tau))\times(0,+\infty)$, where ρ, u and ϕ represent the total density, the mean velocity field and the concentration difference between two fluids, respectively. Moreover, μ denotes the chemical potential, the positive constant δ is related to the thickness of the interfacial region, η is the viscosity coefficient, and the pressure is $P(\rho) = A\rho^\gamma$ with $\gamma > 1$. Without loss of generality, we assume that $A = \delta = 1$. Here, $a(\tau)$ and $b(\tau)$ are the free boundaries satisfying that

$$\begin{cases}\dfrac{\mathrm{d}a(\tau)}{\mathrm{d}\tau} = u(a(\tau),\tau),\\ a(0) = a,\end{cases} \quad \text{and} \quad \begin{cases}\dfrac{\mathrm{d}b(\tau)}{\mathrm{d}\tau} = u(b(\tau),\tau),\\ b(0) = b.\end{cases}$$

We supplement system (1.1) with the initial and boundary value conditions

$$(\rho,\, u,\, \phi)(y,0) = (\rho_0,\, u_0,\, \phi_0)(y), \qquad a \le y \le b, \tag{1.2}$$

$$(\rho^\gamma - \eta u_y,\, \phi_y)(d,\tau) = (0,0), \qquad d = a(\tau), b(\tau),\ \tau \ge 0. \tag{1.3}$$

Moreover, we assume that the viscosity is of the form

$$\eta = \tilde{\eta}\,\rho^\alpha, \tag{1.4}$$

where $\tilde{\eta}$ is a positive constant.

In [22], Liu-Xin-Yang indicated that the Navier-Stokes equations with constant viscosity are not valid at the vacuum states, and they introduced the modified Navier-Stokes equations with a density-dependent viscosity. Later, Okada [23] pointed out that physicists claimed that the viscosity of a gas is proportional to the square root of the temperature. Since then, there have been many efforts investigating the Navier-Stokes equations with a viscosity coefficient like (1.4). Jiang [17] considered the full Navier-Stokes equations with stress-free and fixed boundary conditions on the velocity. He proved that if the viscosity does not decrease to zero too rapidly, i.e., $0 < \alpha < \frac{1}{4}$ for a stress-free boundary condition and $0 < \alpha < \frac{1}{3}$ for a fixed boundary condition, then smooth solutions of initial boundary problem still exist globally in time. Afterwards, Okada-Matušů-Makino [23], Yang-Yao-Zhu [29], Jiang-Xin-Zhang [18] and Guo-Jiang-Xie [14] studied a free boundary problem for isentropic Navier-Stokes equations and assumed that the initial density connected with a vacuum discontinuously. They proved the existence of global weak solutions under the conditions that $0 < \alpha < \frac{1}{3}$, $0 < \alpha < \frac{1}{2}$, $0 < \alpha < 1$ and $0 < \alpha < \min\{3-\gamma, \frac{3}{2}\}$. For when the initial density connects to a vacuum continuously, we refer readers to [11, 12, 19, 21, 26, 30, 31] and the references therein. What attracts our attention here is that, when $0 < \alpha < 1$, Qin-Huang-Yao [24] improved the regularity of the weak solution, which was originally obtained in [18], and proved the H^i-solutions ($i = 2, 4$). Furthermore, Ding-Huang-Liu-Wen raised the regularity of the solutions to C^∞ in [7].

In recent years, the compressible Navier-Stokes/Allen-Cahn system has attracted much attention on account of its distinct physical background, and there have been many valuable investigations of it. Here we recall some results on the 1D compressible NSAC system that are closely related to our study. For the initial boundary value problem (IBVP), for when the viscosity is assumed to be a constant, Ding-Li-Luo [8] and Chen-Guo [2] proved the global strong solutions both without a vacuum and with a vacuum. Later, Zhang [32] improved the regularity, and obtained the H^i-solutions ($i = 2, 4$). Meanwhile, Chen-Zhu [3] assumed that

$$\eta(\rho, \phi) = 1 + \rho^\alpha \phi^\beta,$$

and proved the global existence of strong solutions for when $2 \le \alpha \le \gamma$, $\beta = 0$, and established blow-up criteria for strong solutions for when $\beta \ge 1$ with a vacuum. Under the hypothesis that

$$\eta(\rho) \ge \tilde{\eta} > 0,$$

where $\tilde{\eta}$ is a positive constant, Su [25] established the existence of strong solutions. For 1D compressible NSAC equations with free boundary conditions, Ding-Li-Tang [9] considered a constant viscosity case, proved the existence and uniqueness of local solutions by using the Schauder Fixed Point theorem, and then extended the local solution to a global one by attaining global energy estimates. For when the viscosity depends on the density as

$$\eta(\rho) = 1 + \rho^\alpha,$$

and the initial data is connected to a vacuum continuously, Li-Yan-Ding-Chen [20] obtained the existence of weak solutions by a line method.

For 1D non-isentropic NSAC equations, Chen-He-Huang-Shi [4, 5] simplified the original model and proved the existence of unique strong solutions to the 1D IBVP and the Cauchy problem with a constant viscosity and temperature-dependent heat conductivity

$$\eta = \tilde{\eta}, \qquad \kappa(\theta) = \theta^\beta,$$

where $\beta > 0$. Yan-Ding-Li [27] considered the phase variable dependent viscosity and the temperature-dependent heat conductivity

$$\eta = \chi^\alpha, \qquad \kappa(\theta) = \theta^\beta,$$

where $\beta > 0$. They obtained the well-posedness of strong solutions under the hypothesis that $\alpha \ge 0$ is small. Recently, Dai-Ding-Li [6] studied the density-dependent viscosity and the temperature-dependent heat conductivity

$$\eta(\rho) = 1 + \rho^\alpha, \qquad \kappa(\theta) = \theta^\beta,$$

with $\alpha \ge 0,\ \beta > 0$, and

$$\eta(\rho) = \rho^\alpha, \qquad \kappa(\theta) = \theta^\beta,$$

with $0 \le \alpha < \frac{1}{4},\ \beta \ge 1$. They obtained the existences of unique strong solutions to the initial boundary problem with the above two conditions.

Noticing that the viscosity coefficient in the form $\eta(\rho) = \rho^\alpha$ has been discussed for the non-isentropic case, we want to consider the isentropic case. Thus, in this paper, we deal with the free boundary problem to the compressible NSAC equations with the following viscosity:

$$\eta(\rho) = \tilde{\eta}\rho^\alpha.$$

We will prove the existence and uniqueness of H^2-solutions, then improve these into $H^{2m}(m \in \mathbb{N})$-solutions and C^∞-solutions. Obviously, system (1.5) is of strong coupling and strong nonlinearity, and degeneracy will appear near the vacuum state. In addition, although Li *et al.* [20] dealt with the case of when the initial data is connected to a vacuum continuously, they only obtain the existence of weak solutions, and not the uniqueness. At the cost of the free boundary jump to the vacuum, we can study the case where the viscosity is proportional to the density, and obtain the unique C^∞-solutions.

To solve the free boundary problem (1.1)–(1.3), it is convenient to convert the free boundaries to the fixed boundaries by using Lagrangian coordinates. Set that $x = \int_{a(\tau)}^{y} \rho(\xi,\tau)\mathrm{d}\xi$, $t=\tau$, and assume that $\int_a^b \rho_0(\xi)\mathrm{d}\xi = 1$. Then problem (1.1)–(1.3) becomes

$$\begin{cases} \rho_t + \rho^2 u_x = 0, \\ u_t + (\rho^\gamma)_x = (\rho^{\alpha+1}u_x)_x - \dfrac{1}{2}\left(\rho^2\phi_x^2\right)_x, \\ \rho\phi_t = (\rho\phi_x)_x - (\phi^3-\phi) \end{cases} \tag{1.5}$$

for $(x,t) \in (0,1)\times(0,+\infty)$, with the initial and boundary conditions

$$(\rho,\, u,\, \phi)(x,0) = (\rho_0,\, u_0,\, \phi_0)(x), \qquad 0 \le x \le 1, \tag{1.6}$$

$$(\rho^\gamma - \rho^{\alpha+1}u_x,\, \phi_x)(d,t) = (0,0), \qquad d=0,1,\ t \ge 0. \tag{1.7}$$

The assumptions on the initial data are

(A_1) $0<\alpha<\frac{1}{4}$, $\gamma>1$, $\inf_{[0,1]}\rho_0>0$, $|\phi_0|\le 1$,

(A_2) $\rho_0, u_0 \in H^2([0,1])$, $\phi_0 \in H^3([0,1])$,

(A_3) $\rho_0, u_0 \in H^4([0,1])$, $\phi_0 \in H^5([0,1])$,

(A_4) $\rho_0, u_0 \in H^{2m}([0,1])$, $\phi_0 \in H^{2m+1}([0,1])$, $m \in \mathbb{N}$,

(A_5) $\rho_0, u_0, \phi_0 \in C^\infty([0,1])$.

We now give the definition of an H^i-solution.

