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THE REGULARITY AND UNIQUENESS OF A GLOBAL SOLUTION TO THE ISENTROPIC NAVIER-STOKES EQUATION WITH ROUGH INITIAL DATA*

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Abstract A global weak solution to the isentropic Navier-Stokes equation with initial data around a constant state in the $L^1\cap$ BV class was constructed in [1]. In the current paper, we will continue to study the uniqueness and regularity of the constructed solution. The key ingredients are the Hölder continuity estimates of the heat kernel in both spatial and time variables. With these finer estimates, we obtain higher order regularity of the constructed solution to Navier-Stokes equation, so that all of the derivatives in the equation of conservative form are in the strong sense. Moreover, this regularity also allows us to identify a function space such that the stability of the solutions can be established there, which eventually implies the uniqueness.

Key words compressible Navier-Stokes equation; BV initial data; regularity; uniqueness

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1 Introduction

The compressible Navier-Stokes (NS for short) equation is one of the most classical and important fluid models, and it has been widely studied in various communities, including mathematics, physics, chemistry and engineering, etc.. In this paper, we will focus on the isentropic case in the Lagrangian coordinates, which reads as

$$\begin{cases} v_t - u_x = 0, \\ u_t + \Big(p - \dfrac{\mu u_x}{v}\Big)_x = 0. \end{cases} \tag{1.1}$$

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Here v is the specific volume, u is the velocity, μ is the viscosity of u, and p denotes the pressure. Throughout the paper, we assume the pressure $p(\cdot)$ to be analytic around a constant state $v = \bar{v}$. In particular, the usual polytropic law $p(v) = v^{-\gamma}$ satisfies this assumption, and we will assume that $\bar{v} = 1$, for simplicity. It is also worth mentioning that the Lagrangian formulation can be formally derived from the original compressible NS equation in Euler coordinates; see [2] for details.

The study of the local well-posedness of the compressible NS equation was initiated by Nash [3, 4], and further developed by Itaya in [5], where the initial data was assumed to be Hölder continuous. In these works, the equation for the velocity field was written as a nonlinear parabolic equation with variable diffusion coefficients. Then the solution was sought by an iteration scheme, where the fundamental solution for the variable coefficient parabolic equation played a key role. Based on these pioneer results, *a priori* energy-type estimates were derived to construct the global solutions, see [6, 7] for both Eulerian and Lagrangian coordinates. Later on, for initial data as a regular small perturbation of a constant state, the global existence for the 3-D NS equations using an energy method was obtained by Matsumura-Nishida [8], and then extended to more general hyperbolic-parabolic systems by Kawashima and Shizuta [9, 10]. After these seminal works, the energy methods were extensively developed and applied to various fluid and related models, and the well-posedenss of the compressilbe NS equation with large initial data or vacuum was also studied [11–15].

On the other hand, when the initial data is not regular, the classical local theory is not generally applicable. This is due to the quasi-linear and hyperbolic-parabolic structure of the NS equations. In fact, the initial singularity in a density field will propagate along time, for which one has to deal with the parabolic equation for a velocity filed with a rough diffusion coefficient. Thus, the criteria for the Hölder continuous case in Nash's works cannot be directly applied. It is then natural to focus on the study the weak solution. The approach of Hoff [16, 17] was based on piecewise energy estimates, and a total variation estimate was obtained. The approaches of Lions [18] and Feireisl [19] were applied to more general data. However, in all of these cases, there is neither the well-posedness nor the quantitative structure for a weak solution analogous to that of the classical solution. We refer readers to [20–23] for more theories on the NS equations.

Recently, Liu and Yu [1] considered (1.1) with initial data of small total variation, and established the local-in-time well-posedness and time asymptotic behavior of weak solutions. The key ingredient here was the construction of a fundamental solution for the heat equation with BV conductivity coefficient which completely respects the effects of the singularities of the density field. Therefore, the idea in [3, 5] can be extended to the BV framework, and a weak solution was successfully constructed with a continuous dependence on the initial data. The local well-posedenss result in [1] can be summarized as follows:

Proposition 1.1 ([1]) For polytropic gases with $p(v) = Av^{-\gamma}$ and $1 \leq \gamma < e$, suppose that the initial data (v_0, u_0) satisfies

$$\|v_0 - 1\|_{L^1} + \|v_0\|_{\mathrm{BV}} + \|u_0\|_{L^1} + \|u_0\|_{\mathrm{BV}} \leq \delta \tag{1.2}$$

for $\delta \ll 1$. Then the Navier-Stokes equation (1.1) admit a global weak solution such that the following estimates hold for some constant C:

$$\begin{aligned}&\|(v-1)(\cdot,t)\|_{L^1}+\|\sqrt{t+1}(v-1)(\cdot,t)\|_{L^\infty}+\|(v-1)(\cdot,t)\|_{\mathrm{BV}}\\&+\|u(\cdot,t)\|_{L^1}+\|\sqrt{t+1}u(\cdot,t)\|_{L^\infty}+\|\sqrt{t+1}u_x(\cdot,t)\|_{L^\infty}+\|u(\cdot,t)\|_{\mathrm{BV}}\le C\delta,\quad t>0.\end{aligned}$$

Moreover, suppose that two initial data, (v_0^a,u_0^a) and (v_0^b,u_0^b), satisfy (1.2). Let (v^a,u^a) and (v^b,u^b) be the two weak solutions of the Navier-Stokes equation (1.1) constructed as above. Then, for any positive constant T, there exists a positive constant $C(T)$ such that

$$\sup_{0<t<T}\Big(\|v^a(\cdot,t)-v^b(\cdot,t)\|_{L^1}+\|u^a(\cdot,t)-u^b(\cdot,t)\|_{L^1}\Big)\le C(T)\Big(\|v_0^a-v_0^b\|_{L^1}+\|u_0^a-u_0^b\|_{L^1}\Big).\quad(1.3)$$

As there is a lack of the time derivative estimate or equivalent second order estimates on u, the derivatives in (1.1) are still in the weak sense for the weak solution in Proposition 1.1. Moreover, the uniqueness of the solution with initial condition (1.2) has not been resolved. One may think that (1.3) can imply uniqueness, since it looks like a stability estimate. However, according to the statement in Proposition 1.1, the estimate (1.3) holds only for solutions constructed exactly as in Proposition 1.1, which means that (1.3) depends on the iteration scheme in the proof of the proposition. In other words, if (v^a,u^a) or (v^b,u^b) is constructed in a manner different from the iteration scheme in Proposition 1.1, then it is unknown wether or not (1.3) holds.

As a continuation of [1], in the present paper, we will refine the results in Proposition 1.1 in three respects. As indicated by the above discussions, the motivation for this arises from the following three questions:

- Can we relax the initial requirements of (1.2) and obtain the same results in Proposition 1.1?
- Based on the same initial conditions of (1.2), can we gain a higher regularity than Proposition 1.1 when $t>0$, so that the derivatives in (1.1) are in the strong sense?
- Based on the same initial conditions of (1.2), can we prove the uniqueness of the constructed solution in Proposition 1.1?

In later sections, we will provide affirmative answers to above three questions. In the introduction, we will first give a brief description of our strategies and main results.

First, starting from the small perturbation $v_0-1\in L^1\cap\mathrm{BV}$ and $u_0\in L^1\cap L^\infty$, we successfully construct the weak solution to (1.1). Note that the evolution equation for v is hyperbolic, and the one for u is of a parabolic type. Then, the singularities of v will remain in the evolution, while u will automatically gain regularity when time $t>0$. Therefore, the initial conditions for u can be relaxed compared to v_0. Due to the relaxation, the L^1 estimates of u becomes more singular, and thus we need more careful treatments on the iteration scheme estimates. In particular, we do not require the pressure to be of any special form, such as the Gamma law. Instead, we only require the pressure term to be analytic around a background constant state. Please see Proposition 4.8 in Section 4 for more details.

Second, the detailed regularity of the constructed weak solution is studied for both initial conditions (1.2) and the relaxed one, and it is shown that all of the derivatives in the conservative form (1.1) are in the strong sense. The main difficulty is to gain the estimates of the time derivative of u, which requires the use of higher order estimates on the heat kernel with BV diffusion coefficients. Therefore, we refine the estimates of the heat kernel based on the Laplace

Wave Train criteria in [1], and obtain the pointwise estimates of H_{ty}. Moreover, we prove the spatial Hölder properties of the heat kernels (see the Hölder properties in Lemma 3.1 and Lemma 3.2 for details), which eventually implies that u_t is of Hölder class C^α in x for any $\alpha \in (0,1)$. Then, all of the derivatives in the conservative form (1.1) are in the classical sense. We emphasize that, although u_x and v are not differentiable individually, the flux formed by their combination in (1.1) is much more regular.

Third, if $u_0 \in \mathrm{BV}$, based on the regularity estimates obtained, we are able to give an affirmative answer to the uniqueness of the weak solution. The key point and difficulty is to show that the BV norm of v is continuous with respect to t around $t=0$, so that a proper function space with higher order regularity can be identified; see (1.6). So far, this regularity can be obtained by combining the BV norm of u_0 and the Hölder estimates of heat kernels. Then, all of the solutions with this kind of regularity induce a corresponding heat kernel, which can be used to represent the solution in an integral representation. Finally, thanks to the comparison estimates of the heat kernel, we are able to show the stability of the solutions in this function space, which immediately implies the uniqueness of the solution.

Our main results are stated in the following two theorems:

Theorem 1.2 (Existence and Regularity) Suppose that the pressure $p(v)$ is an analytic function of v around constant 1, and that the initial data (v_0, u_0) satisfies the following condition for a sufficiently small positive number δ:

$$\|v_0-1\|_{L^1}+\|v_0\|_{\mathrm{BV}}+\|u_0\|_{L^1}+\|u_0\|_{L^\infty}\le\delta. \tag{1.4}$$

Then there exists a positive constant $t_\sharp$ such that the Navier-Stokes equation (1.1) admits a weak solution (v,u) for $t<t_\sharp$ in the following function space:

$$\begin{cases} v(x,t)-1\in C\left([0,t_\sharp);L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\right)\cap L^\infty\left(0,t_\sharp;BV(\mathbb{R})\right),\\ u(x,t)\in L^\infty\left(0,t_\sharp;L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\right),\quad \sqrt{t}u_x(x,t)\in L^\infty\left(0,t_\sharp;L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\right),\\ tu_t(x,t)\in L^\infty\left(0,t_\sharp;L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\cap C^{0,\alpha}(\mathbb{R})\right). \end{cases} \tag{1.5}$$

Moreover, if u_0 is further assumed to be of small total variation $\|u_0\|_{\mathrm{BV}}\le\delta$, then the constructed solution is more regular, and in a subspace of (1.5), i.e.,

$$\begin{cases} v(x,t)-1\in C\left([0,t_\sharp);L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\cap \mathrm{BV}(\mathbb{R})\right),\\ u(x,t)\in L^\infty\left(0,t_\sharp;W^{1,1}(\mathbb{R})\cap L^\infty(\mathbb{R})\right),\quad \sqrt{t}u_x(x,t)\in L^\infty\left(0,t_\sharp;L^\infty(\mathbb{R})\right),\\ \sqrt{t}u_t(x,t)\in L^\infty\left(0,t_\sharp;L^1(\mathbb{R})\cap C^{0,\alpha}(\mathbb{R})\right),\quad tu_t(x,t)\in L^\infty\left(0,t_\sharp;L^\infty(\mathbb{R})\cap C^{0,\alpha}(\mathbb{R})\right). \end{cases} \tag{1.6}$$

Theorem 1.3 (Stability and Uniqueness) Suppose that there are two weak solutions (v^a,u^a) and (v^b,u^b) to the Navier-Stokes equation (1.1) in the function space (1.6), subject to initial data (v_0^a,u_0^a) and (v_0^b,u_0^b), with the condition that

$$\|v_0\|_{\mathrm{BV}}+\|v_0-1\|_{L^1}+\|u_0\|_{\mathrm{BV}}+\|u_0\|_{L^1}\le\delta_*,$$

where δ_* is sufficiently small. Then, there exist positive constants t_* and $C_\flat$ such that the following stability holds for $0<t<t_*$:

$$\begin{aligned} &\mathscr{F}\left[v^a-v^b,u^a-u^b\right]\\ \le\ & C_\flat\Big(\|u_0^a-u_0^b\|_{L_x^\infty}+\|u_0^a-u_0^b\|_{L_x^1}+\|v_0^a-v_0^b\|_{L_x^1}+\|v_0^a-v_0^b\|_{L_x^\infty}+\|v_0^a-v_0^b\|_{\mathrm{BV}}\Big). \end{aligned}$$

Here $\mathscr{F}$ is the functional defined in (4.39). Moreover, this immediately implies the uniqueness of the weak solution in the function space (1.6) for $t \in [0, t_*)$.

Moreover, the constant C in Proposition 1.1 is uniform bounded for arbitrarily small δ. Therefore, according to Proposition 1.1, we can choose sufficiently small initial data such that the constructed solution exists globally in time with small L^1 and BV norm. Therefore, our local-in-time results in Theorems 1.2 and 1.3 can be applied to arbitrarily large time, which immediately implies the regularity and uniqueness of the global solution. Then, we have the following corollary:

Corollary 1.4 Suppose that the pressure is $p(v) = Av^{-\gamma}$, that $1 \le \gamma < e$, and that the initial data (v_0, u_0) satisfies the following condition for a sufficiently small positive number δ:

$$\|v_0 - 1\|_{L^1} + \|v_0\|_{\mathrm{BV}} + \|u_0\|_{L^1} + \|u_0\|_{\mathrm{BV}} \le \delta.$$

Then, the NS equation (1.1) admits a unique global solution in the function space

$$\begin{cases} v(x,t) - 1 \in C\left([0,+\infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})\right), \\ u(x,t) \in L^\infty\left(0,+\infty; W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})\right), \quad \sqrt{t} u_x(x,t) \in L^\infty\left(0,+\infty; L^\infty(\mathbb{R})\right), \\ \sqrt{t} u_t(x,t) \in L^\infty\left(0,+\infty; L^1(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})\right), \quad t u_t(x,t) \in L^\infty\left(0,+\infty; L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})\right). \end{cases}$$

Remark 1.5 1. For a weak solution (v, u) constructed in [1] or function space (1.5), if one makes an *a priori* assumption that $\|v(\cdot,t)\|_{\mathrm{BV}}$ is small, then the uniqueness still follows. However, in order to rigorously verify this a priori assumption, we need to require the smallness of $\|u_0\|_{\mathrm{BV}}$.

2. In the above corollary, as we directly apply the global existence results in [1], we have to choose the same setting $p(v) = Av^{-\gamma}$ as in Proposition 1.1. However, according to the proof in [1], the large time behavior is mainly governed by the linearized Green's function of (1.1) around a constant state, and the second order Taylor expansion of $p(v)$ is enough to gain the global existence. Therefore, we can also assume that for the pressure $p(v)$ to be analytic around 1 and for $p'(z) < 0$, we can follow similar arguments as to these in [1] to yield the global unique solution.

The rest of this paper is organized as follows: in Section 2, we will introduce some preliminary concepts and the heat kernel $H(x,t;y,s;\rho)$ with a BV coefficient. The pointwise estimates and comparison principles will be listed in the Appendix, for convenience. In Section 3, we further study the Hölder estimates of the heat kernel in both the time and spatial variables; this plays an important role in the regularity analysis in later sections. Next, in Section 4, we will construct the local solution for relaxed initial assumptions (1.4), and obtain the Hölder continuity of the solution. In Section 5, we will represent the solution by a nonlinear integral formula, and further study the regularity of the solution. The Hölder continuity eventually yields the uniqueness of the solution. The detailed proofs of Theorems 1.2 and 1.3 will also be provided in this section. Finally, Section 6 provides a summary.

2 Preliminary

In this section, we will provide some preliminary concepts and results that will be used in later sections. We first give the definitions of the weak solutions, then we introduce the fundamental solution for the heat equation with the conductivity of bounded variation, which is the kernel for constructing the integral representation of the solution to (1.1). In what follows we denote that

$$\|f(\cdot,t)\|_{\mathrm{BV}} := \int_{\mathbb{R}\setminus\mathscr{D}} |\partial_x f(\cdot,t)|\mathrm{d}x + \sum_{z\in\mathscr{D}} \left| f(\cdot,t)\Big|_{z=x^-}^{z=x^+} \right|,$$

$$|\!|\!|f|\!|\!|_\infty \equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{L^\infty_x}, \quad |\!|\!|f|\!|\!|_1 \equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{L^1_x}, \quad |\!|\!|f|\!|\!|_{\mathrm{BV}} \equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{\mathrm{BV}}.$$

2.1 Concepts and Definitions

As we study the solution with rough initial data, it is very natural to consider the solution in a weak sense. We propose the following definition of the weak solution in the distribution sense:

Definition 2.1 (v,u) is called a weak solution to the equation (1.1) if the following assertions hold:

1. (v,u) satisfy the equation (1.1) in the distribution sense. More precisely, the following equations hold for any test function $\varphi(x,t)\in C_0^1(\mathbb{R}\times[0,+\infty))$:

$$\begin{cases} \displaystyle\int_0^{+\infty}\int_{\mathbb{R}} [\varphi_x u - \varphi_t v]\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \varphi(x,0)v(x,0)\mathrm{d}x, \\ \displaystyle\int_0^{+\infty}\int_{\mathbb{R}} \left[\varphi_x\left(\frac{\mu u_x}{v} - p\right) - \varphi_t u\right]\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \varphi(x,0)u(x,0)\mathrm{d}x. \end{cases}$$

2. The flux of u given by $\frac{\mu u_x}{v} - p$ is continuous with respect to x.

The weak solution of the linearized equation can be similarly defined. Next, we introduce the equivalent definition of a BV function in [1].

Definition 2.2 A function $u(x)$ is BV if the following assertions hold:

- The function $u(x)$ can be represented as the sum of an absolutely continuous function and a step function as follows:

$$u(x) = u_c(x) + u_d(x), \quad u_c \text{ is absolutely continuous}, \quad u_d(x) = \sum_{\alpha\in\mathscr{D},\,\alpha<x} [u](\alpha)H(x-\alpha).$$

Here $\alpha\in\mathscr{D}$, and $\mathscr{D}$ is the discontinuity set of $u(x)$, and $H(\cdot)$ is the Heaviside function.

- The total variation of $u(x)$ is finite, i.e.,

$$\int_{\mathbb{R}\setminus\mathscr{D}} |u_x|\mathrm{d}x + \sum_{\alpha\in\mathscr{D}} \left| u\Big|_{\alpha^-}^{\alpha^+} \right| < +\infty.$$

Note that when $u\in L^1_x$, the total variation is actually a norm, and we will denote it by $\|u\|_{\mathrm{BV}}$. Hereafter, the following notation is frequently used when integrating the derivative of a BV function:

$$\int_{\mathbb{R}\setminus\mathscr{D}} |\partial_x u(x)|\mathrm{d}x \equiv \int_{\mathbb{R}} |\partial_x u_c(x)|\mathrm{d}x. \tag{2.1}$$

Here u_c is the absolutely continuous part of u.

2.2 Heat Kernel with BV Conductivity

We will briefly review the fundamental solution for a heat equation with BV conductivity, which was first introduced in [1]. Consider the following equation for a heat kernel H with the coefficient $\rho(x,t)$ being a BV function with respect to x:

$$\begin{cases} (\partial_t - \partial_x \rho(x,t)\partial_x)\, H(x,t;y,t_0;\rho) = 0, \quad t > t_0, \\ H(x,t_0;y,t_0;\rho) = \delta(x-y). \end{cases} \tag{2.2}$$

Here the BV coefficient $\rho(x,t)$ satisfies the following properties for some positive constants C and δ^*:

$$\begin{cases} \|\rho(\cdot) - C\|_{L^1} \le \delta^*, \quad \|\rho(\cdot,t)\|_{\mathrm{BV}} \le \delta^*, \\ \|\rho_t(\cdot,t)\|_\infty \le \delta^* \max\left(\frac{1}{\sqrt{t}}, 1\right), \quad 0 < \delta^* \ll 1, \\ \mathscr{D} := \{z \mid \rho(z,t) \text{ is not continuous at } z\}. \end{cases} \tag{2.3}$$

We emphasize that there is an implicit requirement in (2.3) that the discontinuity set $\mathscr{D}$ of $\rho(x,t)$ is independent of t. Then, according to [1], the equation (2.2) has a weak solution in the distribution sense, i.e., there exists a solution $H(x,t;y,t_0;\rho)$ such that the following equality holds for all test functions $\phi(x,t) \in C_0^1(\mathbb{R} \times [0,+\infty))$:

$$\begin{aligned} &\int_{t_0}^{+\infty} \int_{\mathbb{R}} (-\phi_t H(x,t;y,t_0;\rho) + \phi_x \rho \partial_x H(x,t;y,t_0;\rho))\, \mathrm{d}x\mathrm{d}t \\ =& -\int_{\mathbb{R}} \phi(x,t_0) H(x,t_0;y,t_0;\rho) = -\phi(y,t_0). \end{aligned}$$

In [1], the pointwise estimates of the heat kernel $H(x,t;y,t_0;\rho)$ were established (we list these estimates in the Appendix). These estimates in [1] are necessary but not sufficient to gain more regularity on the solution to (1.1). Therefore, in the next section, we will introduce new Hölder estimates of the heat kernel; this plays an important role in this paper. We finish this section with a simple remark.

Remark 2.3 According to [1], we make the following remarks:

1. In order to balance the equation (2.2), we actually have that $\rho(x,t)H_x(x,t;y,s;\rho)$ is continuous with respect to x, and also that $\rho(y,s)H_y(x,t;y,s;\rho)$ is continuous with respect to y.

