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THE LOGARITHMIC SOBOLEV INEQUALITY FOR A SUBMANIFOLD IN MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE SECTIONAL CURVATURE*

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Abstract In this note, we prove a logarithmic Sobolev inequality which holds for compact submanifolds without a boundary in manifolds with asymptotically nonnegative sectional curvature. Like the Michale-Simon Sobolev inequality, this inequality contains a term involving the mean curvature.

Key words asymptotically nonnegative sectional curvature; logarithmic Sobolev inequality; ABP method

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1 Introduction

The classical logarithmic Sobolev inequality, first proven by Gross [7], is a very useful tool in analysis and geometric evolution problems (cf. [4, 8, 11]). In 2000, Ecker [6] gave a logarithmic Sobolev inequality which holds for submanifolds in Euclidean space. In 2020, using the ABP technique, Brendle [2] established a sharp logarithmic Sobolev inequality for submanifolds in Euclidean space without a boundary. He [3] also gave several Sobolev inequalities for manifolds with nonnegative curvature by using the same technique. Combining the method in [3] with some comparison theorems, the authors of [5] proved some Sobolev inequalities for manifolds with asymptotically nonnegative curvature. In 2021, Yi and Zheng [10] proved a logarithmic Sobolev inequality for compact submanifolds without a boundary in manifolds with nonnegative sectional curvature. In this paper, we generalize the results of [2, 10] to the case where the ambient space has asymptotically nonnegative sectional curvature. This curvature notion was first introduced by Abresch [1]. We will use some comparison results for these kinds of manifolds in order to prove our results. Complete manifolds with asymptotically nonnegative sectional curvature belong to the class of complete manifolds with radial sectional curvature bounded from

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below. Readers may find more general comparison results for manifolds with radial sectional curvature bounded below in [9, 12].

In this section, we follow closely the exposition of [5]. Let $\lambda(t):[0,+\infty)\to[0,+\infty)$ be a nonnegative and nonincreasing continuous function satisfing that

$$b_0:=\int_0^{+\infty}s\lambda(s)\mathrm{d}s<+\infty, \tag{1.1}$$

which implies that

$$b_1:=\int_0^{+\infty}\lambda(s)\mathrm{d}s<+\infty. \tag{1.2}$$

Recall that a complete noncompact Riemannian manifold (M,g) of dimension $n+p$ is said to have asymptotically nonnegative sectional curvature if there is a base point $o\in M$ such that

$$\mathrm{Sec}_q\geq-\lambda(d(o,q)), \tag{1.3}$$

where $d(o,q)$ is the distance function of M relative to o. In particular, if $\lambda\equiv 0$ in (1.3), then this becomes the case treated in [10].

Let $h(t)$ be the unique solution of

$$\begin{cases} h''(t)=\lambda(t)h(t),\\ h(0)=0, h'(0)=1. \end{cases} \tag{1.4}$$

By ODE theory, the solution $h(t)$ of (1.4) exists for all $t\in[0,+\infty)$. According to [9, Theorem 2.14], the function

$$\frac{|\{q\in M:d(o,q)<r\}|}{(n+p)|B^{n+p}|\int_0^r h^{n+p-1}(t)\mathrm{d}t}$$

is not increasing on $[0,+\infty)$, and thus we may introduce the asymptotic volume ratio of M by

$$\theta:=\lim_{r\to+\infty}\frac{|\{q\in M:d(o,q)<r\}|}{(n+p)|B^{n+p}|\int_0^r h^{n+p-1}(t)\mathrm{d}t}, \tag{1.5}$$

which satisfies that $\theta\leq 1$, by the volume comparison theorem. Moreover, we define a function $P:[0,+\infty)\to(0,1]$ as

$$P(t):=(4\pi)^{-\frac{n+p}{2}}\int_{\mathbb{R}^{n+p}}\mathrm{e}^{-\frac{(|x|+t)^2}{4}}\mathrm{d}x=(4\pi)^{-\frac{n+p}{2}}(n+p)|B^{n+p}|\int_0^{\infty}\tau^{n+p-1}\mathrm{e}^{-\frac{(\tau+t)^2}{4}}\,\mathrm{d}\tau. \tag{1.6}$$

Obviously, $P(0)=1$, and $P(t)$ is a nonnegative decreasing function.