Definition 1.1 For any fixed constant $T>0$, (ρ,u,ϕ) is called a global $H^i([0,1])$-solution ($i=2m$) to problem (1.5)–(1.7) if it satisfies the condition (1.6) and that

$$\begin{aligned} &0 \le c^{-1} \le \rho(x,t) \le c, \quad |\phi(x,t)| \le 1, \quad (x,t) \in [0,1]\times[0,T], \\ &\rho \in L^\infty([0,T];H^i), \quad u \in L^\infty([0,T];H^i)\cap L^2([0,T];H^{i+1}), \quad \phi \in L^\infty([0,T];H^{i+1}). \end{aligned}$$

We now state our main results.

Theorem 1.1 If the initial data (ρ_0,u_0,ϕ_0) is compatible with the boundary conditions and satisfy (A_1), (A_2), then problem (1.5)–(1.7) admits a unique global H^2-solution (ρ,u,ϕ) on $[0,1]\times[0,T]$.

Theorem 1.2 If the initial data (ρ_0,u_0,ϕ_0) is compatible with the boundary conditions and satisfy (A_1), (A_4), then problem (1.5)–(1.7) admits a unique global $H^{2m}(m\in\mathbb{N})$-solution (ρ,u,ϕ) on $[0,1]\times[0,T]$.

Theorem 1.3 If the initial data (ρ_0,u_0,ϕ_0) is compatible with the boundary conditions and satisfy (A_1), (A_5), then problem (1.5)–(1.7) admits a unique global C^∞-solution (ρ,u,ϕ) on $[0,1]\times[0,T]$.

Remark 1.1 In this paper, the initial density is assumed to be connected to a vacuum with discontinuities. Although in [20] the initial density is connected to a vacuum continuously, the form $\eta(\rho) = 1 + \rho^\alpha$ can provide the crucial estimate

$$\int_0^1 \frac{1}{\rho} < \infty.$$

With the help of this estimate, the bounds of $\phi(x,t)$ can be derived directly. Then one can derive the bounds of $\rho(x,t)$ and overcome the most important step in the global *a priori* estimates. However, the form $\eta(\rho) = \rho^\alpha$ considered in this paper can only supply the estimate

$$\int_0^1 \rho^{\alpha-1} < \infty,$$

which is not enough to get the bounds of $\phi(x,t)$. This is the main difficulty encountered in this paper, and is also the reason that we need the hypothesis $0 < \alpha < \frac{1}{4}$. It is evident that this difficulty comes from the coupling of the NS equations and AC equations, and indicates that the phase variable does influence the range of α. We can see that the lower bound of $\rho(x,t)$ and the bounds of $\phi(x,t)$ are obtained simultaneously (Lemma 2.4).

In addition, if one can obtain the bounds on $\phi(x,t)$ before estimating the lower bound of $\rho(x,t)$, the range of α can be released to $0 < \alpha < \frac{1}{3}$.

Remark 1.2 Observing the relations of $u(x,t)$ and $\phi(x,t)$ in the equations (1.5), the global *a priori* estimates and the proof of the uniqueness, we see that the regularity of $\phi(x,t)$ is always one order higher than $u(x,t)$. Thus we retain this character in the definition of the H^i-solution, which is different from [32], and this leads to some difficulties in terms of the induction. Fortunately, because $\rho(x,t)$ satisfies the transport equation, we can improve its regularity a little with the aid of the regularity of $u(x,t)$ and solve the problem (Lemma 3.3 Step 3).

The local existence of unique solutions is known from the standard method, based on the Schauder Fixed Point theorem via the operator defined by the linearization of the problem on a small time interval, as in [9]. We omit the proof and just state the result here.

Lemma 1.1 Suppose that (A_1) and (A_2) hold. Then there exists a small time $T_* > 0$ depending only on initial data such that problem (1.5)–(1.7) admits a unique H^2-solution (ρ, u, ϕ) on $[0,1] \times [0, T_*]$.

The structure of the rest of this paper is as follows: in Section 2, the global existence of unique H^2-solutions to problem (1.5)–(1.7) will be proven by the method of extending the local solutions with respect to time based on *a priori* estimates. In Section 3, we establish higher-order estimates by mathematical induction, then obtain $H^{2m}(m \in \mathbb{N})$-solutions and C^∞-solutions.

2 Proof of Theorem 1.1

In this section, we will provide some *a priori* estimates of the H^2-solutions to (1.5)–(1.7), which enables us to extend the local solution to a global one. First, we give the energy estimate.

Lemma 2.1 For any $0 \le t \le T$, we have the identity

$$\int_0^1 \left(\frac{u^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1} + \frac{\rho\phi_x^2}{2} + \frac{(\phi^2-1)^2}{4} \right)(t) + \int_0^t \int_0^1 \left(\rho^{\alpha+1} u_x^2 + \rho\phi_t^2 \right) = E_0, \tag{2.1}$$

where

$$E_0 = \int_0^1 \left(\frac{u_0^2}{2} + \frac{\rho_0^{\gamma-1}}{\gamma-1} + \frac{\rho_0\phi_{0x}^2}{2} + \frac{(\phi_0^2-1)^2}{4} \right).$$

Proof This estimate can be proven by multiplying $(1.5)_2$ by u and multiplying $(1.5)_3$ by $\frac{1}{\rho}(\rho\phi_x - (\phi^3 - \phi))$. □

Next, we get the upper bound of the density ρ.

Lemma 2.2 For any $(x,t) \in [0,1] \times [0,T]$, it holds that

$$\rho(x,t) \le C. \tag{2.2}$$

Proof Rewriting $(1.5)_1$ as $(\rho^\alpha)_t = -\alpha\rho^{\alpha+1}u_x$ then integrating over $[0,t]$ yields that

$$\rho^\alpha(x,t) = -\alpha \int_0^t \rho^{\alpha+1} u_x(x,s)\mathrm{d}s + \rho_0^\alpha(x). \tag{2.3}$$

On the other hand, integrating $(1.5)_2$ over $[x,1]$, we get that

$$\rho^{\alpha+1}u_x(x,t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_x^1 u(\xi,t)\mathrm{d}\xi + \rho^\gamma(x,t) + \frac{1}{2}\rho^2\phi_x^2(x,t). \tag{2.4}$$

Substituting (2.4) into (2.3) gives that

$$\begin{aligned} &\rho^\alpha(x,t) + \alpha\int_0^t \rho^\gamma(x,s)\mathrm{d}s + \frac{\alpha}{2}\int_0^t \rho^2\phi_x^2(x,s)\mathrm{d}s \\ =\ &\alpha\int_x^1 \left[u(\xi,t)\mathrm{d}\xi - u_0(\xi)\right]\mathrm{d}\xi + \rho_0^\alpha(x) \le C\|u\|_{L^2}^2 + C \le C. \end{aligned}$$

From the nonnegativity of α, we obtain (2.2). This completes the proof of Lemma 2.2. □

Lemma 2.3 For any $0 \le t \le T$, there exists a constant $C > 0$ such that

$$\int_0^1 \rho^{\alpha-1} \le C. \tag{2.5}$$

Proof From equation $(1.5)_1$ and the Cauchy inequality, we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho^{\alpha-1}\mathrm{d}x &= (\alpha-1)\int_0^1 \rho^{\alpha-2}\rho_t\mathrm{d}x = (1-\alpha)\int_0^1 \rho^\alpha u_x\mathrm{d}x \\ &\le C\int_0^1 \rho^{\alpha-1}\mathrm{d}x + C\int_0^1 \rho^{\alpha+1}u_x^2\mathrm{d}x. \end{aligned}$$

Then, by using Grönwall's inequality and (2.1), we arrive at (2.5). □

In what follows, we deal with the lower bound of ρ and the bounds of ϕ. This is the key lemma of this section.

Lemma 2.4 It holds that

$$|\phi(x,t)| \le C, \quad \rho(x,t) \ge C(T), \quad (x,t) \in [0,1] \times [0,T], \tag{2.6}$$

$$\int_0^1 \rho_x^2(t) \le C, \qquad t \in [0,T]. \tag{2.7}$$

Proof Differentiating (2.3) respect to x and using (2.4), we have that

$$(\rho^\alpha)_{xt} = -\alpha\left(u_t + (\rho^\gamma)_x + \frac{1}{2}(\rho^2\phi_x^2)_x\right). \tag{2.8}$$

Multiplying (2.8) by $(\rho^\alpha)_x$ and integrating over $[0,1]$, we have that

$$\begin{aligned}&\frac{\alpha^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho^{2\alpha-2}\rho_x^2 + \alpha^2\gamma\int_0^1 \rho^{\gamma+\alpha-2}\rho_x^2\\ =& -\alpha\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(\rho^\alpha)_x + \alpha\int_0^1 u(\rho^\alpha)_{xt} - \frac{\alpha}{2}\int_0^1 (\rho^2\phi_x^2)_x(\rho^\alpha)_x.\end{aligned}$$