2. The weak solution of the heat equation (2.2) can be defined similarly as to Definition 2.1. In fact, if the equation (2.2) has a source term in the form

$$u_t(x,t) = (\rho(x,t)u_x(x,t) + g(x,t))_x\,, \tag{2.4}$$

then the mild solution constructed by Duhamel's principle is also a weak solution to (2.4) in the distribution sense, provided that $g(x,t)$ is a BV function with respect to x. Furthermore, the flux term $\rho(x,t)u_x(x,t) + g(x,t)$ is continuous with respect to x if one of the following two conditions holds:

(a) $g(x,t)$ is Lipschitz continuous with respect to x, or equivalently,

$$\|g_x(\cdot,t)\|_\infty < +\infty;$$

(b) $g(x,t)$ is Hölder continuous with respect to t in the sense that

$$|g(x,t) - g(x,s)| \le \frac{(t-s)^\alpha}{s^\alpha}, \quad 0 < s < t, \quad 0 < \alpha < 1.$$

3 Hölder Estimates of Heat Kernel

The Hölder continuity in space and time of the heat kernel is very important in this paper. First, the Hölder continuity in time allows us to obtain the smallness of $v-1$ in the initial layer, which is essential for the proof of uniqueness. On the other hand, the Hölder continuity in space yields more regularity regarding the flux of u, and thus we can expect the regularity of u_t according to the balance in (1.1). The next two lemmas contain the Hölder estimates we need in later sections, and we will only prove Lemma 3.2 for simplicity, since the essential idea of the proofs are similar. The interested reader is also referred to [1] for more details about the techniques.

Lemma 3.1 (Hölder continuity in time) Suppose the conditions in (2.3) hold for ρ. Then the following estimates hold when $t_0 < s < t \ll 1$:

$$\begin{aligned}
&\|H_x(\cdot,t;y,t_0;\rho)-H_x(\cdot,s;y,t_0;\rho)\|_\infty \le C_*\frac{(t-s)\,|\log(t-s)|}{(s-t_0)(t-t_0)},\\
&\|H_x(\cdot,t;y,t_0;\rho)-H_x(\cdot,s;y,t_0;\rho)\|_1 \le C_*\frac{(t-s)\,|\log(t-s)|}{(t-t_0)\sqrt{s-t_0}},\\
&\|H_{xy}(\cdot,t;y,t_0;\rho)-H_{xy}(\cdot,s;y,t_0;\rho)\|_\infty \le C_*\frac{(t-s)\,|\log(t-s)|}{(s-t_0)^{3/2}(t-t_0)},\\
&\|H_{xy}(\cdot,t;y,t_0;\rho)-H_{xy}(\cdot,s;y,t_0;\rho)\|_1 \le C_*\frac{(t-s)\,|\log(t-s)|}{(s-t_0)(t-t_0)}.
\end{aligned}$$

Lemma 3.2 (Hölder continuity in space) Suppose that the conditions of ρ in (2.3) hold, that the discontinuity set $\mathscr{D}$ is the same for all $t\ge 0$, and furthermore, that for $0\le t_0<s<t$, ρ satisfies that

$$\begin{cases}
\displaystyle\int_{\mathbb{R}\backslash\mathscr{D}} |\rho_x(x,t)-\rho_x(x,s)|\,\mathrm{d}x \lesssim \delta\frac{(t-s)|\log(t-s)|}{\sqrt{t}},\\
\displaystyle\sum_{z\in\mathscr{D}}\left|\rho(\cdot,t)\Big|_{z^-}^{z^+}-\rho(\cdot,s)\Big|_{z^-}^{z^+}\right| \lesssim \frac{\delta(t-s)}{\sqrt{t}}.
\end{cases}\tag{3.1}$$

Then there exists a positive constant C_* such that $H_{ty}(x,t;y,t_0;\rho)$ satisfies the Hölder continuity in x-variable

$$\begin{aligned}
&|H_t(x+h,t;y,t_0;\rho)-H_t(x,t;y,t_0;\rho)|\\
\lesssim\;& \frac{\delta}{(t-t_0)\sqrt{t}}|h|\big|\log|h|\big|^2+(t-t_0)^{-\frac32}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}}\mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w,\\
&|H_{ty}(x+h,t;y,t_0;\rho)-H_{ty}(x,t;y,t_0;\rho)|\\
\lesssim\;& \frac{\delta}{(t-t_0)^{\frac32}\sqrt{t}}|h|\big|\log|h|\big|^2+(t-t_0)^{-2}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}}\mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w,
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}}|H_t(x+h,t;y,t_0;\rho)-H_t(x,t;y,t_0;\rho)|\,\mathrm{d}y\lesssim\frac{\delta}{\sqrt{t-t_0}\sqrt{t}}|h|\big|\log|h|\big|^2+(t-t_0)^{-\frac32}|h|,\\
&\int_{\mathbb{R}}|H_{ty}(x+h,t;y,t_0;\rho)-H_{ty}(x,t;y,t_0;\rho)|\,\mathrm{d}y\lesssim\frac{\delta}{(t-t_0)\sqrt{t}}|h|\big|\log|h|\big|^2+(t-t_0)^{-2}|h|,
\end{aligned}$$

when $|h|<1$.

Proof Here we only give the proof for H_{ty}; the estimate for H_t can be done similarly. For fixed $t_0 \le T_1 < T_2 \le t$, set

$$\begin{cases} \mu^{T_1}(x) \equiv \rho(x, T_1),\ \mu^{T_2}(x) \equiv \rho(x, T_2), \\ \tilde{H}(x,t;y,\sigma) \equiv \chi\big(\frac{\sigma - t_0}{t - t_0}\big) H(x,t;y,\sigma;\mu^{T_1}) + \Big(1 - \chi\big(\frac{\sigma - t_0}{t - t_0}\big)\Big) H(x,t;y,\sigma;\mu^{T_2}) \text{ for } \sigma \in [t_0, t]. \end{cases} \tag{3.2}$$

Consider the following identity:

$$\int_{t_0}^{t} \int_{\mathbb{R}} \tilde{H}(x,t;z,\sigma) \Big[\partial_\sigma H(z,\sigma;y,t_0;\rho) - \partial_z\big(\rho(z,\sigma)\partial_z H(z,\sigma;y,t_0;\rho)\big)\Big] \mathrm{d}z\mathrm{d}\sigma = 0. \tag{3.3}$$

Performing the integration by parts and using the equations satisfied by the heat kernel, we find the integral representation of $H(x,t;y,t_0;\rho)$ in terms of $\tilde{H}$ as follows:

$$\begin{aligned} &H(x,t;y,t_0;\rho) \\ =& \tilde{H}(x,t;y,t_0) + \int_{t_0}^{t} \int_{\mathbb{R}} \frac{\chi'\big(\frac{\sigma - t_0}{t - t_0}\big)}{t - t_0} \Big[H(x,t;z,\sigma;\mu^{T_1}) - H(x,t;z,\sigma;\mu^{T_2})\Big] H(z,\sigma;y,t_0;\rho) \mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}} \chi\big(\frac{\sigma - t_0}{t - t_0}\big) H_z(x,t;z,\sigma;\mu^{T_1}) \Big[\rho(z,\sigma) - \rho(z,T_1)\Big] H_z(z,\sigma;y,t_0;\rho) \mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}} \Big(1 - \chi\big(\frac{\sigma - t_0}{t - t_0}\big)\Big) H_z(x,t;z,\sigma;\mu^{T_2}) \Big[\rho(z,\sigma) - \rho(z,T_2)\Big] H_z(z,\sigma;y,t_0;\rho) \mathrm{d}z\mathrm{d}\sigma. \end{aligned}$$

Differentiating with respect to t and y, one has the integral representation of H_{ty}. Taking the difference of this evaluated at $x+h$ and x, and setting $T_1 = t_0,\ T_2 = t$, we obtain the following equation:

$$\begin{aligned} &H_{ty}(x+h,t;y,t_0;\rho) - H_{ty}(x,t;y,t_0;\rho) \\ =& \tilde{H}_{ty}(x+h,t;y,t_0) - \tilde{H}_{ty}(x,t;y,t_0) \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}} \frac{\big(-\chi'' \frac{\sigma - t_0}{t - t_0} - \chi'\big)}{(t - t_0)^2} \Big[\big(H(x+h,t;z,\sigma;\mu^{T_1}) - H(x+h,t;z,\sigma;\mu^{T_2})\big) \\ &- \big(H(x,t;z,\sigma;\mu^{T_1}) - H(x,t;z,\sigma;\mu^{T_2})\big)\Big] H_y(z,\sigma;y,t_0;\rho) \mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}} \frac{\chi'}{t - t_0} \Big[\big(H_t(x+h,t;z,\sigma;\mu^{T_1}) - H_t(x+h,t;z,\sigma;\mu^{T_2})\big) \\ &- \big(H_t(x,t;z,\sigma;\mu^{T_1}) - H_t(x,t;z,\sigma;\mu^{T_2})\big)\Big] H_y(z,\sigma;y,t_0;\rho) \mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}} \frac{-\chi' \frac{\sigma - t_0}{t - t_0}}{t - t_0} \Big[\rho(z,\sigma) - \rho(z,T_1)\Big] H_{zy}(z,\sigma;y,t_0;\rho) \\ &\cdot \Big[H_z(x+h,t;z,\sigma;\mu^{T_1}) - H_z(x,t;z,\sigma;\mu^{T_1})\Big] \mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}} \chi\big(\frac{\sigma - t_0}{t - t_0}\big) \Big[\rho(z,\sigma) - \rho(z,T_1)\Big] H_{zy}(z,\sigma;y,t_0;\rho) \\ &\cdot \Big[H_{tz}(x+h,t;z,\sigma;\mu^{T_1}) - H_{tz}(x,t;z,\sigma;\mu^{T_1})\Big] \mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}} \Big[1 - \chi\big(\frac{\sigma - t_0}{t - t_0}\big)\Big] \Big[\rho(z,\sigma) - \rho(z,T_2)\Big] H_{zy}(z,\sigma;y,t_0;\rho) \\ &\cdot \Big[H_{tz}(x+h,t;z,\sigma;\mu^{T_2}) - H_{tz}(x,t;z,\sigma;\mu^{T_2})\Big] \mathrm{d}z\mathrm{d}\sigma =: \sum_{j=1}^{6} \mathcal{I}_j. \end{aligned}$$

Here $T_1 = t_0$, $T_2 = t$, and we may assume that $h > 0$, and for simplicity omit the argument $\frac{\sigma-t_0}{t-t_0}$ of χ. Next, we shall estimate $\mathcal{I}_j,\ j = 1\cdots 6$ one by one.

For $\mathcal{I}_1$, from (3.2), we have that

$$\tilde{H}_{ty}(x,t;y,\sigma) = H_{ty}(x,t;y,\sigma;\mu^{T_2}).$$

It then follows from the estimate of the heat kernel of the time-independent coefficient that

$$\begin{aligned}|\mathcal{I}_1| &\le \left|H_{ty}(x+h,t;y,t_0;\mu^{T_2}) - H_{ty}(x,t;y,t_0;\mu^{T_2})\right| \\ &\lesssim \int_x^{x+h} \frac{\mathrm{e}^{-\frac{(w-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{5/2}}\mathrm{d}w = \int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}} \frac{\mathrm{e}^{-\frac{w^2}{C_*}}}{(t-t_0)^2}\mathrm{d}w.\end{aligned} \tag{3.4}$$

For $\mathcal{I}_2$, we have that

$$\begin{aligned}|\mathcal{I}_2| &\lesssim \int_{t_0+\frac{1}{3}(t-t_0)}^{t_0+\frac{2}{3}(t-t_0)} \int_{\mathbb{R}\setminus\mathscr{D}} \int_x^{x+h} \frac{1}{(t-t_0)^2} \frac{\mathrm{e}^{-\frac{(w-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)} \frac{\mathrm{e}^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)}\mathrm{d}w\mathrm{d}z\mathrm{d}\sigma \\ &\lesssim \int_x^{x+h} \frac{\mathrm{e}^{-\frac{(w-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{5/2}}\mathrm{d}w \lesssim (t-t_0)^{-2}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}} \mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w.\end{aligned} \tag{3.5}$$

The estimate of $\mathcal{I}_3$ is similar, and we have that

$$|\mathcal{I}_3| \lesssim (t-t_0)^{-2}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}} \mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w. \tag{3.6}$$

For $\mathcal{I}_4$, using $|\partial_\sigma \rho(z,\sigma)| \lesssim \frac{\delta}{\sqrt{\sigma}}$, we have that

$$\begin{aligned}|\mathcal{I}_4| &\lesssim \int_{t_0+\frac{1}{3}(t-t_0)}^{t_0+\frac{2}{3}(t-t_0)} \int_{\mathbb{R}\setminus\mathscr{D}} \int_x^{x+h} \frac{1}{(t-t_0)} \frac{\mathrm{e}^{-\frac{(w-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^{3/2}} \frac{\delta(\sigma-t_0)}{\sqrt{\sigma}} \frac{\mathrm{e}^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)^{3/2}}\mathrm{d}w\mathrm{d}z\mathrm{d}\sigma \\ &\lesssim \delta\int_x^{x+h} \frac{\mathrm{e}^{-\frac{(w-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{3/2}\sqrt{t}}\mathrm{d}w \lesssim \frac{\delta}{(t-t_0)\sqrt{t}}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}} \mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w.\end{aligned} \tag{3.7}$$

For $\mathcal{I}_5$, thanks to the cut-off function, the high time singularity $\sigma = t$ does not show up, and we have that

$$\begin{aligned}|\mathcal{I}_5| &\lesssim \int_{t_0}^{t_0+\frac{2}{3}(t-t_0)} \int_{\mathbb{R}} \int_x^{x+h} \frac{\mathrm{e}^{-\frac{(w-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^{5/2}} \frac{\delta(\sigma-t_0)}{\sqrt{\sigma}} \frac{\mathrm{e}^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)^{3/2}}\mathrm{d}w\mathrm{d}z\mathrm{d}\sigma \\ &\lesssim \delta\int_x^{x+h} \frac{\mathrm{e}^{-\frac{(w-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{3/2}\sqrt{t}}\mathrm{d}w \lesssim \frac{\delta}{(t-t_0)\sqrt{t}}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}} \mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w.\end{aligned} \tag{3.8}$$

From the above estimate, we know that $\mathcal{I}_j,\ j = 1,\cdots,5$ are actually differentiable in the x-variable when $t > t_0$.

However, we do not have such differentiability for $\mathcal{I}_6$, because of the high singularity at $\sigma = t$. We want to transfer the z-derivative to the other terms to reduce the time singularity. From the equation of the heat kernel $H(z,\sigma;y,t_0;\rho)$, we can rewrite things as

$$H_{zy}(z,\sigma;y,t_0;\rho) = \frac{1}{\rho(z,\sigma)}\int_{-\infty}^{z} H_{\sigma y}(\zeta,\sigma;y,t_0;\rho)\mathrm{d}\zeta.$$

Using integration by parts and the continuity properties of the derivatives of the heat kernel, we have that

$$
\begin{aligned}
\mathcal{I}_6 =& \int_{t_0}^{t}\int_{\mathbb{R}\backslash\mathscr{D}}\big[1-\chi\big]\Big[H_{tz}(x+h,t;z,\sigma;\mu^{T_2})-H_{tz}(x,t;z,\sigma;\mu^{T_2})\Big]\\
&\cdot\Big[\frac{\rho(z,\sigma)-\rho(z,T_2)}{\rho(z,\sigma)}\Big]\int_{-\infty}^{z}H_{\sigma y}(\zeta,\sigma;y,t_0;\rho)\mathrm{d}\zeta\mathrm{d}z\mathrm{d}\sigma\\
=&-\int_{t_0}^{t}\int_{\mathbb{R}\backslash\mathscr{D}}\big[1-\chi\big]\Big[H_{t}(x+h,t;z,\sigma;\mu^{T_2})-H_{t}(x,t;z,\sigma;\mu^{T_2})\Big]\\
&\frac{\rho_z(z,\sigma)\rho(z,T_2)-\rho_z(z,T_2)\rho(z,\sigma)}{\rho^2(z,\sigma)}\int_{-\infty}^{z}H_{\sigma y}(\zeta,\sigma;y,t_0;\rho)\mathrm{d}\zeta\mathrm{d}z\mathrm{d}\sigma\\
&+\int_{t_0}^{t}\big[1-\chi\big]\sum_{\alpha\in\mathscr{D}}\Big[H_{t}(x+h,t;\alpha,\sigma;\mu^{T_2})-H_{t}(x,t;\alpha,\sigma;\mu^{T_2})\Big]\\
&\cdot\Big[\frac{\rho(z,T_2)}{\rho(z,\sigma)}\Big]_{z=\alpha^-}^{z=\alpha^+}\int_{-\infty}^{\alpha}H_{\sigma y}(\zeta,\sigma;y,t_0;\rho)\mathrm{d}\zeta\mathrm{d}\sigma\\
&-\int_{t_0}^{t}\int_{\mathbb{R}\backslash\mathscr{D}}\big[1-\chi\big]\Big[H_{t}(x+h,t;z,\sigma;\mu^{T_2})-H_{t}(x,t;z,\sigma;\mu^{T_2})\Big]\\
&\cdot\Big[\frac{\rho(z,\sigma)-\rho(z,T_2)}{\rho(z,\sigma)}\Big]H_{\sigma y}(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma\\
=:&\ \mathcal{I}_{61}+\mathcal{I}_{62}+\mathcal{I}_{63}.
\end{aligned}
\tag{3.9}
$$

For $\mathcal{I}_{61}$, we first write

$$
\begin{aligned}
&\rho_z(z,\sigma)\rho(z,T_2)-\rho_z(z,T_2)\rho(z,\sigma)\\
=&\big[\rho_z(z,\sigma)-\rho_z(z,T_2)\big]\rho(z,T_2)+\rho_z(z,T_2)\big[\rho(z,T_2)-\rho(z,\sigma)\big].
\end{aligned}
$$

Then we have that

$$
\begin{aligned}
|\mathcal{I}_{61}|\lesssim&\int_{t_0+\frac13(t-t_0)}^{t}\int_{\mathbb{R}\backslash\mathscr{D}}\int_{x}^{x+h}\frac{\mathrm{e}^{-\frac{(w-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^2}\left|\rho_z(z,T_2)-\rho_z(z,\sigma)\right|\frac{\mathrm{e}^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)^{3/2}}\mathrm{d}w\mathrm{d}z\mathrm{d}\sigma\\
&+\int_{t_0+\frac13(t-t_0)}^{t}\int_{\mathbb{R}\backslash\mathscr{D}}\int_{x}^{x+h}\frac{\mathrm{e}^{-\frac{(w-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^2}\left|\rho_z(z,T_2)\right|\left|\rho(z,T_2)-\rho(z,\sigma)\right|\frac{\mathrm{e}^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)^{3/2}}\mathrm{d}w\mathrm{d}z\mathrm{d}\sigma\\
=:&\ \mathcal{I}_{61;1}+\mathcal{I}_{61;2}.
\end{aligned}
\tag{3.10}
$$

Noticing that

$$
\int_{x}^{x+h}\frac{\mathrm{e}^{-\frac{(w-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^2}\mathrm{d}w=\int_{\frac{x-z}{\sqrt{t-\sigma}}}^{\frac{x-z+h}{\sqrt{t-\sigma}}}\frac{\mathrm{e}^{-\frac{w^2}{C_*}}}{(t-\sigma)^{3/2}}\mathrm{d}w\lesssim\frac{1}{(t-\sigma)^{3/2}}\min\big(1,\ \frac{h}{\sqrt{t-\sigma}}\big),
$$

and recalling the Hölder-in-time assumption (3.1),

$$
\int_{\mathbb{R}\backslash\mathscr{D}}\left|\rho_z(z,T_2)-\rho_z(z,\sigma)\right|\mathrm{d}z\lesssim\delta\frac{(t-\sigma)\big|\log(t-\sigma)\big|}{\sqrt{t}},
$$

we obtain that

$$
\mathcal{I}_{61;1}\lesssim\int_{t_0+\frac13(t-t_0)}^{t}\frac{\delta}{(t-\sigma)^{3/2}}\min\big(1,\ \frac{h}{\sqrt{t-\sigma}}\big)\frac{(t-\sigma)\big|\log(t-\sigma)\big|}{\sqrt{t}}\frac{1}{(\sigma-t_0)^{3/2}}\mathrm{d}\sigma
$$

$$
\begin{aligned}
&= \frac{\delta}{(t-t_0)^{\frac{3}{2}}\sqrt{t}} \int_0^{\frac{t-t_0}{3}} \frac{|\log\sigma|}{\sqrt{\sigma}} \min\big(1,\ \frac{h}{\sqrt{\sigma}}\big)\mathrm{d}\sigma \\
&\lesssim \frac{\delta}{(t-t_0)^{\frac{3}{2}}\sqrt{t}} \left[\int_0^{h^2} \frac{|\log\sigma|}{\sqrt{\sigma}}\mathrm{d}\sigma + \int_{h^2}^{\frac{t-t_0}{3}} \frac{h|\log\sigma|}{\sigma}\mathrm{d}\sigma\right] \\
&\lesssim \frac{\delta}{(t-t_0)^{\frac{3}{2}}\sqrt{t}} \Big[h|\log h| + h|\log h|^2\Big], \qquad \text{for } h<1. \qquad (3.11)
\end{aligned}
$$

Here we assume that $h^2 < \frac{t-t_0}{3}$, and the estimate still holds for other cases, as can easily be seen.