By combining the ABP-method in [2, 3, 10] with some comparison theorems, we obtain a logarithmic Sobolev inequality which holds for submanifolds without a boundary in manifolds with asymptotically nonnegative sectional curvature as follows:

Theorem 1.1 Let M be a complete noncompact $(n+p)$-dimensional manifold of asymptotically nonnegative sectional curvature with respect to a base point $o\in M$. Let Σ be a compact n-dimensional submanifold of M without a boundary, and let f be a positive smooth function on Σ. Then

$$\begin{aligned}&\int_\Sigma f\Big(\log f+n+4n^2b_1^2+\frac{n}{2}\log(4\pi)+(n+p-1)\log(\frac{1+b_0}{\mathrm{e}^{2r_0b_1+b_0}})+\log(\theta P(4nb_1))\Big)\,\mathrm{d\,vol}\\ &-\int_\Sigma\frac{|D^\Sigma f|^2}{f}\,\mathrm{d\,vol}-\int_\Sigma f|H|^2\,\mathrm{d\,vol}\leq\Big(\int_\Sigma f\,\mathrm{d\,vol}\Big)\log\Big(\int_\Sigma f\,\mathrm{d\,vol}\Big),\end{aligned}$$

where $r_0=\max\{d(o,x)|x\in\Sigma\}$, H is the mean curvature vector of Σ, θ is the asymptotic volume ratio of M given by (1.5), b_0 and b_1 are defined as in (1.1) and (1.2).

2 Preliminaries

In this section, we prove some lemmas for later use.

Lemma 2.1 Let h be defined by (1.4). Then we have that

$$\lim_{t\to\infty}\frac{h(tC)}{h(t)}=C,$$

where C is a positive constant.

Proof It is easy to show that

$$\Big(\int_0^t \mathrm{e}^{\int_0^s \tau\lambda(\tau)\mathrm{d}\tau}\mathrm{d}s\Big)''\geq \lambda(t)\int_0^t \mathrm{e}^{\int_0^s \tau\lambda(\tau)\mathrm{d}\tau}\mathrm{d}s.$$

By Lemma 2.1 of [9], we have that $h(t)\leq \int_0^t \mathrm{e}^{\int_0^s \tau\lambda(\tau)\mathrm{d}\tau}\mathrm{d}s\leq t\mathrm{e}^{b_0}$. This gives that

$$h'(t)=h'(0)+\int_0^t h''(s)\mathrm{d}s=1+\int_0^t \lambda h\mathrm{d}s\leq 1+b_0\mathrm{e}^{b_0}.$$

Since h' is nondecreasing and bounded from above, we can find that

$$\lim_{t\to\infty}\frac{h'(tC)}{h'(t)}=1.$$

Thus,

$$\lim_{t\to\infty}\frac{h(tC)}{h(t)}=\lim_{t\to+\infty}\frac{Ch'(tC)}{h'(t)}=C.$$

□

Lemma 2.2 Let P be defined by (1.6). Then we have that

$$P(t)\theta=\lim_{r\to\infty}\Big((4\pi)^{-\frac{n+p}{2}}\frac{1}{rh^{n+p-1}(r)}\int_M \mathrm{e}^{-\frac{(\frac{d(x,o)}{r}+t)^2}{4}}\,\mathrm{d}\,\mathrm{vol}(x)\Big),$$

where θ is the asymptotic volume ratio of M and o is the base point.