Inserting (2.8) into the above equation again, using (2.1) and (2.2), and noting that $\alpha \le \gamma$, we get that

$$\begin{aligned}&\frac{\alpha^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho^{2\alpha-2}\rho_x^2 + \alpha^2\gamma\int_0^1 \rho^{\gamma+\alpha-2}\rho_x^2 + \alpha\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(\rho^\alpha)_x + \frac{\alpha^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u^2\\ =& -\alpha^2\gamma\int_0^1 \rho^{\gamma-1}\rho_x u - \alpha^2\int_0^1 \rho\phi_x(\rho\phi_x)_x u - \alpha\int_0^1 \rho\phi_x(\rho\phi_x)_x(\rho^\alpha)_x\\ \le& C\int_0^1 \rho^{2\alpha-2}\rho_x^2 + C\int_0^1 \rho^{2(\gamma-\alpha)}u^2 + C\int_0^1 |(\rho\phi_x)_x|^2 + C\|\rho\phi_x\|_{L^\infty}^2\left(\int_0^1 u^2 + \int_0^1 \rho^{2\alpha-2}\rho_x^2\right)\\ \le& C\left(\|\rho\phi_x\|_{L^\infty}^2 + 1\right)\int_0^1 \rho^{2\alpha-2}\rho_x^2 + C\int_0^1 |(\rho\phi_x)_x|^2 + \|\rho\phi_x\|_{L^\infty}^2 + C.\end{aligned}$$

Integrating the above equation over $(0,t)$ yields that

$$\begin{aligned}&\frac{\alpha^2}{2}\int_0^1 \rho^{2\alpha-2}\rho_x^2 + \alpha^2\gamma\int_0^t\int_0^1 \rho^{\gamma+\alpha-2}\rho_x^2 + \frac{\alpha^2}{2}\int_0^1 u^2\\ \le& \frac{\alpha^2}{2}\int_0^1 \rho_0^{2\alpha-2}\rho_{0x}^2 - \alpha\int_0^1 u(\rho^\alpha)_x + \alpha\int_0^1 u_0(\rho_0^\alpha)_x + \frac{\alpha^2}{2}\int_0^1 u_0^2\\ &+ C\int_0^t \left(\|\rho\phi_x\|_{L^\infty}^2 + 1\right)\int_0^1 \rho^{2\alpha-2}\rho_x^2 + C\int_0^t\int_0^1 |(\rho\phi_x)_x|^2 + \int_0^t \|\rho\phi_x\|_{L^\infty}^2 + C.\end{aligned}$$

By using Cauchy's inequality, the assumptions on the initial data and (2.1), we arrive at

$$\begin{aligned}&\frac{\alpha^2}{4}\int_0^1 \rho^{2\alpha-2}\rho_x^2 + \alpha^2\gamma\int_0^t\int_0^1 \rho^{\gamma+\alpha-2}\rho_x^2 + \frac{\alpha^2}{2}\int_0^1 u^2\\ \le& C\int_0^t \left(\|\rho\phi_x\|_{L^\infty}^2 + 1\right)\int_0^1 \rho^{2\alpha-2}\rho_x^2 + C\int_0^t\int_0^1 |(\rho\phi_x)_x|^2 + \int_0^t \|\rho\phi_x\|_{L^\infty}^2 + C.\end{aligned} \tag{2.9}$$

In what follows, we deal with the estimates $\int_0^t \|\rho\phi_x\|_{L^\infty}^2$ and $\int_0^t\int_0^1 |(\rho\phi_x)_x|^2$. From the equation $(1.5)_3$ and boundary conditions (1.7), (2.1) and (2.2), we get that

$$\begin{aligned}\|\rho\phi_x\|_{L^\infty}^2 &\le C\left(\int_0^1 |(\rho\phi_x)_x|\right)^2 = C\left(\int_0^1 \left|\rho\phi_t + (\phi^3 + \phi)\right|\right)^2\\ &\le C\int_0^1 \rho\phi_t^2 + \left(\int_0^1 (|\phi|^3 + |\phi|)\right)^2 \le C\int_0^1 \rho\phi_t^2 + C,\end{aligned} \tag{2.10}$$

which, together with (2.1), imply that

$$\int_0^t \|\rho\phi_x\|_{L^\infty}^2 \le C. \tag{2.11}$$

By using (2.1) and (2.5), for any $(x,t)\in[0,1]\times[0,T]$, it follows that

$$|\phi(x,t)|\le C\int_0^1(|\phi|+|\phi_x|)\le C+C\left(\int_0^1\rho\phi_x^2\right)^{1/2}\left(\int_0^1\rho^{-1}\right)^{1/2}$$
$$\le C+C\left(\max_{x\in[0,1]}\rho^{-\alpha}\int_0^1\rho^{\alpha-1}\right)^{1/2}\le C+C\max_{x\in[0,1]}\rho^{-\alpha/2}. \tag{2.12}$$

Then, from (2.1) and (2.12), we get that

$$\int_0^t\int_0^1|(\rho\phi_x)_x|^2\le\int_0^t\int_0^1|\rho\phi_t+(\phi^3-\phi)|^2$$
$$\le C\int_0^t\int_0^1\rho\phi_t^2+\int_0^t\max_{x\in[0,1]}\phi^2\int_0^1(\phi^2-1)^2$$
$$\le C+C\int_0^t\max_{x\in[0,1]}\rho^{-\alpha}. \tag{2.13}$$

For the inequality (2.9), by using Grönwall's inequality, (2.11), (2.13) and the positivity of α, we obtain that

$$\int_0^1\rho^{2\alpha-2}\rho_x^2\le C\int_0^t\max_{x\in[0,1]}\rho^{-\alpha}+C. \tag{2.14}$$

By the mean value theorem, (2.5) and (2.14), for any $(x,t)\in[0,1]\times[0,T]$, we find that

$$\rho^{\alpha-1}(x,t)\le C+C\int_0^1|\rho^{\alpha-2}\rho_x|\le C+C\left(\int_0^1\rho^{2\alpha-2}\rho_x^2\right)^{1/2}\left(\int_0^1\rho^{-2}\right)^{1/2}$$
$$\le C+C\left(\int_0^t\max_{x\in[0,1]}\rho^{-\alpha}+1\right)^{1/2}\max_{x\in[0,1]}\rho^{-\frac{\alpha+1}{2}}$$
$$\le C+C\max_{[0,1]\times[0,t]}\rho^{-\left(\frac12+\alpha\right)}$$
$$\le\frac12\max_{[0,1]\times[0,t]}\rho^{\alpha-1}+C,$$

where we have used the fact that $0<\alpha<\frac14$. This implies that

$$\rho(x,t)\ge C^{-1},\qquad (x,t)\in[0,1]\times[0,T]. \tag{2.15}$$

Inserting (2.15) back into (2.12) and (2.14), we have that

$$|\phi(x,t)|\le C,\qquad \int_0^1\rho_x^2\le C.$$

This completes the proof of Lemma 2.4. □

The rest lemmas are about higher order estimates for (ρ,u,ϕ).

Lemma 2.5 For any $0\le t\le T$, the following inequality holds:

$$\int_0^1(\phi_t^2+\phi_{xx}^2)(t)+\int_0^t\int_0^1\phi_{xt}^2\le C. \tag{2.16}$$

Proof Differentiating $(1.5)_3$ with respect to t, multiplying the result by ϕ_t, then integrating with respect to x over $[0,1]$ and using$(1.5)_1$ and (2.10), we have that

$$\frac12\frac{\mathrm d}{\mathrm dt}\int_0^1\rho\phi_t^2+\int_0^1\rho\phi_{xt}^2=\frac12\int_0^1\rho^2u_x\phi_t^2+\int_0^1\rho^2u_x\phi_x\phi_{xt}-\int_0^1(3\phi^2-1)\phi_t^2$$

$$\begin{aligned}
&\leq C\left\|\rho^{\frac{\alpha+1}{2}}u_x\right\|_{L^2}\|\phi_t\|_{L^\infty}\|\sqrt{\rho}\phi_t\|_{L^2}+C\left\|\rho^{\frac{\alpha+1}{2}}u_x\right\|_{L^2}\|\rho\phi_x\|_{L^\infty}\|\sqrt{\rho}\phi_{xt}\|_{L^2}\\
&\quad+C\|3\phi^2-1\|_{L^\infty}\|\sqrt{\rho}\phi_t\|_{L^2}^2\\
&\leq C\left\|\rho^{\frac{\alpha+1}{2}}u_x\right\|_{L^2}\left(\|\sqrt{\rho}\phi_t\|_{L^2}+\|\sqrt{\rho}\phi_{xt}\|_{L^2}\right)\|\sqrt{\rho}\phi_t\|_{L^2}\\
&\quad+C\left\|\rho^{\frac{\alpha+1}{2}}u_x\right\|_{L^2}\left(\|\sqrt{\rho}\phi_t\|_{L^2}+1\right)\|\sqrt{\rho}\phi_{xt}\|_{L^2}+C\left\|\sqrt{\rho}\phi_t\right\|_{L^2}^2\\
&\leq\frac{1}{2}\int_0^1\rho\phi_{xt}^2+C\left(\int_0^1\rho^{\alpha+1}u_x^2+1\right)\int_0^1\rho\phi_t^2+C\int_0^1\rho^{\alpha+1}u_x^2.
\end{aligned}$$

By using Grönwall's inequality, (2.1) and (2.6), we arrive at

$$\int_0^1\phi_t^2(t)+\int_0^t\int_0^1\phi_{xt}^2\leq C.$$

Recalling $(1.5)_3$, we can derive (2.16) directly. The proof of the lemma is complete. □

Lemma 2.6 For any $0\leq t\leq T$, it holds that

$$\int_0^1(u_t^2+u_{xx}^2+\rho_t^2+\rho_{xt}^2)(t)+\int_0^t\int_0^1(u_{xt}^2+\rho_{tt}^2)\leq C. \tag{2.17}$$