As for $\mathcal{I}_{61;2}$, we have that

$$
\begin{aligned}
\mathcal{I}_{61;2} &\lesssim \int_{t_0+\frac{1}{3}(t-t_0)}^{t} \frac{1}{(t-\sigma)^{3/2}} \min\big(1,\ \frac{h}{\sqrt{t-\sigma}}\big)\Big[\int_{\mathbb{R}\setminus\mathscr{D}} |\rho_z(z,T_2)|\,\mathrm{d}z\Big] \frac{\delta(t-\sigma)}{\sqrt{t}} \frac{1}{(\sigma-t_0)^{\frac{3}{2}}}\mathrm{d}\sigma \\
&\lesssim \frac{\delta^2}{(t-t_0)^{\frac{3}{2}}\sqrt{t}} \int_0^{\frac{t-t_0}{3}} \frac{1}{\sqrt{\sigma}} \min\big(1,\ \frac{h}{\sqrt{\sigma}}\big)\mathrm{d}\sigma \\
&\lesssim \frac{\delta^2}{(t-t_0)^{\frac{3}{2}}\sqrt{t}} \Big[h + h\,|\log h|\Big], \qquad \text{for } h<1. \qquad (3.12)
\end{aligned}
$$

Next, we consider $\mathcal{I}_{62}$. In view of the identity

$$
\begin{aligned}
\left[\frac{\rho(z,T_2)}{\rho(z,\sigma)}\right]_{z=\alpha^-}^{z=\alpha^+} &= \frac{\big(\rho(\alpha^+,T_2)-\rho(\alpha^-,T_2)\big)\big(\rho(\alpha^-,\sigma)-\rho(\alpha^-,T_2)\big)}{\rho(\alpha^+,\sigma)\rho(\alpha^-,\sigma)} \\
&\quad + \frac{\Big[\big(\rho(\alpha^+,T_2)-\rho(\alpha^-,T_2)\big)-\big(\rho(\alpha^+,\sigma)-\rho(\alpha^-,\sigma)\big)\Big]\rho(\alpha^-,T_2)}{\rho(\alpha^+,\sigma)\rho(\alpha^-,\sigma)},
\end{aligned}
$$

the estimate

$$
|\rho(z,T_2)-\rho(z,\sigma)| = \left|\int_\sigma^{T_2} \rho_\tau(z,\tau)\mathrm{d}\tau\right| \lesssim \frac{\delta(t-\sigma)}{\sqrt{t}},
$$

and the Hölder continuity assumption (3.1), namely that

$$
\sum_{\alpha\in\mathscr{D}} \left| \rho(\cdot,T_2)\Big|_{\alpha^-}^{\alpha^+} - \rho(\cdot,\sigma)\Big|_{\alpha^-}^{\alpha^+}\right| \lesssim \frac{\delta^2(t-\sigma)}{\sqrt{t}},
$$

we have that

$$
\begin{aligned}
|\mathcal{I}_{62}| &\lesssim \int_{t_0+\frac{1}{3}(t-t_0)}^{t} \sum_{\alpha\in\mathscr{D}} \int_x^{x+h} \frac{\mathrm{e}^{-\frac{(w-\alpha)^2}{C_*(t-\sigma)}}}{(t-\sigma)^2}\mathrm{d}w \\
&\quad \cdot \left[\left|\rho(\cdot,T_2)\Big|_{\alpha^-}^{\alpha^+}\right| \frac{\delta(t-\sigma)}{\sqrt{t}} + \left|\rho(\cdot,T_2)\Big|_{\alpha^-}^{\alpha^+} - \rho(\cdot,\sigma)\Big|_{\alpha^-}^{\alpha^+}\right|\right] \frac{\mathrm{e}^{-\frac{(\alpha-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)^{3/2}}\mathrm{d}\sigma \\
&\lesssim \int_{t_0+\frac{1}{3}(t-t_0)}^{t} \frac{\min\big(1,\ \frac{h}{\sqrt{t-\sigma}}\big)}{(t-\sigma)^{3/2}} \frac{\delta^2(t-\sigma)}{\sqrt{t}} \frac{1}{(\sigma-t_0)^{3/2}}\mathrm{d}\sigma \\
&\lesssim \frac{\delta^2}{(t-t_0)^{\frac{3}{2}}\sqrt{t}} \int_0^{\frac{t-t_0}{3}} \frac{1}{\sqrt{\sigma}} \min\big(1,\ \frac{h}{\sqrt{\sigma}}\big)\mathrm{d}\sigma \\
&\lesssim \frac{\delta^2}{(t-t_0)^{\frac{3}{2}}\sqrt{t}} \Big[h + h\,|\log h|\Big], \qquad \text{for } h<1. \qquad (3.13)
\end{aligned}
$$

For $\mathcal{I}_{63}$, by Lemma A.1 and property of ρ, we directly have that

$$\begin{aligned}|\mathcal{I}_{63}| &\lesssim \int_{t_0+\frac{1}{3}(t-t_0)}^{t}\int_{\mathbb{R}\setminus\mathscr{D}}\int_x^{x+h}\frac{\mathrm{e}^{-\frac{(w-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^2}\frac{\delta(t-\sigma)}{\sqrt{t}}\frac{\mathrm{e}^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)^2}\mathrm{d}w\mathrm{d}z\mathrm{d}\sigma\\ &\lesssim \frac{\delta}{(t-t_0)^2\sqrt{t}}\int_x^{x+h}\int_{t_0+\frac{1}{3}(t-t_0)}^{t}\frac{\mathrm{e}^{-\frac{(w-y)^2}{C_*(t-t_0)}}}{\sqrt{t-\sigma}}\mathrm{d}\sigma\mathrm{d}w\\ &\lesssim \frac{\delta}{(t-t_0)\sqrt{t}}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}}\mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w. \end{aligned}\tag{3.14}$$

Eventually, combining the estimates (3.5)–(3.14), we can conclude that, when $|h|<1$,

$$\begin{aligned}&|H_{ty}(x+h,t;y,t_0;\rho)-H_{ty}(x,t;y,t_0;\rho)|\\ \lesssim&\frac{\delta}{(t-t_0)^{\frac{3}{2}}\sqrt{t}}|h|\big|\log|h|\big|^2+(t-t_0)^{-2}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}}\mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w.\end{aligned}$$

Now we address the issue of how to adapt the proof to get the Hölder-in-space estimate of $H_t(x,t;y,t_0;\rho)$. Instead of using the approximate heat kernel $\tilde{H}$ in (3.2), we consider the following equation:

$$\int_{t_0}^{t}\int_{\mathbb{R}}H(x,t;z,\sigma;\mu^T)\Big[\partial_\sigma H(z,\sigma;y,t_0;\rho)-\partial_z\big(\rho(z,\sigma)\partial_z H(z,\sigma;y,t_0;\rho)\big)\Big]\mathrm{d}z\mathrm{d}\sigma=0.\tag{3.15}$$

We use integration by parts to yield that

$$\begin{aligned}H(x,t;y,t_0;\rho)=&\,H(x,t;y,t_0;\mu^T)\\ &+\int_{t_0}^{t}\int_{\mathbb{R}}H_z(x,t;z,\sigma;\mu^T)\Big[\rho(z,T)-\rho(z,\sigma)\Big]H_z(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma.\end{aligned}$$

Differentiating with respect to t and setting $T=t$, one obtains the following expression of $H_t(x,t;y,t_0;\rho)$:

$$\begin{aligned}H_t(x,t;y,t_0;\rho)=&\,H_t(x,t;y,t_0;\mu^t)\\ &+\int_{t_0}^{t}\int_{\mathbb{R}}H_{tz}(x,t;z,\sigma;\mu^t)\Big[\rho(z,t)-\rho(z,\sigma)\Big]H_z(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma.\end{aligned}$$

Taking the difference of the result evaluated at $x+h$ and x, we have that

$$\begin{aligned}&H_t(x+h,t;y,t_0;\rho)-H_t(x,t;y,t_0;\rho)\\ =&\,H_t(x+h,t;y,t_0;\mu^t)-H_t(x,t;y,t_0;\mu^t)\\ &+\int_{t_0}^{\frac{1}{2}(t+t_0)}\int_{\mathbb{R}}[H_{tz}(x+h,t;z,\sigma;\mu^t)-H_{tz}(x,t;z,\sigma;\mu^t)][\rho(z,t)-\rho(z,\sigma)]H_z(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma\\ &+\int_{\frac{1}{2}(t+t_0)}^{t}\int_{\mathbb{R}}\Big[H_{tz}(x+h,t;z,\sigma;\mu^t)-H_{tz}(x,t;z,\sigma;\mu^t)\Big]\Big[\rho(z,t)-\rho(z,\sigma)\Big]H_z(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma\\ =:&\,\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.\end{aligned}$$

The estimates of $\mathcal{J}_1$ and $\mathcal{J}_2$ are straightforward, and we have that

$$|\mathcal{J}_1|\lesssim(t-t_0)^{-\frac{3}{2}}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}}\mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w,\qquad |\mathcal{J}_2|\lesssim\frac{\delta}{\sqrt{t-t_0}\sqrt{t}}\int_{\frac{x-y}{\sqrt{t-t_0}}}^{\frac{x-y+h}{\sqrt{t-t_0}}}\mathrm{e}^{-\frac{w^2}{C_*}}\mathrm{d}w.$$

The estimate of $\mathcal{J}_3$ is analogous to the term $\mathcal{I}_6$ in (3.9), and following the similar argument we find that

$$|\mathcal{J}_3| \lesssim \frac{\delta}{(t-t_0)\sqrt{t}} h\,|\log h|^2, \qquad \text{for } h<1.$$

Finally, if we first integrate with respect to the y-variable for $\mathcal{I}_{61}$ in (3.10) and $\mathcal{I}_{62}$ in (3.13), then, following a similar argument, we get that

$$\int_{\mathbb{R}} |H_{ty}(x+h,t;y,t_0;\rho)-H_{ty}(x,t;y,t_0;\rho)|\,\mathrm{d}y \lesssim \frac{\delta}{(t-t_0)\sqrt{t}}|h|\big|\log|h|\big|^2+(t-t_0)^{-2}|h|.$$

Similarly,

$$\int_{\mathbb{R}} |H_{t}(x+h,t;y,t_0;\rho)-H_{t}(x,t;y,t_0;\rho)|\,\mathrm{d}y \lesssim \frac{\delta}{\sqrt{t-t_0}\sqrt{t}}|h|\big|\log|h|\big|^2+(t-t_0)^{-\frac{3}{2}}|h|.$$

We have therefore completed the proof of this lemma. □

Remark 3.3 We remark that in obtaining the integral representation of $H_{ty}(x,t;y,t_0;\rho)$, we first take the derivative with respect to t and y, and next set $T_1=t_0$ and $T_2=t$, so every term is well-defined. If reversing the order, namely, if we first set $T_1=t_0$ and $T_2=t$, and next take the derivative, then one needs to calculate the variation of the heat kernel with respect to the conductivity coefficient, which makes the estimate more complicated. The rationale of our computation can be justified as follows: when we represent $H(x,t;y,t_0;\rho)$ in terms of $\tilde{H}$ in (3.3), we can replace the time integral interval to $[t_1,t_2]$ with $t_0<t_1<t_2<t$, and get the time boundary terms

$$\int_{\mathbb{R}} H(z,t_2;y,t_0;\rho)\tilde{H}(x,t;z,t_2)\mathrm{d}z \quad \text{and} \quad \int_{\mathbb{R}} H(z,t_1;y,t_0;\rho)\tilde{H}(x,t;z,t_1)\mathrm{d}z,$$

rather than

$$H(x,t;y,t_0;\rho) \text{ and } \tilde{H}(x,t;y,t_0)$$

respectively. When the integral does not touch the time singularity t_0 and t, the differentiation with respect to t and y works well. After that, we set $T_1=t_0$ and $T_2=t$, and let t_1 tend to t_0 and t_2 tend to t. Now, because of the choice of T_1 and T_2, the space-time double integrals are all well-defined, while the boundary integral tends to H_{ty} and $\tilde{H}_{ty}$ in the distribution sense. We thus get the integral representation of H_{ty} in terms of $\tilde{H}_{ty}$.

4 Local Solution with Hölder Continuity

In this section, relying on the heat kernel for BV conductivity, we will design an appropriate Picard iteration scheme, and show that it is convergent, which in turn constructs a weak solution that is local in time of system (1.1). Moreover, the Hölder continuity of the iterative approximated solutions will directly imply the Hölder continuity of the limit solution.

For simplicity, we set the constant state of perturbation to be $(v,u)=(1,0)$, and assume that the initial data $(v_0,u_0)=(1+v_0^*,u_0^*)$ satisfies that

$$\|v_0^*\|_{\mathrm{BV}}+\|v_0^*\|_{L^1}+\|u_0^*\|_{L^\infty}+\|u_0^*\|_{L^1}<\delta\ll 1. \tag{4.1}$$

Note that the infinity norm of L^1-integrable functions can be bounded by their BV norm, so we actually know, from (4.1), that the L^∞ norm of v_0^* is bounded by δ. Denote the discontinuity set of v_0^* by $\mathscr{D}$.

4.1 Iteration Scheme

In this part, we will construct the iteration scheme and the corresponding approximate solutions. As the lower order estimates can be constructed similarly as to [1], we will omit the detailed proof. Instead, we mainly focus on the higher order estimates and, in particular, the Hölder estimates; that is new contribution, and plays an important role in the uniqueness criteria. We consider the following standard iteration scheme for $n \geq 0$:

$$\begin{cases} V_t^{n+1} - U_x^{n+1} = 0, \\ U_t^{n+1} - \left(\dfrac{\mu U_x^{n+1}}{1+V^n} \right)_x = -p(1+V^n)_x, \\ (V^{n+1}, U^{n+1})\Big|_{t=0} = (v_0^*, u_0^*). \end{cases} \tag{4.2}$$

Here the initial step is set to be

$$(V^0, U^0) = (0, 0).$$

Next, we estimate (V^1, U^1), which is governed by a homogeneous equation, since the right-handside of (4.2) vanishes due to the choice of the zeroth step. As the estimates are classical, we omit the details and directly provide the following estimates:

Lemma 4.1 Suppose that the initial data (v_0^*, u_0^*) satisfies the condition (4.1). Then, there exists a positive constant $C_\sharp$ such that the following estimates hold for V^1 and U^1:

$$\max\Big\{ \big\|U^1(\cdot,t)\big\|_{L_x^1},\ \big\|U^1(\cdot,t)\big\|_{L_x^\infty},\ \sqrt{t}\,\big\|U_x^1(\cdot,t)\big\|_{L_x^1},\ \sqrt{t}\,\big\|U_x^1(\cdot,t)\big\|_{L_x^\infty},\ t\,\big\|U_t^1(\cdot,t)\big\|_{L_x^\infty} \Big\} \leq C_\sharp \delta,$$

$$\max\left\{ \sqrt{t}\,\big\|V_t^1(\cdot,t)\big\|_{L_x^\infty},\ \int_{\mathbb{R}\setminus\mathscr{D}} |V_x^1(x,t)|\mathrm{d}x,\ \big\|V^1(x,t)\big\|_{L_x^1},\ \big\|V^1(x,t)\big\|_{L_x^\infty} \right\} \leq C_\sharp \delta, \quad 0 < t < 1;$$

$$\left| V^1(\cdot,t)\Big|_{x=z^-}^{x=z^+} \right| = \left| v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+} \right|, \quad z \in \mathscr{D}, \quad 0 < t < 1,$$

$$\int_{\mathbb{R}\setminus\mathscr{D}} |V_x^1(x,t) - V_x^1(x,s)|\mathrm{d}x \leq C_\sharp \frac{t-s}{t}\delta, \quad 0 \leq s \leq t < 1.$$

Now, in order to apply mathematical induction to construct the estimates of the sequence of approximate solutions in (4.2), we need to propose an appropriate ansatz, which is inspired by Lemma 4.1. Suppose that

$$0 < \delta,\ t_\sharp \ll 1,$$

and the following induction hypotheses hold for the solution (V^n, U^n) to (4.2) when $n \leq k$:

$$\begin{cases} 0 < \delta,\ t_\sharp \ll 1, \quad 1 \leq n \leq k, \quad 0 < t < t_\sharp, \\ \max\Big\{ \|U^n(\cdot,t)\|_{L_x^1},\ \|U^n(\cdot,t)\|_{L_x^\infty},\ \sqrt{t}\,\|U_x^n(\cdot,t)\|_{L_x^1},\ \sqrt{t}\,\|U_x^n(\cdot,t)\|_{L_x^\infty} \Big\} \leq 2C_\sharp \delta, \\ \max\left\{ \displaystyle\int_{\mathbb{R}\setminus\mathscr{D}} |V_x^n(x,t)|\mathrm{d}x,\ \|V^n(\cdot,t)\|_{L_x^1},\ \|V^n(\cdot,t)\|_{L_x^\infty},\ \sqrt{t}\,\|V_t^n(\cdot,t)\|_{L_x^\infty} \right\} \leq 2C_\sharp \delta, \\ \displaystyle\int_{\mathbb{R}\setminus\mathscr{D}} |V_x^n(x,t) - V_x^n(x,s)|\mathrm{d}x \leq 2C_\sharp \delta \Big[\frac{t-s}{t} + (t-s)|\log(t-s)| \Big], \quad 0 < s < t, \\ \left| V^n(\cdot,t)\Big|_{x=z^-}^{x=z^+} \right| \leq 2 \left| v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+} \right|, \quad z \in \mathscr{D}. \end{cases} \tag{4.3}$$

By Lemma 4.1, the initial step (V^1, U^1) fulfills the ansatz (4.3). In what follows, we will show that (V^{k+1}, U^{k+1}) also satisfies the ansatz (4.3).

Remark 4.2 Note that in ansatz (4.3), the L^1 norm $\|U_x^1(\cdot,t)\|_{L^1_x}$ has singularity $\frac{1}{\sqrt{t}}$ near $t=0$; this is because we only assume the smallness of $\|u_0^*\|_{L^1\cap L^\infty}$. This singularity causes some difficulty when we close the ansatz (4.3). Moreover, we lose the continuity in time of V^n in BV norm when $t=0$, so that we cannot prove the uniqueness of the solution in this case.

On the other hand, if we further assume that u_0^* satisfies assumptions stronger than (4.1),

$$\|u_0^*\|_{L^1\cap \mathrm{BV}} < \delta \ll 1,$$

then in the initial step, by transferring the spatial derivative to the initial data, we can easily get that $\|U_x^{k+1}(\cdot,t)\|_{L^1} \le 2C_\sharp\delta$. Thus, the Hölder estimate of the BV norm of V^n becomes

$$\int_{\mathbb{R}\backslash\mathscr{D}} |V_x^n(x,t)-V_x^n(x,s)|\mathrm{d}x \le 2C_\sharp\delta\Big[\frac{t-s}{\sqrt{t}}\Big], \quad 0<s<t.$$

Then we take $s=0$ to imply that V^n is Hölder continuous in the BV norm at $t=0$; this plays an important role in the proof of the uniqueness of the solution.

Now we are going to close the ansatz (4.3). For simplicity of presentation, we introduce the notations

$$\mu^k \equiv \frac{\mu}{1+V^k}, \quad \mathcal{N}_1^k(x,t) = -\partial_x p(1+V^k). \tag{4.4}$$

From the initial condition (4.1), the iteration scheme (4.2), and the above ansatz (4.3), we know that U^{k+1} is governed by heat equations with BV conductivities and sources. Notice that the ansatz (4.3) implies that the conductivity $\mu/(1+V^k)$ satisfies (2.3). Therefore, according to Lemma A.1 and Remark 2.3, one can apply Duhamel's principle to construct the weak solution (V^{k+1},U^{k+1}) to equation (4.2) as follows:

$$U^{k+1}(x,t) = \int_{\mathbb{R}} H\left(x,t;y,0;\mu^k\right)u_0^*(y)\mathrm{d}y + \int_0^t\int_{\mathbb{R}\backslash\mathscr{D}} H_y\left(x,t;y,s;\mu^k\right)p(1+V^k)\mathrm{d}y\mathrm{d}s, \tag{4.5}$$

$$V^{k+1}(x,t) = v_0^*(x) + \int_0^t U_x^{k+1}(x,s)\mathrm{d}s. \tag{4.6}$$

We shall prove that, for sufficiently small δ and $t_\sharp$, which are independent of k, the ansatz (4.3) holds for (V^{k+1},U^{k+1}) as well, which implies that the ansatz holds for all $k\ge 1$, by induction. As the proof is lengthy, we will split it into lemmas concerning the estimates of U^{k+1} and V^{k+1}, respectively.

Lemma 4.3 (U^{k+1}) Suppose that the initial data (v_0^*,u_0^*) satisfies the condition (4.1), and that the ansatz (4.3) holds for $n\le k$. Then, the equation (4.2) admits a weak solution (V^{k+1},U^{k+1}) with the flux $\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k)$ being continuous with respect to x. Moreover, for sufficiently small δ and $t_\sharp$, the ansatz (4.3) holds for U^{k+1} when $0<t<t_\sharp$.