Proof By the definition of the asymptotic volume ratio in (1.5), we obtain that

$$\theta=\lim_{r\to\infty}\frac{|\{q\in M:d(o,q)<r\}|}{(n+p)|B^{n+p}|\int_0^r h^{n+p-1}(t)\mathrm{d}t}=\lim_{r\to\infty}\frac{\omega(r)}{(n+p)|B^{n+p}|h^{n+p-1}(r)},$$

where $\omega(r)$ is the area of the sphere of radius r with respect to the base point o. Using Lemma 2.1, it follows that

$$\begin{aligned}
&\lim_{r\to\infty}\Big((4\pi)^{-\frac{n+p}{2}}\frac{1}{rh^{n+p-1}(r)}\int_M \mathrm{e}^{-\frac{(\frac{d(x,o)}{r}+t)^2}{4}}\,\mathrm{d}\,\mathrm{vol}(x)\Big)\\
&=\lim_{r\to\infty}(4\pi)^{-\frac{n+p}{2}}\int_0^{+\infty}\frac{\omega(r\tau)}{h^{n+p-1}(r)}\mathrm{e}^{-\frac{(\tau+t)^2}{4}}\,\mathrm{d}\tau\\
&=\lim_{r\to\infty}(4\pi)^{-\frac{n+p}{2}}\int_0^{+\infty}\frac{\omega(r\tau)}{h^{n+p-1}(r\tau)}\frac{h^{n+p-1}(r\tau)}{h^{n+p-1}(r)}\mathrm{e}^{-\frac{(\tau+t)^2}{4}}\,\mathrm{d}\tau\\
&=(4\pi)^{-\frac{n+p}{2}}(n+p)|B^{n+p}|\theta\int_0^{+\infty}\tau^{n+p-1}\mathrm{e}^{-\frac{(\tau+t)^2}{4}}\,\mathrm{d}\tau=\theta P(t).
\end{aligned}$$

□

Lemma 2.3 Letting P be defined as in (1.6), we have that

$$P(t)\theta=\lim_{r\to\infty}\Big((4\pi)^{-\frac{n+p}{2}}\frac{1}{rh^{n+p-1}(r)}\int_M \mathrm{e}^{-\frac{(\frac{d_{\max}(x,\Sigma)}{r}+t)^2}{4}}\,\mathrm{d}\,\mathrm{vol}(x)\Big),$$

where θ is the asymptotic volume ratio of M and $d_{\max}(x,\Sigma)=\max\{d(x,y)|y\in\Sigma\}$.

Proof Noting that $r_0=\max\{d(y,o)|y\in\Sigma\}$, using the triangle inequality, we get that

$$|d(x,o)-r_0|\le d_{\max}(x,\Sigma)\le d(x,o)+r_0.$$

Following the proof of Lemma 2.2, it is easy to show that

$$\begin{aligned}
&\lim_{r\to\infty}\Big((4\pi)^{-\frac{n+p}{2}}\frac{1}{rh^{n+p-1}(r)}\int_M \mathrm{e}^{-\frac{(\frac{d_{\max}(x,\Sigma)}{r}+t)^2}{4}}\,\mathrm{d}\,\mathrm{vol}(x)\Big)\\
\ge&\lim_{r\to\infty}\Big((4\pi)^{-\frac{n+p}{2}}\frac{1}{rh^{n+p-1}(r)}\int_M \mathrm{e}^{-\frac{(\frac{d(x,o)+r_0}{r}+t)^2}{4}}\,\mathrm{d}\,\mathrm{vol}(x)\Big)\\
=&(4\pi)^{-\frac{n+p}{2}}(n+p)|B^{n+p}|\theta\int_0^{+\infty}\tau^{n+p-1}\mathrm{e}^{-\frac{(\tau+t)^2}{4}}\,\mathrm{d}\tau=\theta P(t).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
&\lim_{r\to\infty}\Big((4\pi)^{-\frac{n+p}{2}}\frac{1}{rh^{n+p-1}(r)}\int_M \mathrm{e}^{-\frac{(\frac{d_{\max}(x,\Sigma)}{r}+t)^2}{4}}\,\mathrm{d}\,\mathrm{vol}(x)\Big)\\
\le&\lim_{r\to\infty}\Big((4\pi)^{-\frac{n+p}{2}}\frac{1}{rh^{n+p-1}(r)}\int_M \mathrm{e}^{-\frac{(\frac{|d(x,o)-r_0|}{r}+t)^2}{4}}\,\mathrm{d}\,\mathrm{vol}(x)\Big)\\
=&(4\pi)^{-\frac{n+p}{2}}(n+p)|B^{n+p}|\theta\int_0^{+\infty}\tau^{n+p-1}\mathrm{e}^{-\frac{(\tau+t)^2}{4}}\,\mathrm{d}\tau=\theta P(t).
\end{aligned}$$