Proof Differentiating $(1.5)_2$ wtih respect to t and multiplying the result by u_t, then integrating over $[0,1]$, we have that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1u_t^2+\int_0^1\rho^{\alpha+1}u_{xt}^2\\
&=-\gamma\int_0^1\rho^{\gamma+1}u_xu_{xt}+(\alpha+1)\int_0^1\rho^{\alpha+2}u_x^2u_{xt}-\int_0^1\rho^3\phi_x^2u_xu_{xt}+\int_0^1\rho^2\phi_x\phi_{xt}u_{xt}\\
&\leq\frac{1}{2}\int_0^1\rho^{\alpha+1}u_{xt}^2+C\int_0^1\rho^{\alpha+1}u_x^2\int_0^1u_t^2+C\int_0^1\rho^{\alpha+1}u_x^2+C\int_0^1\phi_{xt}^2.
\end{aligned}$$

Applying Grönwall's inequality, by (2.1), (2.15) and (2.16), we get that

$$\int_0^1u_t^2+\int_0^t\int_0^1u_{xt}^2\leq C.$$

From $(1.5)_2$, by using the above inequality and (2.1), (2.2), (2.7) and (2.16), we have that

$$\begin{aligned}
\left\|\rho^{\alpha+1}u_{xx}\right\|_{L^2}&\leq\|u_t\|_{L^2}+\|(\rho^\gamma)_x\|_{L^2}+\frac{1}{2}\left\|\left(\rho^2\phi_x^2\right)_x\right\|_{L^2}+\left\|\left(\rho^{\alpha+1}\right)_xu_x\right\|_{L^2}\\
&\leq\|u_t\|_{L^2}+C\|\rho_x\|_{L^2}+\|\rho\phi_x\|_{L^\infty}\|(\rho\phi_x)_x\|_{L^2}+C\|u_x\|_{L^\infty}\|\rho_x\|_{L^2}\\
&\leq\frac{1}{2}\left\|\rho^{\alpha+1}u_{xx}\right\|_{L^2}+C.
\end{aligned}$$

Moreover, by using equation $(1.5)_1$, we get that

$$\begin{aligned}
&\rho_t=-\rho^2u_x,\qquad\rho_{xt}=-\rho^2u_{xx}-2\rho\rho_xu_x,\\
&\rho_{tt}=-\rho^2u_{xt}-2\rho\rho_tu_x=-\rho^2u_{xt}+2\rho^3u_x^2.
\end{aligned}$$

It is easy to prove that (2.17) holds. □

Lemma 2.7 For any $0\leq t\leq T$, it holds that

$$\int_0^1\rho_{xx}^2(t)+\int_0^t\int_0^1(\phi_{xxx}^2+u_{xxx}^2)\leq C. \tag{2.18}$$

Proof From $(1.5)_1$, we see that

$$(\rho^\alpha)_t = -\alpha\rho^{\alpha+1}u_x.$$

Differentiating the above equation with respect to x twice, multiplying the result by $(\rho^\alpha)_{xx}$, then integrating over $[0,1]$, we have that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |(\rho^\alpha)_{xx}|^2 &\le C\int_0^1 |(\rho^\alpha)_{xx}|^2 + C\int_0^1 \left|(\rho^{\alpha+1}u_x)_{xx}\right|^2\\ &\le C\int_0^1 |(\rho^\alpha)_{xx}|^2 + C\left(\int_0^1 |(\rho^\gamma)_{xx}|^2 + \int_0^1 u_{xt}^2 + \int_0^1 \left|(\rho^2\phi_x^2)_{xx}\right|^2\right), \quad (2.19)\end{aligned}$$

where we have used equation $(1.5)_2$. Also, it holds that

$$\begin{aligned}\int_0^1 |(\rho^\gamma)_{xx}|^2 &= \int_0^1 \left|\frac{\gamma}{\alpha}(\rho^{\gamma-\alpha}(\rho^\alpha)_x)_x\right|^2 \le \int_0^1 \left|\frac{\gamma}{\alpha}(\rho^{\gamma-\alpha})_x(\rho^\alpha)_x\right|^2 + \int_0^1 \left|\frac{\gamma}{\alpha}\rho^{\gamma-\alpha}(\rho^\alpha)_{xx}\right|^2\\ &\le C\,\|(\rho^\alpha)_x\|_{L^\infty}^2\int_0^1 |(\rho^\alpha)_x|^2 + C\int_0^1 |(\rho^\alpha)_{xx}|^2\\ &\le C\int_0^1 |(\rho^\alpha)_{xx}|^2 + C, \quad (2.20)\end{aligned}$$

$$\begin{aligned}\int_0^1 \left|(\rho^2\phi_x^2)_{xx}\right|^2 &\le C\int_0^1 |(\rho\phi_x)_x|^4 + \int_0^1 |\rho\phi_x|^2\,|(\rho\phi_x)_{xx}|^2\\ &\le C\int_0^1 \left|\rho\phi_t + (\phi^3-\phi)\right|^4 + \|\rho\phi_x\|_{L^\infty}^2\int_0^1 \left|\rho\phi_{xt} + \rho_x\phi_t + (3\phi^2-1)\phi_x\right|^2\\ &\le C\int_0^1 \phi_{xt}^2 + C. \quad (2.21)\end{aligned}$$

Substituting (2.20) and (2.21) into inequality (2.19), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |(\rho^\alpha)_{xx}|^2 \le C\int_0^1 |(\rho^\alpha)_{xx}|^2 + C\int_0^1 \phi_{xt}^2 + C\int_0^1 u_{xt}^2 + C.$$

Applying Grönwall's inequality, (2.16) and (2.17), we obtain that $\int_0^1 |(\rho^\alpha)_{xx}|^2 \le C$. Moreover, it follows from $(1.5)_{2,3}$ that

$$\begin{aligned}&\rho\phi_{xxx} = (\rho\phi_t)_x + (\phi^3-\phi)_x - \rho_{xx}\phi_x - 2\rho_x\phi_{xx},\\ &\rho^{\alpha+1}u_{xxx} = u_{xt} + (\rho^\gamma)_{xx} + \frac{1}{2}(\rho^2\phi_x^2)_{xx} - (\rho^{\alpha+1})_{xx}u_x - 2(\rho^{\alpha+1})_x u_{xx}.\end{aligned}$$

From this, we can derive (2.18). This completes the proof of Lemma 2.7. □

Lemma 2.8 For any $0 \le t \le T$, it holds that

$$\int_0^1 (\phi_{xt}^2 + \phi_{xxx}^2)(t) + \int_0^t\int_0^1 (\phi_{tt}^2 + \phi_{xxt}^2) \le C. \quad (2.22)$$

Proof The equation$(1.5)_3$ implies that

$$\phi_t = \phi_{xx} + \frac{1}{\rho}\rho_x\phi_x - \frac{1}{\rho}(\phi^3-\phi).$$

Differentiating the above equation with respect to t, multiplying the result by ϕ_{tt} and integrating over $[0,1]$ yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \phi_{xt}^2 + \int_0^1 \phi_{tt}^2 = \int_0^1 \rho_x u_x\phi_x\phi_{tt} + \int_0^1 \frac{1}{\rho}\rho_{xt}\phi_x\phi_{tt} + \int_0^1 \frac{1}{\rho}\rho_x\phi_{xt}\phi_{tt}$$

$$
\begin{aligned}
&-\int_0^1 u_x(\phi^3-\phi)\phi_{tt}-\int_0^1 \frac{1}{\rho}(3\phi^2-1)\phi_t\phi_{tt}\\
\le{}& \frac12\int_0^1 \phi_{tt}^2 + C\|\phi_x\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\|\rho_x\|_{L^2}^2 + C\|\phi_x\|_{L^\infty}^2\|\rho_{xt}\|_{L^2}^2\\
&+C\|\rho_x\|_{L^\infty}^2\int_0^1\phi_{xt}^2 + C\|u_x\|_{L^2}^2 + C\|\phi_t\|_{L^2}^2\\
\le{}& \frac12\int_0^1\phi_{tt}^2 + C\int_0^1\phi_{xt}^2 + C.
\end{aligned}
$$

By using Grönwall's inequality, we get that

$$
\int_0^1 \phi_{xt}^2(t) + \int_0^t\int_0^1 \phi_{tt}^2 \le C.
$$

This, together with $(1.5)_3$, yields (2.22). The proof of Lemma 2.8 is complete. □

Proof of Theorem 1.1 Based on the local existence Lemma 1.1 and Lemmas 2.1 to 2.8, we can finish the proof the existence of the H^2-solutions to problem (1.5)–(1.7) by standard procedure. It remains for us to prove the uniqueness of the solution.