Proof The existence of the solution is guaranteed by Lemma A.1 and Remark 2.3. Note that U^{k+1} is a weak solution to the second equation in (4.2), which can be written in conservative form with the flux $\left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k)\right)$. Therefore, the continuity of the flux follows from Remark (2.3). In what follows, we will only provide the first order estimate in infinity norm, and the other estimates can be constructed similarly. From the representation (4.5), we can derive a representation of U_x^{k+1} as follows:

$$U_x^{k+1} = \int_{\mathbb{R}} H_x\left(x,t;y,0;\mu^k\right)u_0^*(y)\mathrm{d}y + \int_0^t\int_{\mathbb{R}\backslash\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)p(1+V^k)\mathrm{d}y\mathrm{d}s.$$

One applies Lemma A.1 to get that

$$\int_{\mathbb{R}} \left|H_x\left(x,t;y,0;\mu^k\right)\right| |u_0^*(y)|\,\mathrm{d}y \le C_* \int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_* t}}}{t}\mathrm{d}y \|u_0^*\|_{L_x^\infty} = \sqrt{\frac{\pi C_*^3}{t}}\|u_0^*\|_{L_x^\infty} \le \frac{C_\sharp \delta}{\sqrt{t}}, \quad (4.7)$$

where C_* is constructed in Lemma A.1, and $C_\sharp$ is adjusted so that $\sqrt{\pi C_*^3} \le C_\sharp$. One splits the inhomogeneous term into two parts:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}\backslash \mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right) p(1+V^k(y,s))\mathrm{d}y\mathrm{d}s \\
=& \int_0^t \int_{\mathbb{R}\backslash \mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right) p(1+V^k(y,t))\mathrm{d}y\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}\backslash \mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right) \left(p(1+V^k(y,s)) - p(1+V^k(y,t))\right)\mathrm{d}y\mathrm{d}s \\
\equiv& \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \quad (4.8)$$

For the estimate of $\mathcal{I}_1$, one applies integration by parts and takes into account the discontinuities of V^k to yield that

$$\begin{aligned}
\mathcal{I}_1 =& \int_0^t \int_{\mathbb{R}\backslash \mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right) p(1+V^k(y,t))\mathrm{d}y\mathrm{d}s \\
=& \int_0^t \Big[-\int_{\mathbb{R}\backslash \mathscr{D}} H_x\left(x,t;y,s;\mu^k\right) \partial_y\left(p(1+V^k(y,t))\right)\mathrm{d}y \\
&+ \sum_{z\in\mathscr{D}} H_x\left(x,t;y,s;\mu^k\right) p(1+V^k(y,t))\Big|_{y=z^+}^{y=z^-}\Big]\mathrm{d}s = \mathcal{I}_{11} + \mathcal{I}_{12}.
\end{aligned}$$

For $\mathcal{I}_{11}$, one combines the ansatz (4.3) and the estimates of the heat kernel in Lemmas A.1 and A.2 to obtain, for $0 < t < t_\sharp$, that

$$\begin{aligned}
|\mathcal{I}_{11}| =& \left|\int_0^t \int_{\mathbb{R}\backslash \mathscr{D}} H_x\left(x,t;y,s;\mu^k\right) p'(1+V^k(y,t))V_y^k(y,t)\mathrm{d}y\mathrm{d}s\right| \\
=& O(1)\int_{\mathbb{R}\backslash \mathscr{D}} C_* \mathrm{e}^{-\frac{|x-y|^2}{C_*(t-s)}}|V_y^k(y,t)|\mathrm{d}y = O(1)\int_{\mathbb{R}\backslash \mathscr{D}} |V_y^k|\mathrm{d}y \le \frac{C_\sharp \delta}{4\sqrt{t}},
\end{aligned}$$

where the last inequality holds for sufficiently small $t_\sharp$. For $\mathcal{I}_{12}$, noticing that Lemma A.1 shows that $H_x\left(x,t;y,s;\mu^k\right)$ is continuous with respect to y, and that $p'(1+V^k(y,t))$ is uniformly bounded thanks to ansatz (4.3), one combines Lemma A.1, initial condition (4.1), and the ansatz (4.3) to have, for $0 < t < t_\sharp$, that

$$\begin{aligned}
|\mathcal{I}_{12}| =& \left|\sum_{z\in\mathscr{D}} \int_0^t H_x\left(x,t;y,s;\mu^k\right) p(1+V^k(y,t))\Big|_{y=z^+}^{y=z^-}\mathrm{d}s\right| \\
\le& C_* \mathrm{e}^{-\frac{|x-y|^2}{C_*(t-s)}} \sum_{z\in\mathscr{D}} \left|V^k(y,t)\Big|_{y=z^+}^{y=z^-}\right| \le C_* \mathrm{e}^{-\frac{|x-y|^2}{C_*(t-s)}} \sum_{z\in\mathscr{D}} 2\left|v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+}\right| \\
\le& O(1)\|v_0^*\|_{\mathrm{BV}} \le \frac{C_\sharp \delta}{4\sqrt{t}}
\end{aligned}$$

for sufficiently small $t_\sharp$. This finishes the estimates of $\mathcal{I}_1$ in (4.8). Next, for $\mathcal{I}_2$, one applies Lemma A.1 and the ansatz (4.3) to get that

$$|\mathcal{I}_2| = \left|\int_0^t \int_{\mathbb{R}\backslash \mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right) \left(p(1+V^k(y,s)) - p(1+V^k(y,t))\right)\mathrm{d}y\mathrm{d}s\right|$$

$$
\begin{aligned}
&= \left| \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right) \int_s^t \partial_\tau\left(p(1+V^k(y,\tau))\right)\mathrm{d}\tau\mathrm{d}y\mathrm{d}s \right| \\
&= O(1)\int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \int_s^t \left|H_{xy}\left(x,t;y,s;\mu^k\right)\right| \left|V_\tau^k(y,\tau)\right| \mathrm{d}\tau\mathrm{d}y\mathrm{d}s = O(1)\delta\sqrt{t} \le \frac{C_\sharp\delta}{4\sqrt{t}},
\end{aligned}
$$

where the last inequality holds due to the smallness of $t_\sharp$. Now we combine (4.7), (4.8), the estimates of $\mathcal{I}_{11}$, $\mathcal{I}_{12}$, $\mathcal{I}_2$ and the representation of U_x^{k+1} together to yield that

$$
\|U_x^{k+1}\|_{L_x^\infty} \le \frac{C_\sharp\delta}{\sqrt{t}} + |\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3| < \frac{2C_\sharp\delta}{\sqrt{t}}. \tag{4.9}
$$

The other estimates can be obtained in similar manner. □

Lemma 4.4 (V^{k+1}) Suppose that the initial data (v_0^*, u_0^*) satisfies the condition (4.1), and that the ansatz (4.3) holds for $n \le k$. Then, the equation (4.2) admits a weak solution (V^{k+1}, U^{k+1}), and the ansatz (4.3) holds for V^{k+1} when $0 < t < t_\sharp$ for sufficiently small δ and $t_\sharp$. In addition, the following Hölder continuity holds for $0 \le s \le t < t_\sharp$:

$$
\sum_{z\in\mathscr{D}} \left| V_x^n(\cdot,t)\Big|_{z^-}^{z^+} - V_x^n(\cdot,s)\Big|_{z^-}^{z^+} \right| \le O(1)\delta\frac{t-s}{\sqrt{t}}.
$$

Proof We will only show the jump estimates and the Hölder estimates; the other estimates are constructed similarly.

• (Estimate of $\left|V^{k+1}(\cdot,t)\big|_{x=z^-}^{x=z^+}\right|$) We estimate the jump of the specific volume. According to Remark 2.3 and Lemma 4.3, we know that $\left(\frac{\mu U_x^k}{1+V^{k-1}} - p\left(1+V^{k-1}\right)\right)$ is continuous with respect to x for $k \ge 1$, which implies that

$$
\begin{aligned}
V_t^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} = U_x^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} &= \frac{1+V^k}{\mu}\left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k) + p(1+V^k)\right)\Big|_{z^-}^{z^+} \\
&= \frac{1+V^k}{\mu}\Big|_{z^-}^{z^+}\left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k)\right) + \frac{\left(1+V^k\right)\left(p(1+V^k)\right)}{\mu}\Big|_{z^-}^{z^+} \\
&= \frac{V^k}{\mu}\Big|_{z^-}^{z^+}\left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k)\right) + \frac{\left(1+V^k\right)\left(p(1+V^k)\right)}{\mu}\Big|_{z^-}^{z^+}.
\end{aligned}
$$

We integrate the above equality with respect to time from 0 to t, and apply the ansatz (4.3) and the estimates in Lemma 4.3 to yield that

$$
\begin{aligned}
\left|V^{k+1}(\cdot,t)\Big|_{z^-}^{z^+}\right| &\le \left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| + \left|\int_0^t \left(\frac{V^k(\cdot,s)}{\mu}\Big|_{z^-}^{z^+}\left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k)\right)(z,s)\right.\right. \\
&\qquad \left.\left. + \frac{\left(1+V^k(\cdot,s)\right)\left(p(1+V^k(\cdot,s))\right)}{\mu}\Big|_{z^-}^{z^+}\right)\mathrm{d}s\right| \\
&\le \left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| + \frac{1}{\mu}\int_0^t \left(\frac{\mu\left|U_x^{k+1}\right|}{1-\|V^k\|_{L_x^\infty}} + O(1)\right)\mathrm{d}s \sup_{0\le\sigma\le t}\left|V^k(\cdot,\sigma)\Big|_{z^-}^{z^+}\right| \\
&\le \left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| + O(1)\int_0^t \left(1+\frac{1}{\sqrt{s}}\right)\mathrm{d}s\left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| \\
&= \left(1+O(1)t+O(1)\sqrt{t}\right)\left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| \le 2\left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right|, \qquad 0 < t < t_\sharp,
\end{aligned} \tag{4.10}
$$

where the last inequality holds for sufficiently small $t_\sharp$.

• (Hölder continuity of $\int_{\mathbb{R}\setminus\mathscr{D}} |V_x^{k+1}(x,t) - V_x^{k+1}(x,s)|\mathrm{d}x$) We study the Hölder continuity of the specific volume V^{k+1}. Actually, the Hölder continuity of V_x^{k+1} relies on the Hölder continuity of V_x^k, which is assumed to hold in ansatz (4.3). For $x \notin \mathscr{D}$, we can express $V_x^{k+1}(x,t)$ as follows:

$$\begin{aligned} V_x^{k+1}(x,t) =& \ (v_0^*)_x + \int_0^t U_{xx}^{k+1}(x,s)\mathrm{d}s \\ =& \ (v_0^*)_x + \int_0^t \int_{\mathbb{R}} H_{xx}\left(x,s;y,0;\mu^k\right) u_0^*(y)\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_0^s \int_{\mathbb{R}\setminus\mathscr{D}} H_{xxy}\left(x,s;y,\tau;\mu^k\right) p(1+V^k(y,\tau))\mathrm{d}y\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{4.11}$$

Using this, one has the estimate

$$\begin{aligned} &\int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,t) - V_x^{k+1}(x,s)\right| \mathrm{d}x \\ \le& \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_s^t \int_{\mathbb{R}} H_{xx}(x,\sigma;y,0;\mu^k) u_0^*(y)\mathrm{d}y\mathrm{d}\sigma \right| \mathrm{d}x \\ &+ \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_s^t \int_0^\sigma \int_{\mathbb{R}} H_{xxy}(x,\sigma;y,\tau;\mu^k) p(1+V^k)\mathrm{d}y\mathrm{d}\tau\mathrm{d}\sigma \right| \mathrm{d}x \\ \equiv& \ \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{4.12}$$

When $s \le t/2$, by the estimate of $\int_{\mathbb{R}\setminus\mathscr{D}} |V_x^{k+1}(x,t)|\mathrm{d}x$, one directly has that

$$\int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,t) - V_x^{k+1}(x,s)\right| \mathrm{d}x \le \int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,t)\right| \mathrm{d}x + \int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,s)\right| \mathrm{d}x \le O(1)\delta.$$

Therefore, it suffices to consider the case where $t/2 < s < t$. We shall estimate $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively.

$\diamond$ (Estimates of $\mathcal{I}_1$) For the first term $\mathcal{I}_1$ in (4.12), we recall the definition of μ^k in (4.4) and apply Lemma A.1 to obtain, for $x \notin \mathscr{D}$, the following estimates:

$$\begin{aligned} &\int_s^t H_{xx}(x,\sigma;y,0;\mu^k)\mathrm{d}\sigma \\ =& \ V_x^k(x,0) \int_{-\infty}^x \left(H(w,t;y,0;\mu^k) - H(w,s;y,0;\mu^k)\right) \mathrm{d}w \\ &+ V_x^k(x,0) \int_s^t \left(\frac{1}{1+V^k(x,\sigma)} - \frac{1}{1+V^k(x,0)} \right) H_x(x,\sigma;y,0;\mu^k)\mathrm{d}\sigma \\ &+ \left(1+V^k(x,0)\right) \left(H(x,t;y,0;\mu^k) - H(x,s;y,0;\mu^k)\right) \\ &+ \left(1+V^k(x,0)\right) \int_s^t \left(\frac{(1+V^k(x,0)) - (1+V^k(x,\sigma))}{1+V^k(x,0)} \right) \partial_x \left(\frac{H_x(x,\sigma;y,0;\mu^k)}{1+V^k(x,\sigma)} \right) \mathrm{d}\sigma \\ &+ \left(1+V^k(x,0)\right) \int_s^t \partial_x \left(\frac{(1+V^k(x,0)) - (1+V^k(x,\sigma))}{1+V^k(x,0)} \right) \left(\frac{H_x(x,\sigma;y,0;\mu^k)}{1+V^k(x,\sigma)} \right) \mathrm{d}\sigma. \end{aligned}$$

We integrate the above representation with respect to y and x for $t \le t_\sharp$, apply Lemma A.1 and the ansatz (4.3) to obtain the estimates of $\mathcal{I}_1$ in (4.12) as follows:

$$\begin{aligned} \mathcal{I}_1 \le& \int_{\mathbb{R}\setminus\mathscr{D}} \int_{\mathbb{R}} \left| \int_s^t H_{xx}(x,\sigma;y,0;\mu^k)\mathrm{d}\sigma \right| |u_0^*(y)|\mathrm{d}y\mathrm{d}x \\ \le& \ O(1)\delta^2 \int_s^t \frac{1}{\sigma}\mathrm{d}\sigma + O(1)\delta^2 \int_s^t \int_0^\sigma \|V_\tau^k(\cdot,\tau)\|_{L_x^\infty}\mathrm{d}\tau \frac{1}{\sqrt{\sigma}}\mathrm{d}\sigma + \sqrt{\pi C_*^3}(1+2C_\sharp\delta)\delta \int_s^t \frac{1}{\sigma}\mathrm{d}\sigma \end{aligned}$$

$$
\begin{aligned}
&+O(1)\delta\int_s^t\int_0^\sigma \|V_\tau^k(\cdot,\tau)\|_{L_x^\infty}\mathrm{d}\tau\frac{1}{\sigma}\mathrm{d}\sigma+O(1)\delta^2\int_s^t\frac{1}{\sqrt{\sigma}}\mathrm{d}\sigma\\
\le&\left(\sqrt{\pi C_*^3}+O(1)\delta\right)\delta\log(1+\frac{t-s}{s})+O(1)\delta^2\frac{t-s}{\sqrt{t}}\\
\le& C_\sharp\delta\frac{t-s}{t},\quad \frac{t}{2}<s<t\le t_\sharp\ll 1. \qquad (4.13)
\end{aligned}
$$

Here the last inequality holds for properly large $C_\sharp$ and sufficiently small δ.

$\diamond$ (Estimates of $\mathcal{I}_2$) Next we consider the second part $\mathcal{I}_2$ in (4.12). We change the order of the integration, and apply integration by parts to obtain that

$$
\begin{aligned}
\mathcal{I}_2\le&\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\left[\int_s^t H_{xx}(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma\right]\partial_y p\left(1+V^k(y,\tau)\right)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^s\sum_{\alpha\in\mathscr{D}}\left[\int_s^t H_{xx}(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma p\left(1+V^k(y,\tau)\right)\right]_{y=\alpha^-}^{y=\alpha^+}\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_s^t\int_{\mathbb{R}\backslash\mathscr{D}}\left[\int_\tau^t H_{xx}(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma\right]\partial_y p\left(1+V^k(y,\tau)\right)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_s^t\sum_{\alpha\in\mathscr{D}}\left[\int_\tau^t H_{xx}(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma p\left(1+V^k(y,\tau)\right)\right]_{y=\alpha^-}^{y=\alpha^+}\mathrm{d}\tau\right|\mathrm{d}x\\
\equiv& T_1+T_2+T_3+T_4. \qquad (4.14)
\end{aligned}
$$

In what follows, we will denote the pressure term $p\left(1+V^k(y,\tau)\right)$ by $p(y,\tau)$, for short. Now, similarly to before, we will take advantage of the estimates of the time integral of $H_{xx}(x,\sigma;y,\tau;\mu^k)$. Actually, for T_1, we can use the heat equation to represent H_{xx} by H_t and split the integral into several terms as follows:

$$
\begin{aligned}
T_1\le&\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}V_x^k(x,\tau)\left[\int_{-\infty}^x\left(H(z,t;y,\tau;\mu^k)-H(z,s;y,\tau;\mu^k)\right)\mathrm{d}z\right]\partial_y p(y,\tau)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}V_x^k(x,\tau)\left[\int_s^t\frac{\left(\mu^k(x,\sigma)-\mu^k(x,\tau)\right)}{\mu}H_x(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma\right]\partial_y p(y,\tau)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\left(1+V^k(x,\tau)\right)\left[H(x,t;y,\tau;\mu^k)-H(x,s;y,\tau;\mu^k)\right]\partial_y p(y,\tau)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\left(1+V^k(x,\tau)\right)\partial_y p(y,\tau)\right.\\
&\quad\left.\cdot\left[\int_s^t\frac{\left(\mu^k(x,\sigma)-\mu^k(x,\tau)\right)}{\mu^k(x,\sigma)}\partial_x\left(\mu^k(x,\sigma)H_x(x,\sigma;y,\tau;\mu^k)\right)\mathrm{d}\sigma\right]\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\left(1+V^k(x,\tau)\right)\partial_y p(y,\tau)\right.\\
&\quad\left.\cdot\left[\int_s^t\partial_x\left(\frac{\left(\mu^k(x,\sigma)-\mu^k(x,\tau)\right)}{\mu^k(x,\sigma)}\right)\mu^k(x,\sigma)H_x(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma\right]\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
\equiv&\sum_{j=1}^5 T_{1j}, \qquad (4.15)
\end{aligned}
$$

where $\mu^k = \mu/(1+V^k)$ is as defined before. Now, we apply Lemma A.2 and the ansatz (4.3) to obtain the estimates of T_{11} of (4.15) as follows,

$$\begin{aligned}T_{11} &\le O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\int_s^t \left|V_x^k(x,\tau)\right| \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{\sigma-\tau}\left|\partial_y p(y,\tau)\right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\\ &\le O(1)\delta^2(t-s)\left|\log(t-s)\right|.\end{aligned}$$

Here the last inequality holds due to the fact that $0<s<t\le t_\sharp \ll 1$. With similar arguments, we obtain estimates of the other parts as follows:

$$\begin{aligned}T_{12} &\le O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\int_s^t |V_x^k(x,\tau)|\\ &\quad\cdot \sup_{\tau\le h\le\sigma}\left|\sqrt{h}\partial_h V^k(x,h)\right| \frac{\sigma-\tau}{\sqrt{\sigma}}\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{\sigma-\tau}\left|\partial_y p(y,\tau)\right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\\ &\le O(1)\delta^3\frac{(t-s)s}{\sqrt{t}},\\ T_{13} &\le O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\int_s^t \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}}\left|\partial_y p(y,\tau)\right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \le O(1)\delta(t-s)\left|\log(t-s)\right|,\\ T_{14} &\le O(1)\delta\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\int_s^t \frac{\sigma-\tau}{\sqrt{\sigma}}\\ &\quad\cdot\left(\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}} + \left|V_x^k(x,\sigma)\right|\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{\sigma-\tau}\right)\left|\partial_y p(y,\tau)\right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\\ &\le O(1)\delta^2\frac{(t-s)s}{\sqrt{t}},\\ T_{15} &\le O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^s\int_{\mathbb{R}\backslash\mathscr{D}}\int_s^t \left(\left|V_x^k(x,\sigma)\right| + \left|V_x^k(x,\tau)\right|\right)\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{\sigma-\tau}\left|\partial_y p(y,\tau)\right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\\ &\le O(1)\delta^2(t-s)\left|\log(t-s)\right|.\end{aligned}$$

Collecting the above estimates of T_{1j} and combining with the estimate (4.15), we obtain the estimate of T_1 as follows:

$$T_1 \le O(1)\delta\left[(t-s)\left|\log(t-s)\right| + \frac{(t-s)s}{\sqrt{t}}\delta\right]. \tag{4.16}$$

The estimate of the other terms can be obtained in a similar way, therefore we omit the details and directly provide the following estimates:

$$T_2 \le O(1)\delta\left[(t-s)\left|\log(t-s)\right| + \frac{(t-s)s}{\sqrt{t}}\delta\right], \tag{4.17}$$

$$T_3 \le O(1)\delta(t-s), \tag{4.18}$$

$$T_4 \le O(1)\delta(t-s). \tag{4.19}$$

Now we substitute the estimates of T_j in (4.16), (4.17), (4.18) and (4.19) into the estimate (4.14), and apply the fact that $\delta \ll 1$ and $0<s<t\ll 1$ to obtain the estimate of $\mathcal{I}_2$ as follows:

$$\mathcal{I}_2 \le O(1)\delta(t-s)|\log(t-s)|.$$

Now we substitute the estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$ into (4.12) and choose $C_\sharp$ properly to obtain that

$$\int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,t) - V_x^{k+1}(x,s)\right| \mathrm{d}x \le 2C_\sharp\delta\Big[\frac{t-s}{t} + (t-s)|\log(t-s)|\Big].$$

• (Hölder continuity of $\sum_{z\in\mathscr{D}} \left|V_x^{k+1}(\cdot,t)\big|_{z^-}^{z^+} - V_x^{k+1}(\cdot,s)\big|_{z^-}^{z^+}\right|$) In the final step, we will provide the Hölder continuity of the total jump, and thus we know that the BV norm of V^n is Hölder continuous with respect to t. The proof follows directly from the representation in Step 3. Actually, we apply the representation of V^{k+1} and follow the estimates of (4.10) to obtain that

$$\sum_{z\in\mathscr{D}}\left|V_x^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} - V_x^{k+1}(\cdot,s)\Big|_{z^-}^{z^+}\right| \le O(1)\int_s^t\left(1+\frac{1}{\sqrt{\tau}}\right)\mathrm{d}\tau\sum_{z\in\mathscr{D}}\left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| \le O(1)\delta\frac{t-s}{\sqrt{t}}.$$

This finishes the proof of the lemma. □

Remark 4.5 We provide several remarks on the *a priori* estimates.