This completes the proof. □

3 Proof of Theorem 1.1

In this section, we assume that the ambient space M is a complete noncompact $(n+p)$-dimensional Riemannian manifold of asymptotically nonnegative sectional curvature with respect to a base point $o\in M$. Let $\Sigma\subset M$ be a compact submanifold of dimension n without a boundary, and let f be a positive smooth function on Σ. Let $\bar{D}$ denote the Levi-Civita connection of M and let D^Σ denote the induced connection on Σ. The second fundamental form B of Σ is given by

$$\langle B(X,Y),V\rangle=\langle\bar{D}_XY,V\rangle,$$

where X,Y are the tangent vector fields on Σ, and V is a normal vector field along Σ. In particular, the mean curvature vector H is defined as the trace of the second fundamental form B.

We only need to show the proof of Theorem 1.1 in the case where Σ is connected. By scaling, we may assume that

$$\int_\Sigma f\log f\,\mathrm{d}\,\mathrm{vol}-\int_\Sigma\frac{|D^\Sigma f|^2}{f}\,\mathrm{d}\,\mathrm{vol}-\int_\Sigma f|H|^2\,\mathrm{d}\,\mathrm{vol}=0. \tag{3.1}$$

By the connectedness of Σ and (3.1), following the statement in [2], there exists a smooth function $u:\Sigma\to\mathbb{R}$ such that

$$\mathrm{div}_\Sigma(fD^\Sigma u)=f\log f-\frac{|D^\Sigma f|^2}{f}-f|H|^2.$$

For each $r>0$, we denote by A_r the set of all points $(\bar{x},\bar{y})\in T^\perp\Sigma$ satisfing that

$$ru(x)+\frac{1}{2}d(x,\exp_{\bar{x}}(rD^\Sigma u(\bar{x})+r\bar{y}))^2\ge ru(\bar{x})+\frac{1}{2}r^2(|D^\Sigma u(\bar{x})|^2+|\bar{y}|^2)$$

for all $x \in \Sigma$. Define the transport map $\Phi_r : T^{\perp}\Sigma \to M$ by

$$\Phi_r(x, y) = \exp_x(rD^{\Sigma}u(x) + ry)$$

for all $x \in \Sigma, y \in T_x^{\perp}\Sigma$.

The proof of the next lemma is identical to the proof of Lemma 3.2 in [10], so we omit it here.

Lemma 3.1 For each $r > 0$, we have that $\Phi_r(A_r)$=M.

Similarly to the proof of (3.20) in [5], it is easy to get an estimate for the Jacobian determinant of Φ_r, we omit the proof of the next Lemma.