Let (ρ_i,u_i,ϕ_i) $(i=1,2)$ be two H^2-solutions to problem (1.5)–(1.7). For convenience, we set that $v_i(x,t)=\frac{1}{\rho_i(x,t)}$ $(i=1,2)$. From $(1.5)_2$, we get

$$
(u_1-u_2)_t + (\rho_1^\gamma-\rho_2^\gamma)_x = (\rho_1^{\alpha+1}u_{1x}-\rho_2^{\alpha+1}u_{2x})_x - \frac12\left(\rho_1^2\phi_{1x}^2-\rho_2^2\phi_{2x}^2\right)_x.
$$

Multiplying the above equation by u_1-u_2, integrating by parts, using the boundary conditions (1.7) and noticing that $\rho_i^{\alpha+1}\partial_x u_i = -\frac{1}{\alpha}\partial_t\rho_i^\alpha$, $\partial_x u_i=\partial_t v_i$ $(i=1,2)$, we find that

$$
\begin{aligned}
\frac12\frac{\mathrm d}{\mathrm dt}\int_0^1(u_1-u_2)^2 ={}& \int_0^1 (v_1^{-\gamma}-v_2^{-\gamma})(v_1-v_2)_t - \int_0^1 \rho_1^{\alpha+1}(u_{1x}-u_{2x})^2\\
&-\int_0^1\left(v_1^{-(\alpha+1)}-v_2^{-(\alpha+1)}\right)u_{2x}(u_{1x}-u_{2x})\\
&+\frac12\int_0^1\rho_1^2(\phi_{1x}-\phi_{2x})(\phi_{1x}+\phi_{2x})(u_{1x}-u_{2x})\\
&+\frac12\int_0^1(v_1^{-2}-v_2^{-2})\phi_{2x}^2(u_{1x}-u_{2x})\\
\le{}& -\frac{\mathrm d}{\mathrm dt}\int_0^1 a(x,t)(v_1-v_2)^2 + \int_0^1 a_t(x,t)(v_1-v_2)^2\\
&-C_0\int_0^1(u_{1x}-u_{2x})^2 + C\int_0^1 u_{2x}^2(v_1-v_2)^2\\
&+C\int_0^1(\phi_{1x}^2+\phi_{2x}^2)(\phi_{1x}-\phi_{2x})^2 + C\int_0^1\phi_{2x}^4(v_1-v_2)^2,
\end{aligned}
\qquad (2.23)
$$

where C_0 and C are positive constants depending only on the upper and lower bounds of ρ_1 and ρ_2, and where $a(x,t)$ is defined as

$$
a(x,t) := \frac{\gamma}{2}\int_0^1 (v_2+\tau(v_1-v_2))^{-(\gamma+1)}\mathrm d\tau,
$$

which has a positive lower bound on $[0,1]\times[0,T]$. Since

$$
|a_t(x,t)| \le C\left(|v_{2t}|+|v_{1t}-v_{2t}|\right),
$$

we have that

$$\int_0^1 a_t(x,t)(v_1-v_2)^2 \le \frac{C_0}{2}\int_0^1 (v_{1t}-v_{2t})^2 + C\int_0^1 (1+|v_{2t}|)(v_1-v_2)^2.$$

Hence, from (2.23), we get that

$$\begin{aligned}&\frac{1}{2}\int_0^1 (u_1-u_2)^2 + \int_0^1 a(x,t)(v_1-v_2)^2 + \frac{C_0}{2}\int_0^t\int_0^1 (u_{1x}-u_{2x})^2\\ \le{}& C\int_0^t\int_0^1 \left(1+|u_{2x}|^2+|\phi_{2x}|^4\right)(v_1-v_2)^2 + C\int_0^t\int_0^1 \left(|\phi_{1x}|^2+|\phi_{2x}|^2\right)(\phi_{1x}-\phi_{2x})^2. \end{aligned}\tag{2.24}$$

On the other hand, from $(1.5)_3$, we can deduce that

$$(\phi_1-\phi_2)_t - (\phi_1-\phi_2)_{xx} = (\rho_{1x}v_1\phi_{1x} - \rho_{2x}v_2\phi_{2x}) - (v_1\phi_1^3 - v_2\phi_2^3) + (v_1\phi_1 - v_2\phi_2).$$

Multiplying the above equation by $\phi_1-\phi_2$, integrating over $[0,1]$, and integrating by parts, yields that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\phi_1-\phi_2)^2 + \int_0^1 (\phi_{1x}-\phi_{2x})^2\\ ={}&\int_0^1 \rho_{1x}v_1(\phi_{1x}-\phi_{2x})(\phi_1-\phi_2) + \int_0^1 \rho_{1x}(v_1-v_2)\phi_{2x}(\phi_1-\phi_2)\\ &-\int_0^1 (v_1^{-1}-v_2^{-1})v_2\phi_{2x}(\phi_{1x}-\phi_{2x}) - \int_0^1 (v_1^{-1}-v_2^{-1})v_{2x}\phi_{2x}(\phi_1-\phi_2)\\ &-\int_0^1 (v_1^{-1}-v_2^{-1})v_2\phi_{2xx}(\phi_1-\phi_2) - \int_0^1 v_1\left(\phi_1^3-\phi_2^3\right)(\phi_1-\phi_2)\\ &-\int_0^1 (v_1-v_2)\phi_2^3(\phi_1-\phi_2) + \int_0^1 v_1(\phi_1-\phi_2)^2 + \int_0^1 (v_1-v_2)\phi_2(\phi_1-\phi_2)\\ \le{}&\frac{1}{2}\int_0^1 (\phi_{1x}-\phi_{2x})^2 + C\int_0^1 \left(|\rho_{1x}|^2+|\phi_{2xx}|^2+1\right)(\phi_1-\phi_2)^2\\ &+C\int_0^1 \left(|\phi_{2x}|^2+|\rho_{2x}|^2+1\right)(v_1-v_2)^2.\end{aligned}$$

From this, we have that

$$\begin{aligned}&\int_0^1 (\phi_1-\phi_2)^2 + \int_0^t\int_0^1 (\phi_{1x}-\phi_{2x})^2\\ \le{}& C\int_0^t\int_0^1 \left(|\rho_{1x}|^2+|\phi_{2xx}|^2+1\right)(\phi_1-\phi_2)^2 + C\int_0^t\int_0^1 \left(|\phi_{2x}|^2+|\rho_{2x}|^2+1\right)(v_1-v_2)^2.\end{aligned}\tag{2.25}$$

Noticing that $\|\phi_{ix}\|_{L^\infty} \le C$ $(i=1,2)$ and that $a(x,t)\ge C>0$, combining (2.24) and (2.25) together, it holds that

$$\begin{aligned}&\int_0^1 \left[(v_1-v_2)^2+(u_1-u_2)^2+(\phi_1-\phi_2)^2\right]\\ \le{}& C\int_0^t\int_0^1 \left(|\rho_{1x}|^2+|\rho_{2x}|^2+|u_{2x}|^2+|\phi_{2x}|^4+|\phi_{2xx}^2|+1\right)\left[(v_1-v_2)^2+(\phi_1-\phi_2)^2\right].\end{aligned}$$

Since $\int_0^t \|(\rho_{1x},\rho_{2x},u_{2x},\phi_{2xx})\|_{L^\infty}^2 \mathrm{d}s \le C$, applying Grönwall's inequality yields that

$$v_1(x,t)=v_2(x,t),\quad u_1(x,t)=u_2(x,t),\quad \phi_1(x,t)=\phi_2(x,t),\quad \text{a.e. } (x,t)\in[0,1]\times[0,T].$$

The proof of uniqueness is complete. □

3 Proof of Theorem 1.2 and Theorem 1.3

In this section, we will prove Theorems 1.2 and 1.3 by establishing some *a priori* estimates. First, by virtue of the estimates in Section 2, we obtain the following lemma:

Lemma 3.1 For any $0 \le t \le T$, the H^2-solution (ρ, u, ϕ) to the problem (1.5)–(1.7) satisfies that

$$\begin{aligned} &0 < C(T)^{-1} \le \rho(x,t) \le C(T), \quad |\phi(x,t)| \le 1, \quad (x,t) \in [0,1] \times [0,T], \\ &\|\rho_t\|_{H^1}^2 + \|\rho\|_{H^2}^2 + \|u_t\|^2 + \|u\|_{H^2}^2 + \|\phi_t\|_{H^1}^2 + \|\phi\|_{H^3}^2 \\ &+ \int_0^t \left(\|u_t\|_{H^1}^2 + \|u\|_{H^3}^2 + \|\phi_t\|_{H^2}^2 + \|\phi_{tt}\|^2\right) \mathrm{d}\tau \le C_2(T). \end{aligned}$$

As pointed out in [9], we can also obtain

Lemma 3.2 Under the conditions (A_1) and (A_3), there exists a unique H^4-solution (ρ, u, ϕ) to the problem (1.5)–(1.7) such that, for any $0 \le t \le T$, it holds that

$$\begin{aligned} &\|\rho_t\|_{H^2}^2 + \|\rho_{tt}\|^2 + \|\rho\|_{H^4}^2 + \|u_t\|_{H^2}^2 + \|u_{tt}\|^2 + \|u\|_{H^4}^2 + \|\phi_t\|_{H^3}^2 + \|\phi_{tt}\|_{H^1}^2 + \|\phi\|_{H^5}^2 \\ &+ \int_0^t \left(\|u_t\|_{H^3}^2 + \|u_{tt}\|_{H^1}^2 + \|u\|_{H^5}^2 + \|\phi_t\|_{H^4}^2 + \|\phi_{tt}\|_{H^2}^2 + \|\phi_{ttt}\|^2\right) \mathrm{d}\tau \le C_4(T). \end{aligned}$$

Next, we will prove the higher *a priori* estimates by induction. Here we use the notation $\|\cdot\| := \|\cdot\|_{L^2([0,1])}$.