1. The construction of $C_\sharp$ depends only on C_* and the coefficients of the initial step, i.e., μ, κ and c_v. When $\delta < \delta^*$ for some fixed positive number δ^*, we have that $\frac{1}{1+V^n}$ will be uniformly bounded according to the ansatz (4.3). Thus we immediately obtain that the coefficient C_* is uniformly bounded when we apply Lemmas A.1 and A.2 to the estimates of (4.2), because the heat equations in (4.2) have uniform bounded heat conductivity $\frac{1}{1+V^n}$. Therefore, we conclude that $C_\sharp$ and C_* are both uniformly bounded when δ is small. Moreover, as the choice of the small time $t_\sharp$ only depends on $C_\sharp$, we know that $t_\sharp$ is small but uniform when $\delta < \delta^*$.

2. Note that in Lemma 4.4, we obtain that V^{k+1} is BV in the sense of Definition 2.2. Actually, we obtain the expression of U_x^k as follows:

$$U_x^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} = \frac{1+V^k}{\mu}\left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k) + p(1+V^k)\right)\Big|_{z^-}^{z^+}.$$

Since $\left(\frac{\mu U_x^k}{1+V^{k-1}} - p\left(1+V^{k-1}\right)\right)$ are continuous, the above formula immediately implies that V^{k+1} is BV if and only if V^k is BV in the sense of Definition 2.2. Therefore, we only need to show that V^1 is BV. As U^1 is a solution of a homogeneous heat equation with a constant coefficient, we know that U^1 is smooth. Thus, V^1 is BV as v_0^* is BV, and it has the same discontinuities as the initial data v_0^*. Therefore, we can combine the estimates in Lemma 4.4 to conclude that V^{k+1} is BV in the sense of Definition 2.2.

3. From the ansatz (4.3), we have the BV estimate of V^n as follows,

$$\|V^n\|_{\mathrm{BV}} = \int_{\mathbb{R}\mathscr{D}}|\partial_x V^n(\cdot,t)|\mathrm{d}x + \sum_{z\in\mathscr{D}}\left|V^n(\cdot,t)\Big|_{z^-}^{z^+}\right| \le 2C_\sharp\delta + 2\sum_{z\in\mathscr{D}}\left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| \le (2C_\sharp+2)\delta.$$

4. According to the analysis in Section 2, we know that $H_{xy}(x,t;y,t_0)$ is a well defined function when x and y are not at the same discontinuity. Moreover, from the expression of $H_{xx}(x,t;y,t_0)$ in Section 2, the following integrals are also well defined functions for $x\notin\mathscr{D}$:

$$\int_0^t\int_{\mathbb{R}} H_{xx}\left(x,s;y,0;\mu^k\right)u_0^*(y)\mathrm{d}y\mathrm{d}s, \quad \int_0^t\int_0^s\int_{\mathbb{R}\setminus\mathscr{D}} H_{xxy}\left(x,s;y,\tau;\mu^k\right)p(1+V^k)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s.$$

4.2 Convergence of the Scheme

In this part, we will show the convergence of the sequence of approximate solutions (V^n, U^n) constructed from the iteration (4.2). We will mainly focus on the jump error estimates between

V^{n+1} and V^n, in which we carefully take advantage of the analyticity of pressure p to yield the control of the error.

By taking the difference at the $(n+1)$-th and n-th steps, we get the equation for the difference between two consecutive steps:

$$\begin{cases} \partial_t \left(V^{n+1}-V^n\right) - \partial_x \left(U^{n+1}-U^n\right) = 0, \\ \partial_t \left(U^{n+1}-U^n\right) - \partial_x \left(\dfrac{\mu\left(U^{n+1}-U^n\right)_x}{1+V^n} \right) = -\partial_x \left(\dfrac{\mu U^n_x \left(V^n - V^{n-1}\right)}{(1+V^n)(1+V^{n-1})} \right) + \mathcal{N}^n_1 - \mathcal{N}^{n-1}_1, \\ V^{n+1}(x,0) - V^n(x,0) = U^{n+1}(x,0) - U^n(x,0) = 0. \end{cases} \tag{4.20}$$

Here, for simplicity of presentation, we employ the notations

$$\mu^k := \frac{\mu}{1+V^k}, \qquad \mathcal{N}^n_1 - \mathcal{N}^{n-1}_1 = -\partial_x \left(p(1+V^n) - p(1+V^{n-1}) \right). \tag{4.21}$$

Lemma 4.6 $(U^{n+1}-U^n)$ There exists a positive constant $C_\flat$ such that, for sufficiently small δ and $t_\sharp$, the error of velocity between two iteration steps has the following estimates when $0 < t < t_\sharp$:

$$\begin{aligned} \left\| \left(U^{n+1}-U^n\right)(\cdot,t) \right\|_\infty &\le C_\flat \left(\sqrt{t}+\delta\right) \left|\left|\left| V^n - V^{n-1} \right|\right|\right|_\infty, \\ \left\| \left(U^{n+1}-U^n\right)(\cdot,t) \right\|_1 &\le C_\flat \left(\sqrt{t}+\delta\right) \left|\left|\left| V^n - V^{n-1} \right|\right|\right|_1, \\ \frac{\sqrt{t}}{|\log t|} \left\| \left(U^{n+1}_x-U^n_x\right)(\cdot,t) \right\|_\infty &\le C_\flat \left(\sqrt{t}+\delta\right) \Big(\left|\left|\left| V^n - V^{n-1} \right|\right|\right|_\infty + \left|\left|\left| V^n - V^{n-1} \right|\right|\right|_{\mathrm{BV}} \\ &\quad + \left|\left|\left| V^n - V^{n-1} \right|\right|\right|_1 + \left|\left|\left| \frac{\sqrt{\tau}}{|\log\tau|}\left(U^n_x - U^{n-1}_x\right) \right|\right|\right|_\infty \Big), \\ \frac{\sqrt{t}}{|\log t|} \left\| \left(U^{n+1}_x-U^n_x\right)(\cdot,t) \right\|_1 &\le C_\flat \left(\sqrt{t}+\delta\right) \Big(\left|\left|\left| V^n - V^{n-1} \right|\right|\right|_\infty + \left|\left|\left| V^n - V^{n-1} \right|\right|\right|_1 \\ &\quad + \left|\left|\left| V^n - V^{n-1} \right|\right|\right|_{\mathrm{BV}} + \left|\left|\left| \frac{\sqrt{\tau}}{|\log\tau|}\left(U^n_x - U^{n-1}_x\right) \right|\right|\right|_1 \\ &\quad + \left|\left|\left| \frac{\sqrt{\tau}}{|\log\tau|}\left(U^n_x - U^{n-1}_x\right) \right|\right|\right|_\infty \Big). \end{aligned}$$

Proof For simplicity, we will only show the detailed proof of the zeroth order estimate, as the higher order estimates can be obtained similarly. From the equation of difference (4.20), one has the representation

$$\begin{aligned} \left(U^{n+1}-U^n\right)(x,t) &= \int_0^t \int_{\mathbb{R}} H_y(x,t;y,\tau;\mu^n) \frac{\mu U^n_y \left(V^n - V^{n-1}\right)}{(1+V^n)(1+V^{n-1})}(y,\tau)\mathrm{d}y\mathrm{d}\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} H_y(x,t;y,\tau;\mu^n) \left(p(1+V^n) - p(1+V^{n-1}) \right)(y,\tau)\mathrm{d}y\mathrm{d}\tau. \end{aligned} \tag{4.22}$$

By Lemma A.2 and iteration estimates (4.3), we apply the representation (4.22) to obtain the following L^∞ error estimates:

$$\begin{aligned} \left| \left(U^{n+1}-U^n\right)(x,t) \right| &\le O(1) \int_0^t \int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau} \frac{\delta}{\sqrt{\tau}} \left| V^n - V^{n-1} \right| (y,\tau)\mathrm{d}y\mathrm{d}\tau \\ &\quad + O(1) \int_0^t \int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau} \left| V^n - V^{n-1} \right| (y,\tau)\mathrm{d}y\mathrm{d}\tau \end{aligned}$$

$$\leq O(1)\delta\left|\left|\left|V^n - V^{n-1}\right|\right|\right|_\infty + O(1)\sqrt{t}\left|\left|\left|V^n - V^{n-1}\right|\right|\right|_\infty. \tag{4.23}$$

With similar calculations, we find the L^1_x error estimate as follows:

$$\begin{aligned}\int_{\mathbb{R}} \left|\left(U^{n+1} - U^n\right)(x,t)\right| \mathrm{d}x &\leq O(1)\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau}\frac{\delta}{\sqrt{\tau}}\left|V^n - V^{n-1}\right|(y,\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\ &\quad + O(1)\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau}\left|V^n - V^{n-1}\right|(y,\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\ &\leq O(1)\delta\left|\left|\left|V^n - V^{n-1}\right|\right|\right|_1 + O(1)\sqrt{t}\left|\left|\left|V^n - V^{n-1}\right|\right|\right|_1.\end{aligned} \tag{4.24}$$

□

Next we study the iteration error estimates of V^n. Differently from the previous estimates for U^n, as V^n is a BV function, we need to further estimate the evolution of the error for the discontinuities. The estimates are summarized in the following Lemma:

Lemma 4.7 ($V^{n+1} - V^n$) There exists a positive constant $C_\flat$ such that, for sufficiently small δ and $t_\sharp$, the error of velocity between the two iteration steps has the following estimates when $0 < t < t_\sharp$:

$$\left|V^{n+1}(x,t) - V^n(x,t)\right| \leq C_\flat\left(\delta + \sqrt{t}\right)|||V^n - V^{n-1}|||_\infty,$$

$$\|V^{n+1}(\cdot,t) - V^n(\cdot,t)\|_1 \leq C_\flat\left(\delta + \sqrt{t}\right)\left|\left|\left|V^n - V^{n-1}\right|\right|\right|_1,$$

$$\begin{aligned}\left|\left(V^{n+1}(\cdot,t) - V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\right| &\leq C_\flat\left(\delta + \sqrt{t}\right)\sqrt{t}\sup_{0<\tau<t}\left|\left(V^n(\cdot,\tau) - V^{n-1}(\cdot,\tau)\right)\Big|_{z^-}^{z^+}\right| \\ &\quad + C_\flat\delta\Big(\left|\left|\left|V^n - V^{n-1}\right|\right|\right|_\infty + \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1} - U_x^n\right)\right|\right|\right|_\infty\Big),\end{aligned}$$

$$\begin{aligned}\|V^{n+1}(\cdot,t) - V^n(\cdot,t)\|_{\mathrm{BV}} &\leq C_\flat\left(\sqrt{t} + \delta\right)\left(\left|\left|\left|V^n - V^{n-1}\right|\right|\right|_\infty + \left|\left|\left|V^n - V^{n-1}\right|\right|\right|_1\right) \\ &\quad + C_\flat\left(\sqrt{t} + \delta\right)\left(|||V^n - V^{n-1}|||_{\mathrm{BV}} + \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1} - U_x^n\right)\right|\right|\right|_\infty\right).\end{aligned}$$

Proof First, one directly has the equation for $V^{n+1} - V^n$ as follows:

$$\begin{cases}\partial_t\left(V^{n+1} - V^n\right) - \partial_x\left(U^{n+1} - U^n\right) = 0, \\ \partial_t\left(U^{n+1} - U^n\right) - \partial_x\left(\dfrac{\mu\left(U^{n+1} - U^n\right)_x}{1 + V^n}\right) \\ = -\partial_x\left(p(1 + V^n) - p(1 + V^{n-1})\right) + \partial_x\left(\dfrac{\mu\left(V^{n-1} - V^n\right)U_x^n}{(1 + V^n)(1 + V^{n-1})}\right), \\ V^{n+1}(x,0) - V^n(x,0) = U^{n+1}(x,0) - U^n(x,0) = 0.\end{cases} \tag{4.25}$$

Then one applies Duhamel's principle to solve the error of U^n in the second equation of (4.25), and substitutes it into the first equation in (4.25) to obtain that

$$\begin{aligned}&V^{n+1}(x,t) - V^n(x,t) \\ &= \int_0^t\int_{\mathbb{R}\setminus\mathscr{D}}\int_\tau^t H_{xy}(x,s;y,\tau;\mu^n)\left(p(1 + V^n) - p(1 + V^{n-1})\right)(y,\tau)\mathrm{d}s\mathrm{d}y\mathrm{d}\tau \\ &\quad + \int_0^t\int_{\mathbb{R}\setminus\mathscr{D}}\int_\tau^t H_{xy}(x,s;y,\tau;\mu^n)\left(\frac{\mu\left(V^n - V^{n-1}\right)U_y^n}{(1 + V^n)(1 + V^{n-1})}\right)(y,\tau)\mathrm{d}s\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{4.26}$$

Now, for the L^∞ norm and L^1 norm estimates, the equation of the iteration error is used to yield the desired estimates. For the jump estimate, the representation of the jump in (4.10) and the analyticity of pressure p are employed. Moreover, the Hölder continuity proven before plays a crucial role in the estimates of the BV estimates. For simplicity, we will only show the detailed proof of the BV estimates.

• (Step 1: Jump part) First, for the jump estimate, recalling the representation of the jump in (4.10), one has the following representation of the jump error:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(V^{n+1}(\cdot,t)-V^{n}(\cdot,t)\right)\Big|_{z^-}^{z^+}\\
&=\left(\frac{V^n}{\mu}-\frac{V^{n-1}}{\mu}\right)\Big|_{z^-}^{z^+}\left(\frac{\mu U_x^{n+1}}{1+V^n}-p(1+V^n)\right)\\
&\quad+\frac{V^{n-1}}{\mu}\Big|_{z^-}^{z^+}\left(\frac{\mu\left(U_x^{n+1}-U_x^n\right)}{1+V^n}+\frac{\mu U_x^n\left(V^{n-1}-V^n\right)}{\left(1+V^{n-1}\right)\left(1+V^n\right)}+p(1+V^{n-1})-p(1+V^n)\right)\\
&\quad+\left(V^n-V^{n-1}\right)\Big|_{z^-}^{z^+}\frac{p(1+V^n(z^+))}{\mu}+\frac{1+V^{n-1}}{\mu}\Big|_{z^-}^{z^+}\left(p(1+V^n(z^+))-p(1+V^{n-1}(z^+))\right)\\
&\quad+p(1+V^n)\Big|_{z^-}^{z^+}\frac{V^n(z^-)-V^{n-1}(z^-)}{\mu}+\left(p(1+V^n)-p(1+V^{n-1})\right)\Big|_{z^-}^{z^+}\frac{V^{n-1}(z^-)}{\mu}. \qquad (4.27)
\end{aligned}$$

Now, all of the above terms in (4.27) have proper estimates except for the last term, in which we have to take advantage of the good structure of pressure p to yield the desired estimate. Indeed, as $p(s)$ is analytic around $s=1$, without loss of generality, we may assume that $p(s)$ has the following expansion around 1:

$$p(s)=\sum_{k=0}^{+\infty}c_k(s-1)^k,\quad |s-1|\le r_0\ll 1. \qquad (4.28)$$

Then, the jump of the comparison of pressure can be expressed by the expansion (4.28) as

$$\begin{aligned}
&\left(p(1+V^n)-p(1+V^{n-1})\right)\Big|_{z^-}^{z^+}\\
&=\sum_{k=0}^{+\infty}c_k\left[\left(V_+^n-V_-^n\right)-\left(V_+^{n-1}-V_-^{n-1}\right)\right]\sum_{i=1}^{k}(V_+^n)^{k-i}(V_-^n)^{i-1}\\
&\quad+\sum_{k=0}^{+\infty}c_k\left(V_+^{n-1}-V_-^{n-1}\right)\left(\sum_{i=1}^{k}(V_+^n)^{k-i}(V_-^n)^{i-1}-\sum_{i=1}^{k}(V_+^{n-1})^{k-i}(V_-^{n-1})^{i-1}\right)\\
&=\mathcal{I}_1+\mathcal{I}_2. \qquad (4.29)
\end{aligned}$$

Now we assume δ to be sufficiently small such that $2C_\sharp\delta<r_0$. Then, for $\mathcal{I}_1$ in (4.29), we simply apply the ansatzes (4.3) and (4.28) to obtain that

$$|\mathcal{I}_1|\le O(1)\left|\left(V_+^n-V_-^n\right)-\left(V_+^{n-1}-V_-^{n-1}\right)\right|\sum_{k=1}^{+\infty}|c_k|k\delta^{k-1}\le O(1)\left|\left(V^n-V^{n-1}\right)\Big|_{z^-}^{z^+}\right|.$$

Similarly, for $\mathcal{I}_2$ in (4.29), we apply ansatzes (4.3) and (4.28) to yield that

$$\begin{aligned}
|\mathcal{I}_2|&\le\left|V_+^{n-1}-V_-^{n-1}\right|\sum_{k=0}^{+\infty}\sum_{i=1}^{k}|c_k|\left|(V_+^n)^{k-i}(V_-^n)^{i-1}-(V_+^{n-1})^{k-i}(V_-^{n-1})^{i-1}\right|\\
&=\left|V^{n-1}\Big|_{z^-}^{z^+}\right|\sum_{k=0}^{+\infty}\sum_{i=1}^{k}|c_k|
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left|\left((V_+^n)^{k-i} - (V_+^{n-1})^{k-i}\right)(V_-^n)^{i-1} + (V_+^{n-1})^{k-i}\left((V_-^n)^{i-1} - (V_-^{n-1})^{i-1}\right)\right| \\
\leq{}& O(1)\left|V^{n-1}\Big|_{z^-}^{z^+}\right| \sum_{k=0}^{+\infty}\sum_{i=1}^{k}|c_k| \\
&\cdot \left(\left|V_+^n - V_+^{n-1}\right|(k-i)\delta^{k-i-1}\delta^{i-1} + \left|V_-^n - V_-^{n-1}\right|(i-1)\delta^{i-2}\delta^{k-i}\right) \\
\leq{}& O(1)\left|V^{n-1}\Big|_{z^-}^{z^+}\right| |||V^n - V^{n-1}|||_\infty \sum_{k=0}^{+\infty}\sum_{i=1}^{k}|c_k|(k-1)\delta^{k-2} \\
\leq{}& O(1)\left|V^{n-1}\Big|_{z^-}^{z^+}\right| |||V^n - V^{n-1}|||_\infty \sum_{k=0}^{+\infty}|c_k|k(k-1)\delta^{k-2} \\
\leq{}& O(1)\left|V^{n-1}\Big|_{z^-}^{z^+}\right| |||V^n - V^{n-1}|||_\infty.
\end{aligned}
$$

Combining (4.29) and the above estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$, we obtain the following estimates:

$$
\begin{aligned}
&\left|\left(p(1+V^n) - p(1+V^{n-1})\right)\Big|_{z^-}^{z^+}\right| \\
\leq{}& O(1)\left|\left(V^n - V^{n-1}\right)\Big|_{z^-}^{z^+}\right| + O(1)\left|V^{n-1}\Big|_{z^-}^{z^+}\right| |||V^n - V^{n-1}|||_\infty.
\end{aligned} \tag{4.30}
$$

Then, we note that the initial error of the comparison in (4.27) is equal to zero. Therefore, we integrate (4.27) with respect to t and recall (4.30) and the BV estimate in Remark 4.5 to obtain the following estimate:

$$
\begin{aligned}
&\left|\left(V^{n+1}(\cdot,t) - V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\right| \\
={}& O(1)\int_0^t \left|\left(V^n - V^{n-1}\right)\Big|_{z^-}^{z^+}\right| \left(\frac{\delta}{\sqrt{s}} + 1\right)\mathrm{d}s \\
&+ O(1)\int_0^t \left|V^{n-1}\Big|_{z^-}^{z^+}\right| \left(\frac{\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1} - U_x^n\right)\right|\right|\right|_\infty |\log s|}{\sqrt{s}} + \left(\frac{\delta}{\sqrt{s}} + 1\right)|||V^{n-1} - V^n|||_\infty\right)\mathrm{d}s \\
&+ O(1)\int_0^t \left|\left(V^n - V^{n-1}\right)\Big|_{z^-}^{z^+}\right|\mathrm{d}s + O(1)\int_0^t \left(\left|V^n\Big|_{z^-}^{z^+}\right| + \left|V^{n-1}\Big|_{z^-}^{z^+}\right|\right)\mathrm{d}s|||V^n - V^{n-1}|||_\infty \\
\lesssim{}& \left(\delta + \sqrt{t}\right)\sqrt{t}\sup_{0<\tau<t}\left|\left(V^n(\cdot,\tau) - V^{n-1}(\cdot,\tau)\right)\Big|_{z^-}^{z^+}\right| \\
&+ \delta\sqrt{t}|\log t|\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1} - U_x^n\right)\right|\right|\right|_\infty + \delta\sqrt{t}|||V^n - V^{n-1}|||_\infty.
\end{aligned} \tag{4.31}
$$