Lemma 3.2 For each $r > 0$, the Jacobian determinant of Φ_r satisfies that

$$|\det \bar{D}\Phi_r(\bar{x}, \bar{y})| \le (2b_1\sqrt{|D^{\Sigma}u(\bar{x})|^2 + \bar{y}^2} + \frac{1}{r} + \frac{1}{n}(\Delta_{\Sigma}u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle))^n r^{n+p} \mathrm{e}^{(n+p-1)(2r_0b_1+b_0)}$$

for all $(\bar{x}, \bar{y}) \in A_r$. Moreover, we have that

$$(2b_1\sqrt{|D^{\Sigma}u(\bar{x})|^2 + \bar{y}^2} + \frac{1}{r} + \frac{1}{n}(\Delta_{\Sigma}u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle)) \ge 0.$$

Lemma 3.3 For each $r > 0$, the Jacobian determinant of Φ_r satisfies that

$$0 \le \mathrm{e}^{-\frac{(\frac{d(\bar{x},\Phi_r(\bar{x},\bar{y}))}{r}+4nb_1)^2}{4}}|\det \bar{D}\Phi_r(\bar{x}, \bar{y})| \le \mathrm{e}^{(n+p-1)(2r_0b_1+b_0)} r^{n+p} f(\bar{x})\mathrm{e}^{\frac{n}{r}-n-4n^2b_1^2}\mathrm{e}^{-\frac{|2H(\bar{x})+\bar{y}|^2}{4}}$$

for all $(\bar{x}, \bar{y}) \in A_r$.

Proof Similarly to [3, 10], using the identity $\operatorname{div}_{\Sigma}(fD^{\Sigma}u) = f\log f - \frac{|D^{\Sigma}f|^2}{f} - f|H|^2$, we obtain that

$$\Delta_{\Sigma}u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle \le \log f(\bar{x}) + \frac{|D^{\Sigma}u(\bar{x})|^2 + |\bar{y}|^2}{4} - \frac{|2H(\bar{x}) + \bar{y}|^2}{4}.$$

Due to the definition of A_r, we conclude that

$$d(\bar{x}, \Phi_r(\bar{x}, \bar{y})) = r\sqrt{|D^{\Sigma}u(\bar{x})|^2 + \bar{y}^2}.$$

By Lemma 3.2 and the elementary inequality $\lambda \le \mathrm{e}^{\lambda-1}$, it follows that

$$\begin{aligned}
&|\det \bar{D}\Phi_r(\bar{x}, \bar{y})| \\
&\le (2b_1\sqrt{|D^{\Sigma}u(\bar{x})|^2 + \bar{y}^2} + \frac{1}{r} + \frac{1}{n}(\Delta_{\Sigma}u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle))^n r^{n+p}\mathrm{e}^{(n+p-1)(2r_0b_1+b_0)} \\
&\le \mathrm{e}^{(n+p-1)(2r_0b_1+b_0)} r^{n+p} \mathrm{e}^{\frac{n}{r}+2nb_1\sqrt{|D^{\Sigma}u(\bar{x})|^2+\bar{y}^2}+\log f(\bar{x})+\frac{|D^{\Sigma}u(\bar{x})|^2+|\bar{y}|^2}{4}-\frac{|2H(\bar{x})+\bar{y}|^2}{4}-n} \\
&= \mathrm{e}^{(n+p-1)(2r_0b_1+b_0)} r^{n+p} f(\bar{x})\mathrm{e}^{\frac{n}{r}-n-4n^2b_1^2}\mathrm{e}^{-\frac{|2H(\bar{x})+\bar{y}|^2}{4}}\mathrm{e}^{\frac{(\sqrt{|D^{\Sigma}u(\bar{x})|^2+\bar{y}^2}+4nb_1)^2}{4}} \\
&= \mathrm{e}^{(n+p-1)(2r_0b_1+b_0)} r^{n+p} f(\bar{x})\mathrm{e}^{\frac{n}{r}-n-4n^2b_1^2}\mathrm{e}^{-\frac{|2H(\bar{x})+\bar{y}|^2}{4}}\mathrm{e}^{\frac{(\frac{d(\bar{x},\Phi(\bar{x},\bar{y}))}{r}+4nb_1)^2}{4}}.
\end{aligned}$$

Dividing the above inequality by $\mathrm{e}^{-\frac{(\frac{d(\bar{x},\Phi_r(\bar{x},\bar{y}))}{r}+4nb_1)^2}{4}}$ completes the proof. □