Lemma 3.3 Under conditions (A_1) and (A_4), there exists a unique global solution (ρ, u, ϕ) to the problem (1.5)–(1.7) such that, for any $0 \le t \le T$, it holds that

$$\|\rho_{x^{2s}t^{m-s}}\|^2 + \|u_{x^{2s}t^{m-s}}\|^2 + \|\phi_{x^{2s+1}t^{m-s}}\|^2 + \int_0^t \left(\|u_{x^{2s+1}t^{m-s}}\|^2 + \|\phi_{x^{2s}t^{m-s+1}}\|^2\right) \mathrm{d}\tau \le C_{2m}(T), \tag{3.1}$$

where $0 \le s \le m$.

Proof For $m = 1, 2$, (3.1) has been achieved in Lemmas 3.1 and 3.2. Now we suppose that (3.1) is valid for all $1 \le m \le M-1$, and we will prove that (3.1) is also valid for $m = M$.

Step 1 We prove that

$$\|u_{t^M}\|^2 + \|\phi_{xt^M}\|^2 + \int_0^t \left(\|u_{xt^M}\|^2 + \|\phi_{t^{M+1}}\|^2\right) \mathrm{d}\tau \le C_{2M}(T). \tag{3.2}$$

Taking the t^M-order derivative to the equation $(1.5)_2$, multiplying the result by u_{t^M}, integrating over $[0,1]$, then applying integration by parts and the Leibniz formula, we have that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{t^M}\|^2 &= \int_0^1 (\rho^\gamma)_{t^M} u_{xt^M} - \int_0^1 \left(\rho^{\alpha+1}u_x\right)_{t^M} u_{xt^M} + \frac{1}{2}\int_0^1 \left(\rho^2\phi_x^2\right)_{t^M} u_{xt^M} \\ &= \int_0^1 (\rho^\gamma)_{t^M} u_{xt^M} - \int_0^1 \rho^{\alpha+1}u_{xt^M}^2 - \int_0^1 (\rho^{\alpha+1})_{t^M} u_x u_{xt^M} \\ &\quad - \int_0^1 \sum_{k=1}^{M-1} C_M^k (\rho^{\alpha+1})_{t^k} u_{xt^{M-k}} u_{xt^M} + \frac{1}{2}\int_0^1 \left(\rho^2\phi_x^2\right)_{t^M} u_{xt^M} \\ &\le -\frac{1}{2}\int_0^1 \rho^{\alpha+1}u_{xt^M}^2 + C\int_0^1 \left(\rho^{\gamma+1}u_x\right)_{t^{M-1}}^2 + C\|u_x\|_{L^\infty}^2 \int_0^1 (\rho^{\alpha+2}u_x)_{t^{M-1}}^2 \end{aligned}$$

$$+ C\int_0^1 \Big[\sum_{k=1}^{M-1} C_M^k (\rho^{\alpha+1})_{t^k} u_{xt^{M-k}}\Big]^2 + C\int_0^1 \left(\rho^2\phi_x^2\right)_{t^M}^2 . \tag{3.3}$$

By using the Leibniz formula, we calculate that

$$\begin{aligned}\int_0^1 \left(\rho^{\gamma+1}u_x\right)_{t^{M-1}}^2 &= \int_0^1 \Big[\sum_{k=0}^{M-1} C_{M-1}^k (\rho^{\gamma+1})_{t^k} u_{xt^{M-k-1}}\Big]^2 \\ &= \int_0^1 \Big[\rho^{\gamma+1}u_{xt^{M-1}} + \sum_{k=1}^{M-1} C_{M-1}^k (\rho^{\gamma+1})_{t^k} u_{xt^{M-k-1}}\Big]^2 \\ &\le C\|u_{xt^{M-1}}\|^2 + C_{2(M-1)}(T). \end{aligned} \tag{3.4}$$

Similarly, it holds that

$$\int_0^1 (\rho^{\alpha+2}u_x)_{t^{M-1}}^2 \le C\|u_{xt^{M-1}}\|^2 + C_{2(M-1)}(T).$$

Moreover,

$$\begin{aligned}\int_0^1 \left(\rho^2\phi_x^2\right)_{t^M}^2 &= \int_0^1 \Big[\sum_{k=0}^{M} C_M^k (\rho\phi_x)_{t^k}(\rho\phi_x)_{t^{M-k}}\Big]^2 \\ &= \int_0^1 \Big[2\rho\phi_x(\rho\phi_x)_{t^M} + \sum_{k=1}^{M-1} C_M^k (\rho\phi_x)_{t^k}(\rho\phi_x)_{t^{M-k}}\Big]^2 \\ &\le C_2\|\phi_{xt^M}\|^2 + C\|u_{xt^{M-1}}\|^2 + C_{2(M-1)}(T), \end{aligned} \tag{3.5}$$

where we have used that

$$\|\rho_{t^M}\| = \|(-\rho^2u_x)_{t^{M-1}}\| \le C\|u_{xt^{M-1}}\| + C_{2(M-1)}(T).$$

Substituting (3.4)–(3.5) into (3.3), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{t^M}\|^2 + \|u_{xt^M}\|^2 \le C_2\|u_{xt^{M-1}}\|^2 + C_2\|\phi_{xt^M}\|^2 + C_{2(M-1)}(T). \tag{3.6}$$

Taking the t^M-order derivative to equation $(1.5)_3$, multiplying the result by $\phi_{t^{M+1}}$, integrating over $[0,1]$, and applying Leibniz's formula yields that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_{xt^M}\|^2 + \|\phi_{t^{M+1}}\|^2 \\ &\le \int_0^1 ((\ln\rho)_x\phi_x)_{t^M}^2 + \int_0^1 \Big(\frac{1}{\rho}(\phi^3-\phi)\Big)_{t^M}^2 \\ &= \int_0^1 \Big[(\ln\rho)_x\phi_{xt^M} + (\ln\rho)_{xt^M}\phi_x + C_M^{M-1}(\ln\rho)_{xt^{M-1}}\phi_{xt} + \sum_{k=1}^{M-2} C_M^k(\ln\rho)_{xt^k}\phi_{xt^{M-k}}\Big]^2 \\ &\quad + \int_0^1 \Big[\frac{1}{\rho}(\phi^3-\phi)_{t^M} + \Big(\frac{1}{\rho}\Big)_{t^M}(\phi^3-\phi) + \sum_{k=1}^{M-1} C_M^k\Big(\frac{1}{\rho}\Big)_{t^k}(\phi^3-\phi)_{t^{M-k}}\Big]^2 \\ &\le C_2\|\phi_{xt^M}\|^2 + C_2\|\rho_{xt^M}\|^2 + C_2\|\rho_{xt^{M-1}}\|^2 + C\|\phi_{t^M}\|^2 + C\|\rho_{t^M}\|^2 + C_{2(M-1)}(T). \end{aligned} \tag{3.7}$$

From equation $(1.5)_{1,2}$ and the Leibniz formula, we calculate that

$$\begin{aligned}\|\rho_{xt^{M-1}}\| &= \left\|-(\rho^2u_x)_{xt^{M-2}}\right\| \le C_2\left\|\rho_{xt^{M-2}}\right\| + C\left\|u_{x^2t^{M-2}}\right\| + C_{2(M-1)}(T) \\ &\le C_2\left(\left\|\rho_{t^{M-2}}\right\|^{1/2}\left\|\rho_{x^2t^{M-2}}\right\|^{1/2} + \left\|\rho_{t^{M-2}}\right\|\right) + C_{2(M-1)}(T)\end{aligned}$$

$$\leq C_{2(M-1)}(T), \tag{3.8}$$

$$\|\rho_{xt^M}\| = \left\|-(\rho^2 u_x)_{xt^{M-1}}\right\| \leq C_2 \|u_{xt^{M-1}}\| + C\|u_{x^2t^{M-1}}\| + C_{2(M-1)}(T), \tag{3.9}$$

where we have used the Gagliardo-Nirenberg inequality. Now we estimate $\|u_{x^2t^{M-1}}\|$. Rewrite equation $(1.5)_2$ as

$$u_{xx} = \rho^{-(\alpha+1)}u_t + \rho^{-(\alpha+1)}(\rho^\gamma)_x - (\alpha+1)\rho^{-1}\rho_x u_x + \rho^{-\alpha}\phi_x(\rho\phi_x)_x. \tag{3.10}$$

Then we have

$$\|u_{x^2t^{M-1}}\| \leq C\|u_{t^M}\| + C\|u_{xt^{M-1}}\| + C\|\phi_{x^2t^{M-1}}\| + C_{2(M-1)}(T). \tag{3.11}$$

From (3.8), (3.9) and (3.11), inequality (3.7) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_{xt^M}\|^2 + \|\phi_{t^{M+1}}\|^2\\ &\leq C_2\|\phi_{xt^M}\|^2 + C\|u_{t^M}\|^2 + C_2\|u_{xt^{M-1}}\|^2 + C\|\phi_{x^2t^{M-1}}\|^2 + C\|\phi_{t^M}\|^2 + C_{2(M-1)}(T).\end{aligned} \tag{3.12}$$