Taking the summation of z, one then obtains that

$$
\begin{aligned}
&\sum_{z\in\mathscr{D}}\left|\left(V^{n+1}(\cdot,t) - V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\right| \\
\lesssim{}& (\delta + t)\left(|||V^n - V^{n-1}|||_\infty + |||V^n - V^{n-1}|||_{\mathrm{BV}} + \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1} - U_x^n\right)\right|\right|\right|_\infty\right).
\end{aligned} \tag{4.32}
$$

• (Step 2: Absolutely continuous part) Next, for all $x \notin \mathscr{D}$, the derivative of $V^{n+1} - V^n$ can be defined almost everywhere from the representation (4.26). Therefore, we have the following estimate:

$$
\int_{\mathbb{R}\setminus\mathscr{D}}\left|V_x^{n+1}(x,t) - V_x^n(x,t)\right|\mathrm{d}x
$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_0^t \int_{\mathbb{R}} \left[\int_\tau^t H_{xxy}(x,\sigma;y,\tau;\frac{1}{1+V^n}) \mathrm{d}\sigma \right] \left[p^n(y,\tau) - p^{n-1}(y,\tau) \right] \mathrm{d}y\mathrm{d}\tau \right| \mathrm{d}x \\
&\quad + \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_0^t \int_{\mathbb{R}} \left[\int_\tau^t H_{xxy}(x,\sigma;y,\tau;\frac{1}{1+V^n}) \mathrm{d}\sigma \right] \left[\frac{\left(V^n - V^{n-1}\right) U_y^n}{(1+V^n)(1+V^{n-1})} \right] \mathrm{d}y\mathrm{d}\tau \right| \mathrm{d}x \\
&\equiv \mathcal{I}_1 + \mathcal{I}_2. \qquad (4.33)
\end{aligned}
$$

For simplicity, we use the following notation to represent the pressure term for the time being:

$$p^n(y,\tau) = p(v^n)(y,\tau).$$

Then, from the heat kernel estimates in Lemma A.1, we obtain that

$$
\begin{aligned}
&\int_\tau^t H_{xxy}(x,\sigma;y,\tau;\frac{1}{v}) \mathrm{d}\sigma \\
= &- v_x(x,\tau) \left[\int_{-\infty}^x \left(H_y(z,t;y,\tau) + \delta'(z-y) \right) \mathrm{d}z - \int_\tau^t \left(\frac{1}{v(x,\sigma)} - \frac{1}{v(x,\tau)} \right) H_{xy}(x,\sigma;y,\tau) \mathrm{d}\sigma \right] \\
&+ v(x,\tau) \left[H_y(x,t;y,\tau) + \delta'(x-y) - \int_\tau^t \left(\frac{1}{v(x,\sigma)} - \frac{1}{v(x,\tau)} \right) H_{xxy}(x,\sigma;y,\tau;\frac{1}{v}) \mathrm{d}\sigma \right] \\
&- v(x,\tau) \int_\tau^t \partial_x \left(\frac{1}{v(x,\sigma)} - \frac{1}{v(x,\tau)} \right) H_{xy}(x,\sigma;y,\tau;\frac{1}{v}) \mathrm{d}\sigma. \qquad (4.34)
\end{aligned}
$$

Plugging (4.34) into previous expression (4.33), one has that

$$
\begin{aligned}
\mathcal{I}_1 \le\ & O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \int_{\mathbb{R}} |V_x^n(x,\tau)| \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{\sqrt{t-\tau}} \left| p^n(y,\tau) - p^{n-1}(y,\tau) \right| \mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&+ O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t |V_x^n(x,\tau)| \left| p^n(x,\tau) - p^{n-1}(x,\tau) \right| \mathrm{d}\tau\mathrm{d}x \\
&+ O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \int_{\mathbb{R}} \int_\tau^t |V_x^n(x,\tau)| \frac{\delta(\sigma-\tau)}{\sqrt{\sigma}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}} \left| p^n(y,\tau) - p^{n-1}(y,\tau) \right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&+ O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau} \left| p^n(y,\tau) - p^{n-1}(y,\tau) \right| \mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&+ O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \left| p_x^n(x,\tau) - p_x^{n-1}(x,\tau) \right| \mathrm{d}\tau\mathrm{d}x \\
&+ O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \int_{\mathbb{R}} \int_\tau^t \frac{\delta(\sigma-\tau)}{\sqrt{\sigma}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^2} \left| p^n(y,\tau) - p^{n-1}(y,\tau) \right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&+ O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \int_{\mathbb{R}} \int_\tau^t |V_x^n(x,\sigma) - V_x^n(x,\tau)| \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}} \left| p^n(y,\tau) - p^{n-1}(y,\tau) \right| \mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
\equiv & \sum_{j=1}^7 \mathcal{I}_{1j}.
\end{aligned}
$$

For $\mathcal{I}_{15}$, one observes that

$$
\begin{aligned}
&p_x^n(x,\tau) - p_x^{n-1}(x,\tau) \\
&= p'(1+V^{n-1}(x,\tau)) \left(V_x^n - V_x^{n-1} \right)(x,\tau) + V_x^n(x,\tau) \left(p'(1+V^n) - p'(1+V^{n-1}) \right)(x,\tau) \\
&= O(1) \left| V_x^n - V_x^{n-1} \right| + O(1) \left| V^n - V^{n-1} \right| |V_x^n| .
\end{aligned}
$$

Hence we have the estimate

$$\mathcal{I}_{15} \le O(1)t\,|||V^n|||_{\rm BV}\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_\infty + O(1)t\,\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_{\rm BV}.$$

To estimate $\mathcal{I}_{17}$, we apply Hölder estimates in Lemma 4.4 to obtain that

$$\begin{aligned}\mathcal{I}_{17} &\le O(1)\int_0^t\int_\tau^t\left[\int_{\mathbb{R}\backslash\mathscr{D}}|V_x^n(x,\sigma)-V_x^n(x,\tau)|\,{\rm d}x\right]\frac{1}{(\sigma-\tau)^{3/2}}{\rm d}\sigma{\rm d}\tau\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1\\ &\le O(1)\delta\sqrt{t}\big(1+t|\log t|\big)\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1.\end{aligned}$$

The other terms can be directly estimated by using similar calculations, so we omit the details and provide the following estimates:

$$\begin{aligned}&\mathcal{I}_{11} \le O(1)\sqrt{t}|||V^n|||_{\rm BV}\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1, \quad \mathcal{I}_{12} \le O(1)t|||V^n|||_{\rm BV}\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_\infty,\\ &\mathcal{I}_{13} \le O(1)t\delta|||V^n|||_{\rm BV}\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_\infty, \quad \mathcal{I}_{14} \le O(1)\sqrt{t}\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1,\\ &\mathcal{I}_{16} \le O(1)\delta t\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1.\end{aligned}$$

Now, combining the above estimates, and in view of (4.3), we have the following estimate of $\mathcal{I}_1$:

$$\mathcal{I}_1 \le O(1)\sqrt{t}\Big(\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_\infty + \big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1 + \big|\big|\big|V^n - V^{n-1}\big|\big|\big|_{\rm BV}\Big). \tag{4.35}$$

Similarly, we can obtain the estimates of $\mathcal{I}_2$ as

$$\mathcal{I}_2 \le O(1)\delta\left(\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1 + \big|\big|\big|V^n - V^{n-1}\big|\big|\big|_\infty + \left|\left|\left|\frac{\sqrt{\tau}}{\log\tau}(U_x^n - U_x^{n-1})\right|\right|\right|_\infty\right).$$

Together with (4.33) and (4.35), we conclude the error estimates of the BV norm of V^n as follows:

$$\begin{aligned}\int_{\mathbb{R}\backslash\mathscr{D}}\big|V_x^{n+1}(x,t)-V_x^n(x,t)\big|\,{\rm d}x \le O(1)\left(\sqrt{t}+\delta\right)\Big(&\big|\big|\big|V^n - V^{n-1}\big|\big|\big|_\infty + \big|\big|\big|V^n - V^{n-1}\big|\big|\big|_1\\ &+ \big|\big|\big|V^n - V^{n-1}\big|\big|\big|_{\rm BV} + \left|\left|\left|\frac{\sqrt{\tau}}{\log\tau}(U_x^n - U_x^{n-1})\right|\right|\right|_\infty\Big).\end{aligned} \tag{4.36}$$

□

Now we are ready to provide the main result of this section, i.e., the local-in-time existence of the weak solution to the nonlinear Navier-Stokes equation (1.1).

Proposition 4.8 Suppose that initial data (v_0^*, u_0^*) satisfies the condition (4.1) for small δ. Then there exists a sufficiently small positive constant $t_\sharp$ such that equation (1.1) admits a weak solution in the sense of Definition 2.1, $(v,u)=(v^*+1,u^*),\ t<t_\sharp$. Moreover, the solution has the properties

$$\begin{cases}\delta>0, \quad 0<t<t_\sharp \ll 1,\\ \max\Big\{\|u(\cdot,t)\|_{L_x^1},\ \|u(\cdot,t)\|_{L_x^\infty},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L_x^1},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L_x^\infty}\Big\} \le 2C_\sharp\delta,\\ \max\Big\{\displaystyle\int_{\mathbb{R}\backslash\mathscr{D}}|v_x(x,t)|{\rm d}x,\ \|v(\cdot,t)-1\|_{L_x^1},\ \|v(\cdot,t)-1\|_{L_x^\infty},\ \sqrt{t}\,\|v_t(\cdot,t)\|_{L_x^\infty}\Big\} \le 2C_\sharp\delta,\\ v^*=v_c^*+v_d^*, \quad v_d^*(x,t)=\displaystyle\sum_{z<x,z\in\mathscr{D}}v^*\Big|_{z^-}^{z^+}H(x-z), \quad v_c^* \text{ is AC},\\ \left|v(\cdot,t)\Big|_{x=z^-}^{x=z^+}\right| \le 2\left|v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+}\right|, \quad z\in\mathscr{D},\end{cases} \tag{4.37}$$

where $H(x)$ is the Heaviside step function, and AC means "absolutely continuous". In particular, the total variation of $v(x,t)$ is Hölder continuous in time for any $t>0$ in the following sense:

$$\|v(\cdot,t)-v(\cdot,s)\|_{\mathrm{BV}} \le 2C_\sharp\delta\Big[\frac{t-s}{t}+(t-s)|\log(t-s)|\Big], \quad 0<s<t. \tag{4.38}$$

Proof We will split the proof into four steps.

• (Step 1: Strong convergence) For simplicity of notation, we define the following functional of the iteration error:

$$\begin{aligned}&\mathscr{F}\left[V^{n+1}-V^n, U^{n+1}-U^n\right]\\ \equiv\ &\left|\!\left|\!\left|V^{n+1}-V^n\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|V^{n+1}-V^n\right|\!\right|\!\right|_1+\left|\!\left|\!\left|V^{n+1}-V^n\right|\!\right|\!\right|_{\mathrm{BV}}+\left|\!\left|\!\left|U^{n+1}-U^n\right|\!\right|\!\right|_\infty\\ &+\left|\!\left|\!\left|U^{n+1}-U^n\right|\!\right|\!\right|_1+\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)\right|\!\right|\!\right|_1.\end{aligned} \tag{4.39}$$

Here we denote that

$$\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)\right|\!\right|\!\right|_\infty=\sup_{0<\tau<t_\sharp}\left\|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)(\cdot,\tau)\right\|_{L_x^\infty},$$

and similarly for other $|\!|\!|\cdot|\!|\!|$ norms.

Then we combine Lemma 4.6 and Lemma 4.7 to obtain that the iteration error has the following contraction property for sufficiently small δ and $t_\sharp$:

$$\mathscr{F}\left[V^{n+1}-V^n, U^{n+1}-U^n\right]\le C_\flat\left(\delta+\sqrt{t_\sharp}\right)\mathscr{F}\left[V^n-V^{n-1}, U^n-U^{n-1}\right].$$

From the analysis performed in Remark 4.5, we know that $C_\flat$ is uniformly bounded when δ is sufficiently small. Thus, for enough small δ, $\mathscr{F}\left[V^{n+1}-V^n, U^{n+1}-U^n\right]$ is a Cauchy sequence. As the L^∞ and L^1 spaces are complete, we immediately obtain that the iteration scheme admits a strong limit (v^*,u^*) in the following functional spaces:

$$\begin{cases} v^*(x,t)\in L^\infty\left(0,t_\sharp;L^1(\mathbb{R})\right),\\ u^*(x,t)\in L^\infty\left(0,t_\sharp;L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\right), \quad \sqrt{t}u_x^*(x,t)\in L^\infty\left(0,t_\sharp;L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\right).\end{cases} \tag{4.40}$$

Now we let $(v,u)=(v^*+1,u^*)$. Then, the strong convergence immediately implies that (v,u) is a weak solution to the original Navier-Stokes equation (1.1) in the distribution sense. Indeed, we can multiply a test function to equation (4.2) to get the weak formulation for the iteration equation. Then, by strong convergence, we can pass to limit to prove that these are weak solutions to the original nonlinear equation.

• (Step 2: Regularity) Second, for the regularity of the solutions, according to Lemma 4.3 and Lemma 4.4, we know that $\|V^n\|_{\mathrm{BV}}$ and $\sqrt{t}\|U_x^n\|_{L^1\cap L^\infty}$ are uniformly bounded for n. Moreover, from the above analysis, V^n and U^n are convergent in the L^1 sense. Therefore, we apply Helly's selection Theorem and the estimates in Lemma 4.3 and Lemma 4.4 to conclude that the limit (v^*,u^*) has the following properties:

$$\|v^*(\cdot,t)\|_{\mathrm{BV}}\le 2C_\sharp\delta, \quad \sqrt{t}\|u_x^*(\cdot,t)\|_{L^1\cap L^\infty}\le 2C_\sharp\delta. \tag{4.41}$$

Similarly, as the Hölder continuity estimates of $V^n(x,t)$ in Lemma 4.4 are uniform, we apply the convergence of the iteration scheme and Helly's theorem to conclude the Hölder continuity estimate of the limit solution $v(x,t)$.

• (Step 3: v is BV) Third, we need the jump estimate. According to Remark 4.5, we know that V^n is a BV function, i.e., it can be decomposed as follows:

$$V^n = V_c^n + V_d^n,\ V_d^n(x,t) = \sum_{z<x, z\in\mathscr{D}} d^n(z,t)H(x-z),\ d^n(z,t)\Big|_{z\in\mathscr{D}} = V^n(\cdot,t)\Big|_{z^-}^{z^+},\ V_c^n \text{ is AC.}$$

Here AC means "absolutely continuous", and $H(x)$ is the Heaviside function. From Lemma 4.7, we can actually show that $\left|\left(V^{n+1}(\cdot,t) - V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\right|$ is also a Cauchy sequence, and thus the jump at time t admits a limit $d(z,t)$ for $z \in \mathscr{D}$. Now we follow [1] to construct the step function

$$v_d^*(x,t) := \sum_{z<x} d(z,t)H(x-z), \quad |d(z,t)| \le 2\left|v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+}\right|, \quad z\in\mathscr{D}, \tag{4.42}$$

where the estimate of $|d(z,t)|$ is due to the strong convergence and the uniform boundedness of the jump at each iteration step. From the above analysis, we know that V_d^n converges to v_d^* pointwisely, so we need to show that v_d^* contains exactly all of the discontinuity of v^*. Indeed, according to (4.11) in Lemma 4.4, we have the following estimates:

$$|\partial_x\left(V_c^n(x,t)\right)| = |V_x^n(x,t)| \le |(v_0^*(x))_x| + 2C_\sharp\delta, \quad 0<t<t_\sharp, \quad \text{a.e. } x\notin\mathscr{D}.$$

As the right-hand side in the above formula is independent of t and n, we conclude that V_c^n are equi-continuous and uniformly bounded in any bounded closed interval, which yields an absolutely continuous limit $v_c^*(x,t)$, by Ascoli-Arzela. Finally, we conclude that

$$v^* = \lim_{n\to+\infty} V^n = \lim_{n\to+\infty} V_c^n + \lim_{n\to+\infty} V_d^n = v_c^* + v_d^*. \tag{4.43}$$

Thus, the solution v^* is a BV function, and has the exactly the same discontinuity as the initial data.

• (Step 4: Flux continuity) Finally, we show the continuity of the flux of u. In fact, we can substitute v into the equation of u and obtain an inhomogeneous linear equation of $\bar{u}$ as follows:

$$\bar{u}_t = \left(\frac{\mu\bar{u}_x}{v} - p\right)_x.$$

This is of the same form as in Remark 2.3. By virtue of the regularity of u and $v_t = u_x$, we infer that v is Lipschitz continuous with respect to t, which also follows from the Lipshitz continuity of $p(v)$, since $p(v)$ is analytic.

Therefore, we apply Remark 2.3 and immediately conclude the existence of the solution $\bar{u}$, and the continuity of the flux for the linear equation. Again, as the equation is linear for $\bar{u}$, it thus has a unique weak solution. Therefore, u must coincide with $\bar{u}$ and thus has a continuous flux.

Now, we combine the above assertions, i.e., the strong convergence in the sense of (4.40), the estimates in (4.41), and the property of v in (4.42) and (4.43), to finish the proof of the desired properties of the solution. □

Remark 4.9 We close this section with the following remarks:

1. As pointed out in Remark 1.5, if u_0^* satisfies the stronger assumption $\|u_0^*\|_{L^1\cap \mathrm{BV}} \ll 1$, then the constructed weak solution satisfes that

$$\|u_x(\cdot,t)\|_{L_x^1} \le 2C_\sharp\delta.$$

2. From Proposition 4.8, the solution has the following regularity with respect to x:

$$\begin{cases} v^*(x,t) \in L^\infty\left(0,t_\sharp; L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\cap \mathrm{BV}\right), \\ u^*(x,t) \in L^\infty\left(0,t_\sharp; L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\right), \quad \sqrt{t}u_x^*(x,t)\in L^\infty\left(0,t_\sharp; L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\right). \end{cases}$$

However, we have not shown that u is also regular with respect to t.

3. Proposition 4.8 states that v^* is a BV function in the sense of that it can be represented as a sum of the step function and an absolutely continuous function. Moreover, v^* has the same discontinuity as initial data v_0^*. Moreover, according to (4.38), the total variation of $v(x,t)$ may not be Hölder continuous at $t=0$.

5 Regularity and Uniqueness

By Proposition 4.8 and Remark 4.9, for the weak solution constructed in Section 3 as the limit of iteration scheme (4.2), we already have the first order regularity with respect to x and the continuity of the flux. In this section, we further investigate the regularity of the solution; for instance, the regularity with respect to t. As a consequence, we will show that the weak solution we constructed is actually the unique weak solution in the function space it belongs to, and thus finish the proof of Theorems 1.2 and 1.3.

5.1 Regularity and Proof of Theorem 1.2

According to Proposition 4.8, the constructed solutions (v,u) are small in the short time. In particular, the smallness of the BV estimates of the specific volume v allows us to apply the results in Section 2 to construct the corresponding heat kernel $H(x,t;y,t_0;\frac{1}{v})$ when $t<t_\sharp$. Then, we can take advantage of the integral representation of u to study the regularity of u with respect to t.

Lemma 5.1 Suppose that the initial data satisfies (4.1), and that (v,u) is the weak solution constructed in Proposition 4.8. Then $u_t(x,t)$ is well-defined for any x when $t>0$. Moreover, it has the following property:

$$\begin{cases} \|u_t(\cdot,t)\|_{L^\infty(\mathbb{R})} \le O(1)\dfrac{\delta}{t}, \qquad \|u_t(\cdot,t)\|_{L^1(\mathbb{R})} \le O(1)\dfrac{\delta}{t}, \\ |u_t(x+h,t)-u_t(x,t)| \le O(1)\dfrac{\delta}{t^{3/2}}|h| + O(1)\Big(\dfrac{\delta^2}{t^{3/2}}+\dfrac{\delta}{t}\Big)|h|\big|\log|h|\big|^2 \quad \text{for } |h|<1. \end{cases}$$

Proof We proceed with the proof in three steps.