Proof of Theorem 1.1 Let $x = \Phi_r(\bar{x}, \bar{y})$ for $(\bar{x}, \bar{y}) \in T^{\perp}\Sigma$. Using Lemma 3.3 and the formula for the change of variables in multiple integrals, we find that

$$\begin{aligned}
&\int_M \mathrm{e}^{-\frac{(\frac{d_{\max}(x,\Sigma)}{r}+4nb_1)^2}{4}}\mathrm{d}\,\mathrm{vol}(x) \\
&\le \int_{\Sigma}\Big(\int_{T_{\bar{x}}^{\perp}\Sigma}\mathrm{e}^{-\frac{(\frac{d(\bar{x},\Phi_r(\bar{x},\bar{y}))}{r}+4nb_1)^2}{4}}|\det \bar{D}\Phi_r(\bar{x}, \bar{y})|1_{A_r}(\bar{x}, \bar{y})\mathrm{d}\bar{y}\Big)\mathrm{d}\,\mathrm{vol}(\bar{x})
\end{aligned}$$

$$\leq \mathrm{e}^{(n+p-1)(2r_0b_1+b_0)}r^{n+p}\mathrm{e}^{\frac{n}{r}-n-4n^2b_1^2}\int_\Sigma f(\bar{x})\Big(\int_{T_{\bar{x}}^\perp\Sigma}\mathrm{e}^{-\frac{|2H(\bar{x})+\bar{y}|^2}{4}}\Big)\mathrm{d\,vol}(\bar{x})$$
$$= \mathrm{e}^{(n+p-1)(2r_0b_1+b_0)}r^{n+p}\mathrm{e}^{\frac{n}{r}-n-4n^2b_1^2}(4\pi)^{\frac{p}{2}}\int_\Sigma f(\bar{x})\mathrm{d\,vol}(\bar{x}). \tag{3.2}$$

Moreover, by (1.4), we obtain $h(t)\geq t$ and

$$\lim_{t\to\infty}h'(t)=1+\int_0^\infty h(s)\lambda(s)\mathrm{d}s\geq 1+\int_0^\infty s\lambda(s)\mathrm{d}s=1+b_0.$$

Dividing (3.2) by $rh^{n+p-1}(r)$ and sending $r\to+\infty$, and using Lemma 2.3, we have that

$$(4\pi)^{\frac{n+p}{2}}\theta P(4nb_1)\leq\Big(\frac{\mathrm{e}^{(2r_0b_1+b_0)}}{1+b_0}\Big)^{n+p-1}\mathrm{e}^{-n-4n^2b_1^2}(4\pi)^{\frac{p}{2}}\int_\Sigma f(\bar{x})\ \mathrm{d\,vol}(\bar{x}).$$

Thus,

$$n+4n^2b_1^2+\frac{n}{2}\log(4\pi)+(n+p-1)\log(\frac{1+b_0}{\mathrm{e}^{2r_0b_1+b_0}})+\log(\theta P(4nb_1))\leq\log\Big(\int_\Sigma f\ \mathrm{d\,vol}\Big).$$

Combining the above inequality with (3.1), we obtain that

$$\int_\Sigma f\Big(\log f+n+4n^2b_1^2+\frac{n}{2}\log(4\pi)+(n+p-1)\log(\frac{1+b_0}{\mathrm{e}^{2r_0b_1+b_0}})+\log(\theta P(4nb_1))\Big)\mathrm{d\,vol}$$
$$-\int_\Sigma\frac{|D^\Sigma f|^2}{f}\ \mathrm{d\,vol}-\int_\Sigma f|H|^2\ \mathrm{d\,vol}\leq\Big(\int_\Sigma f\ \mathrm{d\,vol}\Big)\log\Big(\int_\Sigma f\ \mathrm{d\,vol}\Big).$$

This completes the proof of Theorem 1.1. □

Conflict of Interest The authors declare no conflict of interest.

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