Adding (3.6) and (3.12) together yields that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_{t^M}\|^2 + \|\phi_{xt^M}\|^2\right) + \|u_{xt^M}\|^2 + \|\phi_{t^{M+1}}\|^2\\ &\leq C_2\left(\|u_{t^M}\|^2 + \|\phi_{xt^M}\|^2\right) + C_2\left(\|u_{xt^{M-1}}\|^2 + \|\phi_{x^2t^{M-1}}\|^2 + \|\phi_{t^M}\|^2\right) + C_{2(M-1)}(T).\end{aligned}$$

Applying Grönwall's inequality, we obtain that

$$\begin{aligned}&\|u_{t^M}\|^2 + \|\phi_{xt^M}\|^2 + \int_0^t \left(\|u_{xt^M}\|^2 + \|\phi_{t^{M+1}}\|^2\right)\mathrm{d}\tau\\ &\leq C_2(\|u_{t^M}(0)\|^2 + \|\phi_{xt^M}(0)\|^2) + C_{2(M-1)}(T).\end{aligned} \tag{3.13}$$

In what follows, we will handle $\|u_{t^M}(0)\|$ and $\|\phi_{xt^M}(0)\|$. First, taking x^{2M-2}-order derivative to equation $(1.5)_2$ gives that

$$\begin{aligned}\|u_{tx^{2M-2}}\| &= \left\|-(\rho^\gamma)_{x^{2M-1}} + (\rho^{\alpha+1}u_x)_{x^{2M-1}} - \frac{1}{2}(\rho^2\phi_x^2)_{x^{2M-1}}\right\|\\ &\leq C\left(\|\rho\|_{H^{2M-1}} + \|u\|_{H^{2M}} + \|\phi\|_{H^{2M}}\right).\end{aligned}$$

Next, rewrite the equation $(1.5)_3$ as

$$\phi_t = \phi_{xx} + (\ln\rho)_x\phi_x - \frac{1}{\rho}(\phi^3 - \phi). \tag{3.14}$$

Taking the x^{2M-1}-order derivative to equation (3.14), we get that

$$\begin{aligned}\|\phi_{tx^{2M-1}}\| &= \left\|\phi_{x^{2M+1}} + ((\ln\rho)_x\phi_x)_{x^{2M-1}} - \left(\frac{1}{\rho}(\phi^3 - \phi)\right)_{x^{2M-1}}\right\|\\ &\leq C\left(\|\phi\|_{H^{2M+1}} + \|\rho\|_{H^{2m}}\right).\end{aligned}$$

Then taking the tx^{2M-4}-order derivative to equation $(1.5)_2$, it holds that

$$\begin{aligned}\|u_{t^2x^{2M-4}}\| &= \left\|-(\rho^\gamma)_{tx^{2M-3}} + (\rho^{\alpha+1}u_x)_{tx^{2M-3}} - \frac{1}{2}(\rho^2\phi_x^2)_{tx^{2M-3}}\right\|\\ &\leq C\|(\rho^{\gamma+1}u_x)_{x^{2M-3}}\| + C\|u_{tx^{2M-2}}\| + C\|\phi_{tx^{2M-2}}\| + C\\ &\leq C\left(\|\rho\|_{H^{2M}} + \|u\|_{H^{2M}} + \|\phi\|_{H^{2M}}\right).\end{aligned}$$

Taking the tx^{2M-3}-order derivative to equation (3.14), we have that

$$\begin{aligned}\|\phi_{t^2x^{2M-3}}\| &= \left\|\phi_{tx^{2M-1}} + ((\ln\rho)_x\phi_x)_{tx^{2M-3}} - \left(\frac{1}{\rho}(\phi^3-\phi)\right)_{tx^{2M-3}}\right\| \\ &\le C\|\phi_{tx^{2M-1}}\| + C\|\phi_{tx^{2M-2}}\| + C\|(\rho^2u_x)_{x^{2M-2}}\| + C \\ &\le C\left(\|\rho\|_{H^{2M}} + \|u\|_{H^{2M}} + \|\phi\|_{H^{2M+1}}\right).\end{aligned}$$

Applying the same method $M-1$ times and recalling the assumption (A_4), we conclude that

$$\|u_{t^M}(0)\| + \|\phi_{xt^M}(0)\| \le C_{2M}(T).$$

From this and (3.13), we obtain that

$$\|u_{t^M}\|^2 + \|\phi_{xt^M}\|^2 + \int_0^t \left(\|u_{xt^M}\|^2 + \|\phi_{t^{M+1}}\|^2\right)\mathrm{d}\tau \le C_{2M}(T).$$

Step 2 We prove

$$\|u_{x^{2s}t^{M-s}}\| + \|\phi_{x^{2s+1}t^{M-s}}\| \le C_{2M}(T), \qquad 0\le s\le M. \tag{3.15}$$

Taking the t^{M-1}-order derivative to equation $(1.5)_2$, it holds that

$$u_{t^M} + (\gamma\rho^{\gamma-1}\rho_x)_{t^{M-1}} = \left((\rho^{\alpha+1})_xu_x + \rho^{\alpha+1}u_{xx}\right)_{t^{M-1}} - (\rho\phi_x(\rho\phi_x)_x)_{t^{M-1}}.$$

Applying Leibniz's formula, we get that

$$\begin{aligned}u^2_{x^2t^{M-1}} = \rho^{-2(\alpha+1)}\Bigg[&u_{t^M} + \gamma\sum_{k=0}^{M-1}C^k_{M-1}(\rho^{\gamma-1})_{t^k}\rho_{xt^{M-1-k}} - \sum_{k=0}^{M-1}C^k_{M-1}(\rho^{\alpha+1})_{xt^k}u_{xt^{M-1-k}} \\ &- \sum_{k=1}^{M-1}C^k_{M-1}(\rho^{\alpha+1})_{t^k}u_{x^2t^{M-1-k}} + \sum_{k=0}^{M-1}C^k_{M-1}(\rho\phi_x)_{t^k}(\rho\phi_x)_{xt^{M-1-k}}\Bigg]^2.\end{aligned}$$

Integrating the above equality over $[0,1]$, using (3.2), (3.8) and applying the Gagliardo-Nirenberg inequality yields that

$$\begin{aligned}\|u_{x^2t^{M-1}}\| &\le C\left(\|u_{t^M}\| + \|u_{xt^{M-1}}\| + \|\phi_{x^2t^{M-1}}\|\right) + C_{2(M-1)}(T) \\ &\le \varepsilon\|u_{x^2t^{M-1}}\| + \varepsilon\|\phi_{x^3t^{M-1}}\| + C_{2M}(T).\end{aligned} \tag{3.16}$$

Taking the xt^{M-1}-order derivative to (3.14), we have that

$$\begin{aligned}\|\phi_{x^3t^{M-1}}\| &= \left\|\phi_{xt^M} - ((\ln\rho)_x\phi_x)_{xt^{M-1}} + \left(\frac{1}{\rho}(\phi^3-\phi)\right)_{xt^{M-1}}\right\| \\ &\le \|\phi_{xt^M}\| + C\|\rho_{x^2t^{M-1}}\| + C\|\phi_{x^2t^{M-1}}\| + C_{2(M-1)}(T) \\ &\le \|\phi_{xt^M}\| + C\|\rho_{x^2t^{M-1}}\| + \varepsilon\|\phi_{x^3t^{M-1}}\| + C_{2(M-1)}(T).\end{aligned} \tag{3.17}$$

Moreover, by $(1.5)_1$, we deduce that

$$\|\rho_{x^2t^{M-1}}\| = \left\|-(\rho^2u_x)_{x^2t^{M-2}}\right\| \le C\,\|u_{x^3t^{M-2}}\| + C_{2(M-1)}(T). \tag{3.18}$$

Recalling (3.10) and applying the Gagliardo-Nirenberg inequality, it holds that

$$\begin{aligned}\|u_{x^3t^{M-2}}\| &\le C\|u_{xt^{M-1}}\| + C\|\rho_{x^2t^{M-2}}\| + C\|\phi_{x^3t^{M-2}}\| + C_{2(M-1)}(T) \\ &\le \varepsilon\|u_{x^2t^{M-1}}\| + C_{2(M-1)}(T).\end{aligned} \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17) gives that

$$\|\phi_{x^3t^{M-1}}\| \le \varepsilon\|u_{x^2t^{M-1}}\| + \varepsilon\|\phi_{x^3t^{M-1}}\| + C_{2M}(T). \tag{3.20}$$

Adding (3.16) and (3.20) together and choosing ε small enough, we arrive at

$$\|u_{x^2t^{M-1}}\| + \|\phi_{x^3t^{M-1}}\| \le C_{2M}(T).$$

Using the same method s times, we conclude that

$$\|u_{x^{2s}t^{M-s}}\| + \|\phi_{x^{2s+1}t^{M-s}}\| \le C_{2M}(T).$$

Step 3 We prove that

$$\|\rho_{x^{2s}t^{M-s}}\|^2 + \int_0^t \left(\|u_{x^{2s+1}t^{M-s}}\|^2 + \|\phi_{x^{2s}t^{M-s+1}}\|^2\right) \mathrm{d}\tau \le C_{2M}(T), \quad 0 \le s \le M. \tag{3.21}$$