• (Step 1: L^∞ estimate) We can follow the arguments in Lemma 4.3 to use the heat kernel to construct the representation

$$\begin{aligned} u_t(x,t) =& \int_{\mathbb{R}} H_t\Big(x,t;y,0;\frac{1}{v}\Big)u_0(y)\mathrm{d}y + \int_{\mathbb{R}\backslash\mathscr{D}} H_y\Big(x,t;y,t;\frac{1}{v}\Big)p(v(y,t))\mathrm{d}y \\ &+ \int_0^{\frac{t}{2}}\int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\Big(x,t;y,s;\frac{1}{v}\Big)p(v(y,s))\mathrm{d}y\mathrm{d}s \\ &+ \int_{\frac{t}{2}}^t\int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\Big(x,t;y,s;\frac{1}{v}\Big)\left(p(v(y,s))-p(v(y,t))\right)\mathrm{d}y\mathrm{d}s \\ &+ \int_{\mathbb{R}\backslash\mathscr{D}}\Big(\int_{\frac{t}{2}}^t H_{ty}\Big(x,t;y,s;\frac{1}{v}\Big)\mathrm{d}s\Big)p(v(y,t))\mathrm{d}y \end{aligned}$$

$$= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5. \tag{5.1}$$

Now, according to the estimate of $H_t(x,t;y,t_0)$ and $H_y(x,t;y,t) = -\delta'(x-y)$ in Section 2, Lemma A.1, we directly obtain the estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$ in (5.1) as follows:

$$|\mathcal{I}_1| \le O(1)\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*t}}}{t^{\frac{3}{2}}} u_0(y)\mathrm{d}y \le O(1)\frac{\delta}{t},$$
$$\mathcal{I}_2 = \int_{\mathbb{R}\setminus\mathscr{D}} H_y\left(x,t;y,t;\frac{1}{v}\right)p(v(y,t))\mathrm{d}y = -\partial_x\left(p(v(x,t))\right).$$

For $\mathcal{I}_3$ in (5.1), we note the vanishing of the integration of H_{ty} with respect to y. Then we apply the estimate of H_{ty} in Lemma A.1 in Section 2 and the zeroth order estimates of v in Proposition 4.8 to obtain that

$$|\mathcal{I}_3| = \int_0^{\frac{t}{2}}\int_{\mathbb{R}\setminus\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)\left(p(v(y,s)) - p(1)\right)\mathrm{d}y\mathrm{d}s$$
$$\le O(1)\delta\int_0^{\frac{t}{2}}\int_{\mathbb{R}\setminus\mathscr{D}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*(t-s)}}}{(t-s)^2}\mathrm{d}y\mathrm{d}s \le O(1)\frac{\delta}{\sqrt{t}}.$$

For $\mathcal{I}_4$ in (5.1), again by the estimate of H_{ty} and the time derivative estimate of v in Proposition 4.8, one obtains that

$$|\mathcal{I}_4| \le O(1)\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}\setminus\mathscr{D}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*(t-s)}}}{(t-s)^2}\frac{\delta(t-s)}{\sqrt{s}}\mathrm{d}y\mathrm{d}s \le O(1)\delta\int_{\frac{t}{2}}^{t}\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\mathrm{d}s \le O(1)\delta.$$

Next, for $\mathcal{I}_5$ in (5.1), we apply integration by parts with respect to y to obtain the following representation:

$$\mathcal{I}_5 = \int_{\mathbb{R}\setminus\mathscr{D}}\left(\int_{\frac{t}{2}}^{t} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)\mathrm{d}s - \delta'(x-y)\right)p(v(y,t))\mathrm{d}y + \int_{\mathbb{R}\setminus\mathscr{D}}\delta'(x-y)p(v(y,t))\mathrm{d}y$$
$$= -\int_{\mathbb{R}\setminus\mathscr{D}}\left(\int_{\frac{t}{2}}^{t} H_{t}\left(x,t;y,s;\frac{1}{v}\right)\mathrm{d}s + \delta(x-y)\right)p'(v(y,t))v_y(y,t)\mathrm{d}y$$
$$-\sum_{z\in\mathscr{D}}\left(\int_{\frac{t}{2}}^{t} H_{t}\left(x,t;\cdot,s;\frac{1}{v}\right)\mathrm{d}s + \delta(x-\cdot)\right)p(v(\cdot,t))\Big|_{z^-}^{z^+} + \partial_x p(v(x,t))$$
$$= \mathcal{I}_{51} + \mathcal{I}_{52} + \mathcal{I}_{53}.$$

Then we apply the time integral estimate of H_t with respect to s in Lemma A.1 and the BV estimate of v in Proposition 4.8 to obtain that

$$\mathcal{I}_{51} \le O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*t}}}{\sqrt{t}}|v_y(y,t)|\mathrm{d}y \le O(1)\frac{\delta}{\sqrt{t}},$$
$$\mathcal{I}_{52} \le O(1)\frac{1}{\sqrt{t}}\sum_{z\in\mathscr{D}}\left|p(v(\cdot,t))\Big|_{z^-}^{z^+}\right| \le O(1)\frac{\delta}{\sqrt{t}}.$$

Note that in $\mathcal{I}_2$ and $\mathcal{I}_{53}$, p_x only appears for $x\notin\mathscr{D}$, and it has no global L^∞ bound. Fortunately, when we combine $\mathcal{I}_2$ and $\mathcal{I}_{53}$, the two terms p_x cancel out. Therefore, we combine the estimates above to conclude that $\mathcal{I}_1$ is dominant in (5.1), and that

$$\|u_t(\cdot,t)\|_{L^\infty} \le O(1)\frac{\delta}{t}, \quad 0<t<t_\sharp.$$

• (Step 2: L^1 estimate) The L^1 estimate of u_t can be constructed in a similar manner as to L^∞ estimate. Actually, according to the cancellation in Step 1, we have the following estimate:

$$\begin{aligned}\int_{\mathbb{R}}|u_t(x,t)|\mathrm{d}x \leq & \int_{\mathbb{R}}\left|\int_{\mathbb{R}} H_t\left(x,t;y,0;\frac{1}{v}\right)u_0(y)\mathrm{d}y\right|\mathrm{d}x \\ & + \int_{\mathbb{R}}\left|\int_0^{\frac{t}{2}}\int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)p(v(y,s))\mathrm{d}y\mathrm{d}s\right|\mathrm{d}x \\ & + \int_{\mathbb{R}}\left|\int_{\frac{t}{2}}^t\int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)\left(p(v(y,s))-p(v(y,t))\right)\mathrm{d}y\mathrm{d}s\right|\mathrm{d}x \\ & + \int_{\mathbb{R}}\left|\int_{\mathbb{R}\backslash\mathscr{D}}\left(\int_{\frac{t}{2}}^t H_t\left(x,t;y,s;\frac{1}{v}\right)\mathrm{d}s+\delta(x-y)\right)\partial_y p(v(y,t))\mathrm{d}y\right|\mathrm{d}x \\ & + \int_{\mathbb{R}}\left|\sum_{z\in\mathscr{D}}\left(\int_{\frac{t}{2}}^t H_t\left(x,t;\cdot,s;\frac{1}{v}\right)\mathrm{d}s+\delta(x-\cdot)\right)p(v(\cdot,t))\Big|_{z^-}^{z^+}\right|\mathrm{d}x \\ = & \,\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5. \end{aligned} \tag{5.2}$$

For the estimates of the last four terms, we only need to add the integration of x to the terms in the L^∞ estimates in Step 1. More precisely, for the estimate of $\mathcal{I}_2$ in (5.2), we have that

$$\begin{aligned}|\mathcal{I}_2| &= \int_{\mathbb{R}}\left|\int_0^{\frac{t}{2}}\int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)\left(p(v(y,s))-1\right)\mathrm{d}y\mathrm{d}s\right|\mathrm{d}x \\ &\leq O(1)\int_0^{\frac{t}{2}}\int_{\mathbb{R}\backslash\mathscr{D}}\int_{\mathbb{R}}\frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*(t-s)}}}{(t-s)^2}\mathrm{d}x\,|(p(v(y,s))-1)|\,\mathrm{d}y\mathrm{d}s \leq O(1)\frac{\delta}{\sqrt{t}}.\end{aligned}$$

For the estimate of $\mathcal{I}_3$ in (5.2), we recall from the estimates in Proposition 4.8 that $v_t = u_x$ and that $\|u_x\|_{L^1}$ is of the order $\delta/\sqrt{t}$. Therefore, we obtain the following estimates:

$$\begin{aligned}\mathcal{I}_3 &= \int_{\mathbb{R}}\left|\int_{\frac{t}{2}}^t\int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)\left(p(v(y,s))-p(v(y,t))\right)\mathrm{d}y\mathrm{d}s\right|\mathrm{d}x \\ &\leq O(1)\int_{\frac{t}{2}}^t\int_{\mathbb{R}\backslash\mathscr{D}}\int_{\mathbb{R}}\frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*(t-s)}}}{(t-s)^2}\mathrm{d}x\left(\left|\int_s^t v_\sigma(y,\sigma)\mathrm{d}\sigma\right|\right)\mathrm{d}y\mathrm{d}s \\ &\leq O(1)\int_{\frac{t}{2}}^t\frac{1}{(t-s)^{\frac{3}{2}}}\left(\int_s^t\int_{\mathbb{R}\backslash\mathscr{D}}|u_y(y,\sigma)|\,\mathrm{d}y\mathrm{d}\sigma\right)\mathrm{d}s \\ &\leq O(1)\int_{\frac{t}{2}}^t\frac{1}{(t-s)^{\frac{3}{2}}}\frac{\delta(t-s)}{\sqrt{t}}\mathrm{d}s \leq O(1)\delta.\end{aligned}$$

For the estimates of $\mathcal{I}_4$ and $\mathcal{I}_5$ in (5.2), the integration of the heat kernel with respect to x yields a $\sqrt{t}$ factor. Thus, we have that

$$\begin{aligned}\mathcal{I}_4 &\leq O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_{\mathbb{R}}\frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*t}}}{\sqrt{t}}\mathrm{d}x\,(|v_y(y,t)|)\,\mathrm{d}y \leq O(1)\delta, \\ \mathcal{I}_5 &\leq O(1)\sum_{z\in\mathscr{D}}\int_{\mathbb{R}}\frac{\mathrm{e}^{-\frac{(x-y)^2}{C^*t}}}{\sqrt{t}}\mathrm{d}x\left|p(v(\cdot,t))\Big|_{z^-}^{z^+}\right| \leq O(1)\delta.\end{aligned}$$

Finally, for the estimate of $\mathcal{I}_1$ in (5.2), by the estimate of H_t in Lemma A.1, we have that

$$\mathcal{I}_1 \le O(1)\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C^* t}}}{t^{3/2}}|u_0(y)|\mathrm{d}y\mathrm{d}x \le O(1)\frac{\|u_0\|_{L^1}}{t} \le O(1)\frac{\delta}{t}. \tag{5.3}$$

Combining all of the estimates of $\mathcal{I}_i$ in (5.2) above, we find that $\mathcal{I}_1$ and $\mathcal{I}_2$ are dominant, hence we conclude that $\|u_t(\cdot,t)\|_{L^1(\mathbb{R})} \le O(1)\frac{\delta}{t}$.

• (Step 3: Hölder estimate of $u_t(x,t)$ in x) Starting with the representation u_t in (5.1), and evaluating at x and $x+h$ and taking the difference, we obtain that

$$\begin{aligned}
&u_t(x+h,t) - u_t(x,t)\\
&= \int_{\mathbb{R}}\Big[H_t\big(x+h,t;y,0;\frac{1}{v}\big) - H_t\big(x,t;y,0;\frac{1}{v}\big)\Big]u_0(y)\mathrm{d}y\\
&\quad + \int_{\mathbb{R}\setminus\mathscr{D}}\Big[H_y\big(x+h,t;y,t;\frac{1}{v}\big) - H_y\big(x,t;y,t;\frac{1}{v}\big)\Big]p(v(y,t))\mathrm{d}y\\
&\quad + \int_0^{\frac{t}{2}}\int_{\mathbb{R}\setminus\mathscr{D}}\Big[H_{ty}\big(x+h,t;y,s;\frac{1}{v}\big) - H_{ty}\big(x,t;y,s;\frac{1}{v}\big)\Big]p(v(y,s))\mathrm{d}y\mathrm{d}s\\
&\quad + \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}\setminus\mathscr{D}}\Big[H_{ty}\big(x+h,t;y,s;\frac{1}{v}\big) - H_{ty}\big(x,t;y,s;\frac{1}{v}\big)\Big]\left(p(v(y,s)) - p(v(y,t))\right)\mathrm{d}y\mathrm{d}s\\
&\quad + \int_{\mathbb{R}\setminus\mathscr{D}}\Big[\int_{\frac{t}{2}}^{t} H_{ty}\big(x+h,t;y,s;\frac{1}{v}\big)\mathrm{d}s - \int_{\frac{t}{2}}^{t} H_{ty}\big(x,t;y,s;\frac{1}{v}\big)\mathrm{d}s\Big]p(v(y,t))\mathrm{d}y\\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5.
\end{aligned} \tag{5.4}$$

In what follows, we will only show the detailed estimates on $\mathcal{I}_4$; the other estimates can be obtained in a similar manner.

For $\mathcal{I}_4$, using integration by parts, we transfer the y-derivative to pressure p to get that

$$\begin{aligned}
\mathcal{I}_4 = &\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}\setminus\mathscr{D}}\Big[H_t\big(x+h,t;y,s;\frac{1}{v}\big) - H_t\big(x,t;y,s;\frac{1}{v}\big)\Big]\left[\partial_y p(v(y,t)) - \partial_y p(v(y,s))\right]\mathrm{d}y\mathrm{d}s\\
&+ \int_{\frac{t}{2}}^{t}\sum_{\alpha\in\mathscr{D}}\Big[H_t\big(x+h,t;\alpha,s;\frac{1}{v}\big) - H_t\big(x,t;\alpha,s;\frac{1}{v}\big)\Big]\Big[p(v(\cdot,t)) - p(v(\cdot,s))\Big]_{\alpha^-}^{\alpha^+}\mathrm{d}s.
\end{aligned} \tag{5.5}$$

To take care of the singularity of the heat kernel when s is close to t, we need to gain some factor of $(t-s)$. Thanks to the Hölder estimate (4.38), we have that

$$\begin{aligned}
&\int_{\mathbb{R}\setminus\mathscr{D}} |\partial_y p(v(y,t)) - \partial_y p(v(y,s))|\,\mathrm{d}y\\
&\lesssim \int_{\mathbb{R}\setminus\mathscr{D}}\Big[|p'(v(y,t)) - p'(v(y,s))|\,|v_y(y,t)| + |p'(v(y,s))|\,|v_y(y,t) - v_y(y,s)|\Big]\mathrm{d}y\\
&\lesssim \frac{\delta^2(t-s)}{\sqrt{t}} + \delta\frac{t-s}{t} + \delta(t-s)|\log(t-s)|.
\end{aligned} \tag{5.6}$$

For the second term in (5.5), direct calculations show that

$$\begin{aligned}
&\Big[p(v(\cdot,t)) - p(v(\cdot,s))\Big]_{\alpha^-}^{\alpha^+}\\
&= \int_s^t \Big[p'(v(\alpha^+,\sigma))v_\sigma(\alpha^+,\sigma) - p'(v(\alpha^-,\sigma))v_\sigma(\alpha^-,\sigma)\Big]\mathrm{d}\sigma\\
&= \int_s^t \Big[\big(p'(v(\alpha^+,\sigma)) - p'(v(\alpha^-,\sigma))\big)v_\sigma(\alpha^+,\sigma) + p'(v(\alpha^-,\sigma))\big(v_\sigma(\alpha^+,\sigma) - v_\sigma(\alpha^-,\sigma)\big)\Big]\mathrm{d}\sigma.
\end{aligned}$$

In view of $v_\sigma(z,\sigma)=u_z(z,\sigma)$ and the continuity of flux $p+\frac{\mu u_x}{v}$, one can rewrite things as

$$
\begin{aligned}
&v_\sigma(\alpha^+,\sigma)-v_\sigma(\alpha^-,\sigma)\\
=&\,u_x(\alpha^+,\sigma)-u_x(\alpha^-,\sigma)\\
=&\,\frac{v(\alpha^+,\sigma)}{\mu}\big(p(v(\alpha^+,\sigma))-p(v(\alpha^-,\sigma))\big)+\frac{v(\alpha^+,\sigma)-v(\alpha^-,\sigma)}{v(\alpha^-,\sigma)}u_x(\alpha^-,\sigma).
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\sum_{\alpha\in\mathscr{D}}\left|\Big[p(v(\cdot,t))-p(v(\cdot,s))\Big]_{\alpha^-}^{\alpha^+}\right|\\
\lesssim&\int_s^t\sum_{\alpha\in\mathscr{D}}\left|\big(p'(v(\alpha^+,\sigma))-p'(v(\alpha^-,\sigma))\big)v_\sigma(\alpha^+,\sigma)\right|\mathrm{d}\sigma\\
&+\int_s^t\sum_{\alpha\in\mathscr{D}}\left|p'(v(\alpha^-,\sigma))\frac{v(\alpha^+,\sigma)}{\mu}\big(p(v(\alpha^+,\sigma))-p(v(\alpha^-,\sigma))\big)\right|\mathrm{d}\sigma\\
&+\int_s^t\sum_{\alpha\in\mathscr{D}}\left|p'(v(\alpha^-,\sigma))\frac{v(\alpha^+,\sigma)-v(\alpha^-,\sigma)}{v(\alpha^-,\sigma)}u_x(\alpha^-,\sigma)\right|\mathrm{d}\sigma\\
\lesssim&\int_s^t\Big[\delta\frac{\delta}{\sqrt{\sigma}}+\delta+\delta\frac{\delta}{\sqrt{\sigma}}\Big]\mathrm{d}\sigma\lesssim\delta(t-s)+\frac{\delta^2(t-s)}{\sqrt{t}}.
\end{aligned}\tag{5.7}
$$

Plugging (5.6) and (5.7) into (5.5) and using Lemma 3.2, we obtain that

$$
\begin{aligned}
|\mathcal{I}_4|\lesssim&\int_{\frac{t}{2}}^t\frac{\delta}{(t-s)\sqrt{t}}|h|\,|\log|h||^2\,\delta\Big[\frac{t-s}{t}+(t-s)|\log(t-s)|\Big]\mathrm{d}s\\
&+\int_{\frac{t}{2}}^t\min\left(1,\frac{|h|}{\sqrt{t-s}}\right)\frac{1}{(t-s)^{3/2}}\delta\Big[\frac{t-s}{t}+(t-s)|\log(t-s)|\Big]\mathrm{d}s\\
\lesssim&\,\delta^2|h|\,|\log|h||^2\left[\frac{1}{\sqrt{t}}+\sqrt{t}|\log t|\right]+\frac{\delta}{t}|h||\log|h||+\delta|h||\log|h||^2.
\end{aligned}\tag{5.8}
$$

Then, we apply similar criteria for $\mathcal{I}_1$, $\mathcal{I}_3$ and $\mathcal{I}_5$, and obtain the following estimates:

$$
|\mathcal{I}_1|\lesssim\frac{\delta}{t^{3/2}}|h|+\frac{\delta^2}{t^{3/2}}|h|\,|\log|h||^2,\tag{5.9}
$$

$$
|\mathcal{I}_3|\lesssim\frac{\delta^2}{t}|h|\big|\log|h|\big|^2+\frac{\delta}{t}|h|,\tag{5.10}
$$

$$
|\mathcal{I}_5+\mathcal{I}_2|\lesssim\frac{\delta^2}{t}|h|\,|\log|h||^2+\delta\frac{|h|}{t^{\frac{3}{2}}}+\delta\frac{|h||\log|h||^2}{t}.\tag{5.11}
$$

Now, we combine (5.4), (5.8), (5.9), (5.10) and (5.11) to conclude that, for $0<t\ll1$ and $|h|<1$,

$$
|u_t(x+h,t)-u_t(x,t)|\lesssim\frac{\delta}{t^{3/2}}|h|+\left(\frac{\delta^2}{t^{3/2}}+\frac{\delta}{t}\right)|h|\big|\log|h|\big|^2.
$$

□

Now we are ready to give a rigorous proof of Theorem 1.2.

Proof of Theorem 1.2 The existence and regularity of u has been investigated in Proposition 4.8 and Lemma 5.1. Therefore, we only need to take care of the regularity of v. For the first assertion, we combine the strong convergence in Proposition 4.8 and the Hölder estimate of the total variation of V^n in Lemma 4.1 to imply that

$$\int_{\mathbb{R}\setminus\mathscr{D}} |v_x(x,t) - v_x(x,s)|\mathrm{d}x \le 2C_\sharp\delta\Big[\frac{t-s}{t}\Big], \quad 0<s<t.$$

This, together with the regularity obtained in Proposition 4.8 and Lemma 5.1, gives that the constructed solution with initial condition (1.4) lies in the function space (1.5).

Next, for the second assertion, u_0^* satisfies the stronger assumption $\|u_0^*\|_{L^1\cap \mathrm{BV}} \ll 1$. Therefore, by a similar argument as to that of Remark 4.2, one can obtain the following estimates of u:

$$\|u_x(\cdot,t)\|_{L^1_x} \le O(1)\delta, \quad \|u_t(\cdot,t)\|_{L^1_x} \le O(1)\frac{\delta}{\sqrt{t}}.$$

Thus the solution gains more regularity in t than Proposition 4.8 and Lemma 5.1. Then, with similar arguments as to those of Remark 4.2, we can refine the estimate of the total variation of v and obtain the key estimate

$$\int_{\mathbb{R}\setminus\mathscr{D}} |v_x(x,t) - v_x(x,s)|\mathrm{d}x \le 2C_\sharp\delta\Big[\frac{t-s}{\sqrt{t}}\Big], \quad 0<s<t,$$

which is Hölder continuous in the time variable in $[0,t_\sharp]$. Thus the solution belongs to (1.6), and we have finished the proof of the theorem. □

5.2 Uniqueness and Proof of Theorem 1.3

In this part, we will continue to study the stability of the weak solution constructed in Proposition 4.8 and Theorem 1.2; this will imply the continuous dependence on initial data and the uniqueness of the solution. We first state that the weak solution continuously depends on the initial data without proof, provided that it satisfies the smallness condition (4.37).

Lemma 5.2 Let (v_0^a, u_0^a) and (v_0^b, u_0^b) be the initial data satisfying (4.1). Let (v^a, u^a) and (v^b, u^b) be two weak solutions, in the sense of Definition 2.1, to the Navier-Stokes equation (1.1), and assume that they satisfy the smallness properties (4.37) in Proposition 4.8.

Then, there exists a positive constant $C_\flat$ such that we have the following error estimates showing the stability of the solution:

$$\begin{aligned}&\mathscr{F}\left[v^a - v^b, u^a - u^b\right]\\ \le\, & C_\flat\Big(\|u_0^a - u_0^b\|_{L^\infty_x} + \|u_0^a - u_0^b\|_{L^1_x} + \|v_0^a - v_0^b\|_{L^1_x} + \|v_0^a - v_0^b\|_{L^\infty_x} + \|v_0^a - v_0^b\|_{\mathrm{BV}}\Big).\end{aligned}$$

Here $\mathscr{F}$ is the functional defined in (4.39) of Proposition 4.8. Moreover, the following L^1 stability holds:

$$\big|\!\big|\!\big|v^a - v^b\big|\!\big|\!\big|_1 + \big|\!\big|\!\big|u^a - u^b\big|\!\big|\!\big|_1 \le C_\flat\Big(\|v_0^a - v_0^b\|_{L^1_x} + \|u_0^a - u_0^b\|_{L^1_x}\Big).$$

Remark 5.3 The proof of this Lemma is similar to that in [1], where a stronger assumption $u_0 \in L^1 \cap \mathrm{BV}$ is assumed. Indeed, thanks to the smallness requirement, one can construct an associated BV coefficient heat kernel and obtain integral representations of the solutions (v^a, u^a) and (v^b, u^b), and then follow the arguments in Lemma 4.6 and Lemma 4.7 to complete the proof. In particular, the weak solution constructed in Proposition 4.8 automatically satisfies the smallness condition (4.37), which implies that it depends on initial data continuously.

Lemma 5.2 shows the continuous dependence on the initial data of the solution, if it satisfies the smallness condition (4.37). However, given initial data satisfying (4.1), and a weak solution

(v, u) in the function space (1.5), we do not know in advance whether or not the weak solution is small, thus do not know whether or not our constructed solution is unique in function space (1.5). In the next lemma, by further assuming that $\|u_0^*\|_{\rm BV} < \delta$, we prove that all possible solutions to (1.1) will be small in the function space (1.6).

Lemma 5.4 Let the initial data satisfy the following smallness condition for small δ_*:

$$\|v_0\|_{\rm BV} + \|v_0 - 1\|_{L^1} + \|u_0\|_{\rm BV} + \|u_0\|_{L^1} < \delta_* \ll 1.$$

Let (v, u) be any weak solution to (1.1) in the function space (1.6). Moreover, let $C_\sharp$ and δ be the parameters given in Proposition 4.8 and Theorem 1.2. Then, if δ_* is sufficiently small, there exists a small positive constant t_* such that the following smallness properties hold for the solution:

$$\max\Big\{\|u(\cdot,t)\|_{L^1_x},\ \|u(\cdot,t)\|_{L^\infty_x},\ \|u_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L^\infty_x}\Big\} \le 2C_\sharp\delta, \quad 0 < t < t_*.$$

Proof According to Proposition 4.8 and Theorem 1.2, at least one weak solution exists in the space (1.6), provided that the initial data is sufficiently small. If the solution is exactly the one we constructed before, then the smallness of the solution immediately follows from Theorem 1.2 and we are done. In general, suppose that the solution satisfies condition (1.6) with sufficiently small initial data. Then, we first note in Remark 1.5 that $v(x, t)$ is continuous in the short time. Therefore, there exists a small t_* such that

$$\|v(\cdot,t) - 1\|_{L^1_x} \le C\delta_*, \quad \|v(\cdot,t) - 1\|_{L^\infty_x} \le C\delta_*, \quad \|v(\cdot,t) - 1\|_{\rm BV} \le C\delta_*, \quad 0 \le t < t_* \ll 1,$$

which provides the smallness of $v(x, t)$ in the short time. Then, as δ_* and t_* are sufficiently small, we can follow the arguments in Section 2 to construct the heat kernel $H(x, t; y, \tau; \frac{\mu}{v})$. Then, mulitiplying $H(x, t; y, \tau; \frac{\mu}{v})$ to the second equation in (1.1) and using integration by parts to yield an integral representation of u,

$$u(x,t) = \int_{\mathbb{R}} H(x,t;y,0;\frac{\mu}{v})u(y,0){\rm d}y + \int_0^t\int_{\mathbb{R}} H_y(x,t;y,\tau;\frac{\mu}{v})p(y,\tau){\rm d}y{\rm d}\tau. \tag{5.12}$$

In what follows, we will show the smallness of $u(x, t)$ in short time. Actually, the estimates of $u(x, t)$ are very similar to Lemma 4.3. First, for the zeroth order estimates, we directly apply the representation (5.12), the properties of H, and the regularity of the solution to obtain, for $t < t_* \ll 1$, that

$$\begin{aligned}
|u(x,t)| &\le \int_{\mathbb{R}} \Big|H(x,t;y,0;\frac{\mu}{v})\Big|\,|u(y,0)|{\rm d}y + \int_0^t\int_{\mathbb{R}} \Big|H_y(x,t;y,\tau;\frac{\mu}{v})\Big|\,|p(y,\tau) - p(1)|{\rm d}y{\rm d}\tau \\
&\le O(1)\delta_* + O(1)\sqrt{t}, \\
\|u(x,t)\|_{L^1} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \Big|H(x,t;y,0;\frac{\mu}{v})\Big|\,|u(y,0)|{\rm d}y{\rm d}x \\
&\quad + \int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} \Big|H_y(x,t;y,\tau;\frac{\mu}{v})\Big|\,|p(y,\tau) - p(1)|{\rm d}y{\rm d}\tau{\rm d}x \\
&\le O(1)\delta_* + O(1)\sqrt{t}.
\end{aligned}$$

Then, for the first order estimates of $u(x, t)$, we deal with the L^∞ estimate first. Actually, we follow the representation (5.12) to obtain that

$$u_x(x,t) = \int_{\mathbb{R}} H_x(x,t;y,0;\frac{\mu}{v})u(y,0){\rm d}y + \int_0^t\int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})p(y,\tau){\rm d}y{\rm d}\tau. \tag{5.13}$$

For the homogeneous term, the initial data will provide the small factor δ, and we follow the proof in Lemma 4.3 to get that

$$\Big|\int_{\mathbb{R}} H_x(x,t;y,0;\frac{\mu}{v})u(y,0)\mathrm{d}y\Big| \le O(1)\frac{\delta_*}{\sqrt{t}}. \tag{5.14}$$

Then, for the inhomogeneous term, we also follow (4.8) in the proof of Lemma 4.3 to obtain the estimates. Actually, as the estimates are very similar to the estimates of (4.8), we omit the details and provide the following estimates:

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})p(y,\tau)\mathrm{d}y\mathrm{d}\tau\Big| \\
&\le \Big|\int_0^t\int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})p(y,t)\mathrm{d}y\mathrm{d}\tau\Big| + \Big|\int_0^t\int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})\left(p(y,\tau)-p(y,t)\right)\mathrm{d}y\mathrm{d}\tau\Big| \\
&\le O(1) + O(1)\int_0^t\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}\left(\int_\tau^t |v_\sigma(y,\sigma)|\mathrm{d}\sigma\right)\mathrm{d}y\mathrm{d}\tau \\
&\le O(1) + O(1)\int_0^t \frac{1}{(t-\tau)}\left(\int_\tau^t \frac{1}{\sqrt{\sigma}}\mathrm{d}\sigma\right)\mathrm{d}\tau \le O(1)(1+\sqrt{t}).
\end{aligned} \tag{5.15}$$

Now, since $t < t_* \ll 1$, we combine (5.13), (5.14) and (5.15), and let t_* be sufficiently small, to obtain that

$$|u_x(x,t)| \le O(1)\frac{\delta_*}{\sqrt{t}} + O(1) \le O(1)\frac{\delta_* + \sqrt{t_*}}{\sqrt{t}}, \quad 0 < t < t_*.$$

Similarly, we have the L^1 estimates as follows:

$$\|u_x(\cdot,t)\|_{L^1} \le O(1)(\delta_* + \sqrt{t_*}), \quad 0 < t < t_*.$$

Now, we combine all of the above zeroth and first order estimates to conclude that, for sufficiently small δ_* and t_*, the following estimates hold:

$$\max\left\{\|u(\cdot,t)\|_{L^1_x},\ \|u(\cdot,t)\|_{L^\infty_x},\ \|u_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L^\infty_x}\right\} \le 2C_\sharp\delta, \quad 0 < t < t_*.$$

Here $C_\sharp$ and δ are given as in Proposition 4.8 and Theorem 1.2. □

Lemma 5.4 shows that if the initial data is sufficiently small, then for any weak solution in the function space (1.6), we can find a small time t_* such that the weak solution will be as small as the solution constructed in Proposition 4.8 and Theorem 1.2 in the short time period $t \in [0, t_*]$. Now, with Lemma 5.2 and Lemma 5.4 in hand, we are ready to show the proof of the second main result, Theorem 1.3, regarding the local-in-time stability and uniqueness of the weak solution.

Proof of Theorem 1.3 Taking all of the above, Proposition 4.8 and Theorem 1.2 guarantee the existence of the two weak solutions if $\delta_* < \delta$ and $t < t_\sharp$, where δ and $t_\sharp$ are given as in Proposition 4.8 and Theorem 1.2. Next, Lemma 5.4 guarantees the smallness of the two solutions for $t \in [0, t_*)$. Therefore, for $t \in [0, t_*)$, we can apply Lemma 5.2 to obtain the error estimates

$$\begin{aligned}
&\mathscr{F}\left[v^a - v^b, u^a - u^b\right] \\
&\le C_\flat\Big(\|u_0^a - u_0^b\|_{L^\infty_x} + \|u_0^a - u_0^b\|_{L^1_x} + \|v_0^a - v_0^b\|_{L^1_x} + \|v_0^a - v_0^b\|_{L^\infty_x} + \|v_0^a - v_0^b\|_{\mathrm{BV}}\Big),\ 0 < t < t_*.
\end{aligned}$$

This finishes the proof of the first part. Then, if the two solutions have common initial data, one immediately has that $\mathscr{F}\left[v^a - v^b, u^a - u^b\right] = 0$, which shows that the two solutions coincide with each other almost everywhere. □

6 Summary

In the present paper, we have studied the regularity and uniqueness of the weak solution to the isentropic compressible Navier-Stokes equation (1.1). The key point is the representation of the solution by heat kernels with variable BV coefficients and the corresponding Hölder estimates. The delicate estimates of the heat kernels allowed us to gain the existence, regularity and uniqueness of the solution to (1.1) in local time. Finally, combining the estimates of Green's function to (1.1) linearized around a constant state, we have followed the criteria in [1] to conclude the global existence of the solution. We refer to [1, 24–27] for details regarding the Green's function and global existence. Moreover, this method can further yield a pointwise structure of the solution, and can also be naturally extended to the non-isentropic case, which will be left to our future works.

Conflict of Interest The authors declare no conflict of interest.

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Appendix

1. Estimates of Heat Kernel

Lemma A.1 (Pointwise estimate of heat kernel, [1]) Suppose that the conditions of ρ in (2.3) hold. Then the heat equation (2.2) admits a weak solution in the distribution sense.

Moreover, there exist positive constants $t_\sharp \ll 1$ and C_* such that the following pointwise estimates hold for the heat kernel when $t_0 < t < t_0 + t_\sharp$:

$$|H(x,t;y,t_0;\rho)| \le C_* \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \qquad t_0 < t < t_0 + t_\sharp,$$

$$|H_x(x,t;y,t_0;\rho)| + |H_y(x,t;y,\mu)| \le C_* \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}, \qquad t_0 < t < t_0 + t_\sharp,$$

$$|H_{xy}(x,t;y,t_0;\rho)| + |H_t(x,t;y,t_0;\rho)| \le C_* \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{\frac{3}{2}}}, \qquad t_0 < t < t_0 + t_\sharp,$$

$$|H_{ty}(x,t;y,t_0;\rho)| \le C_* \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^2}, \qquad t_0 < t < t_0 + t_\sharp.$$

Furthermore, we have the following estimates for the time integration of the heat kernel when $t_0 < t < t_0 + t_\sharp$:

$$\left|\int_{t_0}^t H_x(x,\tau;y,t_0;\rho)\mathrm{d}\tau\right|, \left|\int_{t_0}^t H_x(x,t;y,s;\rho)\mathrm{d}s\right| \le C_* \mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}},$$

$$\left|\int_{t_0}^t H_{xy}(x,\tau;y,t_0;\rho)\mathrm{d}\tau - \frac{\delta(x-y)}{\rho(x,t_0)} - \int_{t_0}^t \frac{\rho(x,t_0)-\rho(x,\tau)}{\rho(x,t_0)} H_{xy}(x,\tau;y,t_0;\rho)\mathrm{d}\tau\right| \le C_* \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}},$$

$$\left|\int_{t_0}^{t} H_{xy}(x,t;y,s;\rho)\mathrm{d}s + \frac{\delta(x-y)}{\rho(y,t)} - \int_{t_0}^{t} \frac{\rho(y,t)-\rho(y,s)}{\rho(y,t)} H_{xy}(x,t;y,s;\rho)\mathrm{d}s\right| \le C_* \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}},$$

$$\int_{t_0}^{t} H_{xx}(x,\tau;y,t_0;\rho)\mathrm{d}\tau = -\frac{\delta(x-y)}{\rho(x,t_0)} - \frac{1}{\rho(x,t_0)}\partial_x\left[\int_{t_0}^{t}(\rho(x,\tau)-\rho(x,t_0))\,H_x(x,\tau;y,t_0;\rho)\mathrm{d}\tau\right]$$
$$+O(1)\left(|\partial_x\rho(x,t_0)|\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}} + \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\right), \quad \text{for } x \notin \mathscr{D},$$

$$\int_{t_0}^{t} H_{xxy}(x,\tau;y,t_0;\rho)\mathrm{d}\tau = \frac{1}{\rho(x,t_0)}\left[\delta'(x-y) - \int_{t_0}^{t}\partial_x\Big[\big(\rho(x,\tau)-\rho(x,t_0)\big)H_{xy}(x,\tau;y,t_0;\rho)\Big]\mathrm{d}\tau\right]$$
$$-\frac{\partial_x\rho(x,t_0)}{\rho^2(x,t_0)}\left[\delta(x-y) - \int_{t_0}^{t}\big(\rho(x,\tau)-\rho(x,t_0)\big)H_{xy}(x,\tau;y,t_0;\rho)\mathrm{d}\tau\right]$$
$$+O(1)\left(|\partial_x\rho(x,t_0)|\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}} + \frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\right), \quad \text{for } x \notin \mathscr{D},$$

$$\int_{t_0}^{t} H_t(x,t;y,s;\rho)\mathrm{d}s = H(x,t-t_0;y;\mu^t) - \delta(x-y) + O(1)\delta_*\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}, \quad \mu^t \equiv \rho(\cdot,t),$$

$$\int_{t_0}^{t} H_{ty}(x,t;y,s;\rho)\mathrm{d}s = \int_{t_0}^{t}\frac{\rho(y,t)-\rho(y,s)}{\rho(y,t)}H_{ty}(x,t;y,s;\rho)\mathrm{d}s$$
$$+\frac{1}{\rho(y,t)}\int_{-\infty}^{y} H_t(x,t;\xi,t_0;\rho)\mathrm{d}\xi - H_y(x,t;y,t;\rho).$$

Note that the estimates for the terms involving twice x-derivatives do not hold when $x \in \mathscr{D}$, which is due to the appearance of the Dirac delta functions in H_{xx} if $x \in \mathscr{D}$. Moreover, the zeroth order estimate can actually be extended to global time, while the higher order estimates have only been obtained for the local time so far.

In addition to the estimate of fundamental solution itself, we also need the comparison estimate of two fundamental solutions to heat equation (2.2), associated with different heat conductivities ρ^a and ρ^b.

Lemma A.2 (Comparison estimates [1]) Suppose that the conditions in (2.3) hold for ρ^a and ρ^b. Then there exist positive constants $t_\sharp \ll 1$ and C_* such that the following estimates hold when $t_0 < t < t_0 + t_\sharp$:

$$\big|H(x,t;y,t_0;\rho^b) - H(x,t;y,t_0;\rho^a)\big| \le C_*\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_\infty,$$

$$\big|H_x(x,t;y,t_0;\rho^a) - H_x(x,t;y,t_0;\rho^b)\big|,\ \big|H_y(x,t;y,t_0;\rho^a) - H_y(x,t;y,t_0;\rho^b)\big|$$
$$\le C_*\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\Bigg[|\log(t-t_0)|\,\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_\infty + \big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_{\mathrm{BV}}$$
$$+\sqrt{t-t_0}\Big(\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_1 + |\log t|\,\Big|\!\Big|\!\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\left[\rho^a-\rho^b\right]\Big|\!\Big|\!\Big|_\infty\Big)\Bigg],$$

$$\big|H_{xy}(x,t;y,t_0;\rho^a) - H_{xy}(x,t;y,t_0;\rho^b)\big|,\ \big|H_t(x,t;y,t_0;\rho^a) - H_t(x,t;y,t_0;\rho^b)\big|$$
$$\le C_*\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{3/2}}\Bigg[|\log(t-t_0)|\,\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_\infty + \big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_{\mathrm{BV}}$$

$$+\sqrt{t-t_0}\Big(\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_1+|\log t|\,\Big|\!\Big|\!\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\left[\rho^a-\rho^b\right]\Big|\!\Big|\!\Big|_\infty\Big)\Big].$$

Furthermore, we also have comparison estimates for time integrals of the heat kernel as follows:

$$\left|\int_{t_0}^t\left[H_x(x,\tau;y,t_0;\rho^a)-H_x(x,\tau;y,t_0;\rho^b)\right]\mathrm{d}\tau\right|,\ \left|\int_{t_0}^t\left[H_y(x,t;y,s;\rho^a)-H_y(x,t;y,s;\rho^b)\right]\mathrm{d}s\right|$$
$$\le C_*\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}\left[\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_\infty+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_{\mathrm{BV}}+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_1+\Big|\!\Big|\!\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\Big|\!\Big|\!\Big|_\infty\right],$$

$$\left|\int_{t_0}^t\left[H_x(x,t;y,s;\rho^a)-H_x(x,t;y,s;\rho^b)\right]\mathrm{d}s\right|,\ \left|\int_{t_0}^t\left[H_y(x,\tau;y,t_0;\rho^a)-H_y(x,\tau;y,t_0;\rho^b)\right]\mathrm{d}\tau\right|$$
$$\le C_*\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}\left[\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_\infty+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_{\mathrm{BV}}+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_1+\Big|\!\Big|\!\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\Big|\!\Big|\!\Big|_\infty\right],$$

$$\int_{t_0}^t\left[H_{xy}(x,\tau;y,t_0;\rho^a)-H_{xy}(x,\tau;y,t_0;\rho^b)\right]\mathrm{d}\tau$$
$$=\left[\frac{1}{\rho^a(x,t_0)}-\frac{1}{\rho^b(x,t_0)}\right]\delta(x-y)-\int_{t_0}^t\Big[\frac{\rho^a(x,\tau)-\rho^a(x,t_0)}{\rho^a(x,t_0)}H_{xy}(x,\tau;y,t_0;\rho^a)$$
$$-\frac{\rho^b(x,\tau)-\rho^b(x,t_0)}{\rho^b(x,t_0)}H_{xy}(x,\tau;y,t_0;\rho^b)\Big]\mathrm{d}\tau+O(1)\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\Big[|\log(t-t_0)|\,\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_\infty$$
$$+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_{\mathrm{BV}}+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_1+\Big|\!\Big|\!\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\Big|\!\Big|\!\Big|_\infty\Big],$$

$$\int_{t_0}^t\left[H_{xy}(x,t;y,s;\rho^a)-H_{xy}(x,t;y,s;\rho^b)\right]\mathrm{d}s$$
$$=\left[\frac{1}{\rho^a(y,t)}-\frac{1}{\rho^b(y,t)}\right]\delta(x-y)+\int_{t_0}^t\Big[\frac{\rho^a(y,t)-\rho^a(y,s)}{\rho^a(y,t)}H_{xy}(x,t;y,s;\rho^a)$$
$$-\frac{\rho^b(y,t)-\rho^b(y,s)}{\rho^b(y,t)}H_{xy}(x,t;y,s;\rho^b)\Big]\mathrm{d}s+O(1)\frac{\mathrm{e}^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\Big[|\log(t-t_0)|\,\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_\infty$$
$$+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_{\mathrm{BV}}+\big|\!\big|\!\big|\rho^a-\rho^b\big|\!\big|\!\big|_1+\Big|\!\Big|\!\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\Big|\!\Big|\!\Big|_\infty\Big].$$

Next, according to the symmetry of the heat equation, we can obtain the following identities of the heat kernel (as the proof can be constructed directly from proper integration of the equation (2.2), we omit the details):

Lemma A.3 For the heat equation in conservative form (2.2), the solution has the following properties:

$$\begin{cases}\displaystyle\int_{\mathbb{R}}H(x,t;y,\tau;\rho)\mathrm{d}x=\int_{\mathbb{R}}H(x,t;y,\tau;\rho)\mathrm{d}y=1,\\ \displaystyle\int_{\mathbb{R}}H_x(x,t;y,\tau;\rho)\mathrm{d}x=\int_{\mathbb{R}}H_x(x,t;y,\tau;\rho)\mathrm{d}y=0,\\ \displaystyle\int_{\mathbb{R}}H_y(x,t;y,\tau;\rho)\mathrm{d}x=\int_{\mathbb{R}}H_y(x,t;y,\tau;\rho)\mathrm{d}y=0,\\ \displaystyle\int_{\mathbb{R}}H_t(x,t;y,\tau;\rho)\mathrm{d}x=\int_{\mathbb{R}}H_t(x,t;y,\tau;\rho)\mathrm{d}y=0,\\ \displaystyle\int_{\mathbb{R}}H_\tau(x,t;y,\tau;\rho)\mathrm{d}x=\int_{\mathbb{R}}H_\tau(x,t;y,\tau;\rho)\mathrm{d}y=0.\end{cases}$$