Taking the x^{2M-1}-order derivative to $(1.5)_1$ yields that

$$\rho_{tx^{2M-1}} = -(\rho^2 u_x)_{x^{2M-1}}.$$

Multiplying this by $\rho_{x^{2M-1}}$, integrating over $[0,1]$ and using Leibniz's formula, we have that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_{x^{2M-1}}\|^2 &= \| -(\rho^2 u_x)_{x^{2M-1}}\rho_{x^{2M-1}}\| \\
&= \left\|\rho^2 u_{x^{2M}}\rho_{x^{2M-1}} + u_x\left(\rho^2\right)_{x^{2M-1}}\rho_{x^{2M-1}} + \sum_{k=1}^{2M-2} C_{2M-1}^k\left(\rho^2\right)_{x^k} u_{x^{2M-k}}\rho_{x^{2M-1}}\right\| \\
&\le C_2\|\rho_{x^{2M-1}}\|^2 + C_{2M}(T).
\end{aligned}$$

Then Grönwall's inequality and (A_5) imply that

$$\|\rho_{x^{2M-1}}\| \le C_{2M}(T).$$

Using equation $(1.5)_1$ and (3.15) gives that

$$\|\rho_{tx^{2M-1}}\| = \| -(\rho^2 u_x)_{x^{2M-1}}\| \le C\|\rho_{x^{2M-1}}\| + C\|u_{x^{2M}}\| \le C_{2M}(T). \tag{3.22}$$

Then it follows that

$$\|\rho_{tx^{2M-2}}\| \le C_{2M}(T).$$

From this and $(1.5)_1$ we derive that

$$\begin{aligned}
\|\rho_{t^2x^{2M-4}}\| &= \| -(\rho^2 u_x)_{tx^{2M-4}}\| \le C\|\rho_{tx^{2M-4}}\| + \|u_{tx^{2M-3}}\| \le C_{2M}(T), \\
\|\rho_{t^3x^{2M-6}}\| &= \| -(\rho^2 u_x)_{t^2x^{2M-6}}\| \le C\|\rho_{t^2x^{2M-6}}\| + \|u_{t^2x^{2M-5}}\| \le C_{2M}(T).
\end{aligned}$$

Applying the same procedure s times, we conclude that

$$\|\rho_{x^{2s}t^{M-s}}\| \le C_{2M}(T), \qquad 0 \le s \le M-1. \tag{3.23}$$

In fact, from (3.22), by a similar procedure, we can also obtain that

$$\|\rho_{x^{2s+1}t^{M-s}}\| \le C_{2M}(T), \qquad 0 \le s \le M-1. \tag{3.24}$$

Taking the xt^{M-1}-order derivative to equation (3.14) and using (3.23), we have that

$$\begin{aligned}
\|\phi_{x^3t^{M-1}}\| &= \left\|\phi_{xt^M} + ((\ln\rho)_x\phi_x)_{xt^{M-1}} + \left(\frac{1}{\rho}(\phi^3-\phi)\right)_{xt^{M-1}}\right\| \\
&\le \|\phi_{xt^M}\| + C\|\rho_{x^2t^{M-1}}\| + C\|\phi_{x^2t^{M-1}}\| + C_{2(M-1)}(T) \\
&\le C_{2M}(T).
\end{aligned}$$

Taking the xt^{M-1}-order derivative to equation $(1.5)_2$ yields that

$$u_{xt^M}+(\rho^\gamma)_{x^2t^{M-1}}=(\rho^{\alpha+1}u_x)_{x^2t^{M-1}}-\frac{1}{2}(\rho^2\phi_x^2)_{x^2t^{M-1}}.$$

From this, along with (3.2), (3.15) and (3.23), we have that

$$\begin{aligned}\int_0^t\|u_{x^3t^{M-1}}\|^2\mathrm{d}\tau&\le C\int_0^t\|u_{xt^M}\|^2\mathrm{d}\tau+\|\rho_{x^2t^{M-1}}\|^2+\|\phi_{x^3t^{M-1}}\|^2+C_{2(M-1)}(T)\\&\le C_{2M}(T).\end{aligned}$$

Taking the t^M-order derivative to equation (3.14) and using (3.2), (3.15) and (3.24), we have that

$$\begin{aligned}\int_0^t\|\phi_{x^2t^M}\|^2\mathrm{d}\tau&=\int_0^t\left\|\phi_{t^{M+1}}-((\ln\rho)_x\phi_x)_{t^M}-\Big(\frac{1}{\rho}(\phi^3-\phi)\Big)_{t^M}\right\|^2\mathrm{d}\tau\\&\le C\int_0^t\|\phi_{t^{M+1}}\|^2\mathrm{d}\tau+C\|\rho_{xt^M}\|^2+C\|\phi_{xt^M}\|^2+C_{2(M-1)}(T)\\&\le C_{2M}(T).\end{aligned}$$

Applying a similar procedure s times, we conclude that

$$\int_0^t\left(\|u_{x^{2s+1}t^{M-s}}\|^2+\|\phi_{x^{2s}t^{M-s+1}}\|^2\right)\mathrm{d}\tau\le C_{2M}(T),\quad 0\le s\le M-1. \tag{3.25}$$

Finally, we deal with the case $s=M$. Multiplying $(1.5)_1$ by $\rho_{x^{2M}}$, integrating over $[0,1]$ and using Leibniz's formula yields that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_{x^{2M}}\|^2&=\|-(\rho^2u_x)_{x^{2M}}\rho_{x^{2M}}\|\\&=\left\|\rho^2u_{x^{2M+1}}\rho_{x^{2M}}+u_x\left(\rho^2\right)_{x^{2M}}\rho_{x^{2M}}+\sum_{k=1}^{2M-1}C_{2M}^k\left(\rho^2\right)_{x^k}u_{x^{2M-k+1}}\rho_{x^{2M}}\right\|\\&\le C_2\|\rho_{x^{2M}}\|^2+C\|u_{x^{2M+1}}\|^2+C_{2M}(T).\end{aligned} \tag{3.26}$$

Taking the x^{2M-1}-order derivative to equation $(1.5)_2$ and using (3.15) yields that

$$\begin{aligned}\|u_{x^{2M+1}}\|&\le C\|u_{x^{2M-1}t}\|+C_2\|\rho_{x^{2M}}\|+C\|\phi_{x^{2M+1}}\|+C_{2(M-1)}(T)\\&\le C\|u_{x^{2M-1}t}\|+C_2\|\rho_{x^{2M}}\|+C_{2M}(T).\end{aligned} \tag{3.27}$$

Inserting (3.27) into (3.26) yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_{x^{2M}}\|^2\le C_2\|\rho_{x^{2M}}\|^2+C\|u_{x^{2M-1}t}\|^2+C_{2M}(T).$$

Then it follows from Grönwall's inequality, (A_5) and (3.25) that

$$\|\rho_{x^{2M}}\|\le C_{2M}(T). \tag{3.28}$$

Substituting (3.28) back into (3.27), integrating over $[0,t]$ and using (3.25), we obtain that

$$\int_0^t\|u_{x^{2M+1}}\|\mathrm{d}\tau\le C\int_0^t\|u_{x^{2M-1}t}\|\mathrm{d}\tau+C_2\|\rho_{x^{2M}}\|+C_{2M}(T)\le C_{2M}(T). \tag{3.29}$$

Finally, taking the $x^{2M-2}t$-order derivative to equation (3.14) and using (3.15), (3.24) and (3.25) yields that

$$\int_0^t\|\phi_{x^{2M}t}\|^2\mathrm{d}\tau=\int_0^t\left\|\phi_{x^{2M-2}t^2}-((\ln\rho)_x\phi_x)_{x^{2M-2}t}-\Big(\frac{1}{\rho}(\phi^3-\phi)\Big)_{x^{2M-2}t}\right\|^2\mathrm{d}\tau$$

$$\leq C\int_0^t \|\phi_{x^{2M-2}t^2}\|^2 \mathrm{d}\tau + C\|\rho_{x^{2M-1}t}\|^2 + C\|\phi_{x^{2M-1}t}\|^2 + C_{2(M-1)}(T)$$
$$\leq C_{2M}(T).$$

This together with (3.23), (3.25), (3.28) and (3.29), implies (3.21).

Collecting together (3.15) and (3.21) finishes the proof of Lemma 3.3. □

Noticing that equation $(1.5)_1$ can be written as $(\frac{1}{\rho})_t = u_x$, with the aid of (3.21), it is easy to obtain the following estimate:

Lemma 3.4 Under conditions (A_1) and (A_4), there exists a unique global solution (ρ, u, ϕ) to problem (1.5)–(1.7) such that, for any $0 \leq t \leq T$, it holds that

$$\int_0^t \|\rho_{x^{2s}t^{M+1-s}}\|^2 \mathrm{d}\tau \leq C_{2M}(T), \qquad 0 \leq s \leq M.$$

Proof of Theorem 1.2 Applying Lemmas 3.1–3.4 and the interpolation inequality readily proves the theorem. □

Proof of Theorem 1.3 The embedding theorem easily proves the theorem. □

Conflict of Interest The authors declare no conflict of interest.

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